PANM 15

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In: Jan Chleboun and Petr Přikryl and Karel Segeth and Jakub Šístek (eds.): Programs and Algorithms of Numerical Mathematics, Proceedings of Seminar. Dolní Maxov, June 6-11, 2010. Institute of Mathematics AS CR, Prague, 2010. pp. 158--163.

Persistent URL: http://dml.cz/dmlcz/702754

A POSTERIORI ERROR ESTIMATES OF THE DISCONTINUOUS GALERKIN METHOD FOR PARABOLIC PROBLEM*

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Abstract

We deal with a posteriori error estimates of the discontinuous Galerkin method applied to the nonstationary heat conduction equation. The problem is discretized in time by the backward Euler scheme and a posteriori error analysis is based on the Helmholtz decomposition.

1 Introduction

Our aim is to develop a sufficiently accurate and efficient numerical method for simulations of unsteady flows. A promising technique is a combination of the discontinuous Galerkin finite element method (DGFEM) for the space discretization and the backward difference formula for the time discretization, see [1], [2]. In order to both apply an adaptive algorithm and assess the discretization error, a posteriori error estimates have to be developed.

Within this paper, we focus on simplified model problem, represented by the heat equation, which is discretized by the high order DGFEM and the backward Euler method. We develop a posteriori error estimates based on the Helmholtz decomposition of the gradient of the error, see [4]. Therefore, this paper represents an extension of results from [6] where low order DGFEM was considered.

2 Problem definition

Let $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) be a bounded simply connected polyhedral Lipschitz domain with a boundary $\partial\Omega$, $T > 0$ and $Q_T = \Omega \times (0, T)$. Let us consider the problem:

$$\begin{aligned} \partial u/\partial t - \Delta u &= f && \text{in} \quad Q_T, \\ u &= 0 && \text{on} \quad \partial\Omega \times (0, T), \\ u(x, 0) &= u^0(x) && \text{in} \quad \Omega. \end{aligned} \tag{1}$$

We use a standard notation for the Lebesgue, Sobolev and Bochner spaces. We introduce a weak formulation of (1).

*This work is a part of the research projects MSM 0021620839 (V. Dolejší). The research of I. Šebestová was supported by grant No. 10209/B-MAT/MFF of the Grant Agency of Charles University Prague.

Definition 1. *The function $u : Q_T \to \mathbb{R}$ such that $u \in L^2(0,T;H_0^1(\Omega))$ and $\partial u/\partial t \in L^2(0,T;H_0^1(\Omega))$ is the weak solution of the problem* (1) *if*

$$\begin{array}{rcl} \langle \partial u(t)/\partial t, v\rangle + \int_\Omega \nabla u(t)\cdot\nabla v\,dx &=& \langle f(t), v\rangle \quad \forall v\in H_0^1(\Omega),\ \text{for a.a. } t\in(0,T), \\ u(x,0) &=& u^0(x) \quad \text{in } \Omega, \end{array} \tag{2}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$ and where we assume $f \in C(0,T;H^{-1}(\Omega))$ and $u^0 \in L