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1 Introduction

Meshless methods are a group of numerical algorithms that serve for solving boundary value problems. These methods are alternative to the popular and efficient FEM. The greatest advantage of meshless methods is that they need no connectivity condition, like the FEM, in the beginning of computation.

We can specify meshless methods as Galerkin methods where basis functions are replaced by shape functions built in a special way. The construction of the shape functions differs for different meshless methods. Some shape functions are approximations of the kernel in the integral transform

$$u(x) = \int_\Omega K(x,y)u(y)\,\mathrm{d}y \tag{1}$$

(see [10], [4]). Some are constructed by means of the moving least squares method (see [3], [8]). Shape functions based on the idea of partition of unity, that are a composition of an extrinsic and an intrinsic basis form, form the next specific group (see [2], [12]).

The meshless methods have received their place among numerical techniques. They were used for instance in solving problems from mechanic of solid body (see [4]), biomechanics (see [1]) or structural dynamic (see [9]). They are successfully used in the modelling of large deformations, crack propagation or moving boundary. A serious limitation is the fact that the meshless methods do not reproduce the Dirichlet, more generally essential boundary conditions.

Several attempts to solve the problem involving Dirichlet conditions are discussed in this contribution. Our attention will be focused only on one of meshless methods – the reproducing kernel particle method (briefly the RKP method or the RKPM). The construction of the RKP-shape functions and an application of the RKPM to an elliptic boundary value problem are presented in Section 2. Methods that enable to realize the Dirichlet condition in the RKPM are introduced in Section 3.

2 RKPM approximation

Consider the problem

$$-\Delta u(x) = f(x) \quad \text{in} \quad \Omega \subset \mathbb{R}^n, \tag{2}$$

$$\frac{\partial u}{\partial n}(x) = g(x) \quad \text{on} \quad \partial\Omega_1, \tag{3}$$

$$u(x) = u^0(x) \quad \text{on} \quad \partial\Omega_0. \tag{4}$$

Denote

$$V = \{v \in W^{1,2}(\Omega) |\ v(x) = 0 \text{ on } \partial\Omega_0 \text{ in the sense of traces}\}$$

and find a weak solution $u \in W^{1,2}(\Omega)$ of the problem (2)–(4) such that

$$u - u^0 \in V,$$

$$\int_\Omega \sum_{i=1}^{n} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} \,\mathrm{d}x = \int_\Omega fv \,\mathrm{d}x + \int_{\partial\Omega_1} gv \,\mathrm{d}s,\ \forall v \in V. \tag{5}$$

The numerical solution of the problem (2)–(4) will be constructed at points $x^1, \ldots, x^N \in \Omega$. At first it is necessary to choose the monomial basis p of degree s and some one-dimensional weight function Φ_1.

Definition 1 The points $x^1, \ldots, x^N \in \Omega$, which are used for construction of the RKP approximation, are called particles.

Remark 1 The particles $x^1, \ldots, x^N$ differ mutually, they can be distributed uniformly or nonuniformly.

Remark 2 For example, $p(x) = (1, x_1, x_2, x_3, x_1x_2, x_1x_3, x_2x_3, x_1^2, x_2^2, x_3^2)^{\mathrm{T}}$ is the second degree monomial basis in $\mathbb{R}^3$.

The space generated by monomials of degree less than or equal to s will be denoted by $\mathbb{P}^s$.

Remark 3 The most often weight functions chosen are the *Gaussian function*

$$\Phi_1(x) = \begin{cases} e^r(x^2-1)/(1-e^r) & \text{if} \quad |x| \le 1, \\ 0 & \text{if} \quad |x| > 1, \end{cases}$$

with $r > 0$, the *cubic spline*

$$\Phi_1(x) = \begin{cases} \frac{2}{3} - 4x^2 + 4|x|^3 & \text{if} \quad |x| \le \frac{1}{2}, \\ \frac{4}{3} - 4|x| + 4x^2 - \frac{4}{3}|x|^3 & \text{if} \quad \frac{1}{2} < |x| \le 1, \\ 0 & \text{if} \quad |x| > 1 \end{cases}$$

and the *conic function*

$$\Phi_1(x) = \begin{cases} (1-x^2)^k & \text{if} \quad |x| \le 1, \\ 0 & \text{if} \quad |x| > 1, \end{cases}$$

such that $k > 1$. See [2], [5], [4].

The n-dimensional weight function can be constructed from a one-dimensional weight function Φ_1 by putting

$$\Phi(x) = \prod_{i=1}^{n} \Phi_1(x_i), \text{ where } x = (x_1, x_2, \ldots, x_n).$$

Definition 2 Let the particles $x^1, \dots, x^N \in \Omega$, the degree s of the monomial basis p and the weight function Φ_1 be given. Interpolants constructed by means of the RKPM are the linear combinations

$$\tilde{u}(x) = \sum_{I=1}^{N} \Psi_I(x) u_I \tag{6}$$

of the RKP shape-functions Ψ_I with coefficients u_I. The shape functions are of the form

$$\Psi_I(x) = p^T \left(\frac{x^I - x}{\rho} \right) b(x)\, \Phi \left(\frac{x^I - x}{\rho} \right) \Delta V_I. \tag{7}$$

Here $\rho > 0$ is a dilatation parameter[1], ΔV_I is the quadrature weight and the function $b(x)$ is the solution of the linear equations

$$M(x) b(x) = p(0) \tag{8}$$

with the *moment matrix*

$$M(x) = \sum_{I=1}^{N} p \left(\frac{x^I - x}{\rho} \right) p^T \left(\frac{x^I - x}{\rho} \right) \Phi \left(\frac{x^I - x}{\rho} \right) \Delta V_I. \tag{9}$$

Remark 4 Because the RKPM is based on the approximation of the kernel in the integral transformation (1) and this integral is discretized by means of numerical quadrature, the quadrature weight ΔV_I occurs in (7).

Remark 5 There are some principles how to choose the particles $x^1, \dots, x^N$ to receive suitable results. Especially, the necessary condition for the unique solvability of (8) is that

$$\operatorname{card} \left\{ x_I |\, x \in \operatorname{supp} \Phi \left(\frac{x^I - x}{\rho} \right) \right\} \geq \dim \mathbb{P}^s$$

$\forall x \in \mathbb{R}^n$, see [2].

If we put $v = \Psi_K$ and insert the form (6) of the approximate solution into the weak formulation (5), we receive

$$\int_\Omega \sum_{i=1}^{n} \left(\sum_{I=1}^{N} u_I \frac{\partial \Psi_I}{\partial x_i} \right) \frac{\partial \Psi_K}{\partial x_i} \, \mathrm{d}x = \int_\Omega f \Psi_K \, \mathrm{d}x + \int_{\partial\Omega_1} g \Psi_K \, \mathrm{d}s, \;\; K = 1, 2, \dots, N.$$

The matrix form of these equations for an unknown vector $u = (u_1, \dots, u_N)^{\mathrm{T}}$ is

$$Au = b,$$

where

$$A = (A_{IK})_{I,K=1}^{N}, \quad A_{IK} = \int_\Omega \sum_{i=1}^{n} \frac{\partial \Psi_I}{\partial x_i} \frac{\partial \Psi_K}{\partial x_i} \, \mathrm{d}x, \tag{10}$$

[1] The role of ρ is to specify the size of supp Φ.

$$b = (b_1, \dots, b_N)^T, \quad b_K = \int\limits_{\Omega} f\Psi_K \,\mathrm{d}x + \int\limits_{\partial\Omega_1} g\Psi_K \,\mathrm{d}s.$$

3 Methods for problems involving the Dirichlet boundary condition

As soon as we compute the components of the vector u, the approximation $\tilde{u}(x) = \sum_{I=1}^{N} \Psi_I(x) u_I$ is known. But there is a problem – there can be particles $x^J \in \partial\Omega_0$ such that $\Psi_I(x^J) \neq \delta_{IJ}$. Consequently,

$$\tilde{u}(x^J) = \sum_I \Psi_I(x^J) u_I \neq u_J.$$

It is the reason why the imposition of Dirichlet boundary conditions is not trivial. We will deal with the question how to remove this trouble.

3.1 Method of weight functions

The first idea how to satisfy the Dirichlet condition (for instance, see the article [11]) is to multiply the weight function Φ by a smooth function w that is equal to one on Ω_0 and declines to zero near $\partial\Omega_0$ successively. This correction is then reflected in the relations (9) and (7). The new moment matrix is

$$M(x) = \sum_{I=1}^{N} p\left(\frac{x^I - x}{\rho}\right) p^T\left(\frac{x^I - x}{\rho}\right) w(x)\Phi\left(\frac{x^I - x}{\rho}\right) \Delta V_I$$

and the new shape functions

$$\Psi_I(x) = p^T\left(\frac{x^I - x}{\rho}\right) b(x)\, w(x)\Phi\left(\frac{x^I - x}{\rho}\right) \Delta V_I.$$

Both the idea of the method and its implementation are simple. But, because the smoothness of the RKP-approximation depends on the smoothness of $w\Phi$, the smoothness of the approximate solution may become worse.

3.2 Transform method

The approximation (6) has to satisfy the Dirichlet condition. But the Dirichlet boundary conditions are prescribed for the real nodal values $\tilde{u}(x^I)$ and not for the unknowns u_I. This discrepancy can be removed in the following way: If we denote $\tilde{u}(x^I) = \tilde{u}_I$ and $\Psi_I(x^J) = T_{IJ}$, then the approximation (6) can be written in the form

$$\tilde{u}_J = \sum_{I=1}^{N} T_{IJ} u_I$$

now, or shortly $\tilde{u} = Tu$. If the matrix T is non-singular, there exists an inverse matrix T^{-1} such that $T^{-1}\tilde{u} = u$, i.e.

$$\sum_{J=1}^{N} T_{JI}^{-1} \tilde{u}_J = u_I.$$

Returning to the formula (6), we obtain

$$\tilde{u}(x) = \sum_{I=1}^{N} \Psi_I(x) u_I = \sum_{I=1}^{N} \Psi_I(x) \sum_{J=1}^{N} T_{JI}^{-1} \tilde{u}_J = \sum_{J=1}^{N} \left(\sum_{I=1}^{N} \Psi_I(x) T_{JI}^{-1} \right) \tilde{u}_J$$
$$= \sum_{J=1}^{N} \tilde{\Psi}_J(x) \tilde{d}_J.$$

The transformed functions $\tilde{\Psi}_J(x)$ have the Kronecker delta property now.

The transform method is based on manipulation with the matrix of values of the shape functions at given particles. In general, this matrix is full and, moreover, it is required to be non-singular. These facts belong to the disadvantages of the method. On the other hand, the Dirichlet condition is satisfied exactly at the particles from $\partial\Omega_0$. An application of the transform method can be found in the article [4].

3.3 Method of Lagrange multipliers

This method is based on a modification of the weak formulation of the problem given. The main idea is to minimize the functional

$$I(u, \lambda) = \frac{1}{2} \int_\Omega \sum_{i=1}^{n} \left(\frac{\partial u}{\partial x_i} \right)^2 \mathrm{d}x - \int_\Omega f u \, \mathrm{d}x - \int_{\partial\Omega_1} g u \, \mathrm{d}s + \int_{\partial\Omega_0} \lambda (u - u^0) \, \mathrm{d}s$$

with respect to u and λ. We put

$$\tilde{u} = \sum_{I=1}^{N} \Psi_I u_I, \qquad \tilde{\lambda} = \sum_{I=1}^{N_0} \theta_I \lambda_I$$

in this case. The shape functions Ψ_I are the same as in Section 2, θ_I are the linear Lagrange basis functions and N_0 is the number of points discretizing $\partial\Omega_0$. The method leads to the system of linear equations $Au = b$ such that

$$A = \begin{pmatrix} H & G \\ G^T & 0 \end{pmatrix}, \; u = (u_1, \ldots, u_N, \lambda_1, \ldots, \lambda_{N_0})^T, \; b = (b_1, \ldots, b_N, c_1, \ldots, c_{N_0})^T,$$

$$H_{IK} = \int_\Omega \sum_{i=1}^{n} \frac{\partial \Psi_I}{\partial x_i} \frac{\partial \Psi_K}{\partial x_i} \, \mathrm{d}x, \; I = 1, \ldots, N, \; K = 1, \ldots, N,$$

$$G_{IK} = \int_{\partial\Omega_0} \Psi_I \theta_K \, \mathrm{d}s, \; I = 1, \ldots, N, \; K = 1, \ldots, N_0,$$

$$b_K = \int_\Omega f \Psi_K \, \mathrm{d}x + \int_{\partial\Omega_1} g \Psi_K \, \mathrm{d}s, \; K = 1, \ldots, N, \; c_K = \int_{\partial\Omega_0} u^0 \theta_K \, \mathrm{d}s, \; K = 1, \ldots, N_0.$$

In this method the matrix A is more complicated than the matrix (10). It is also necessary to compute more unknown parameters. The advantage of this method is that it is general and accurate. The method was used for instance in the article [4].

4 Conclusions

This contribution deals with the question how to discretize the Dirichlet boundary conditions in the RKPM that occurs when the Dirichlet condition has to be realized in the RKPM. Three approaches – the method of weight functions, the transform method and the method of Lagrange multipliers – are described and their essential properties are discussed.

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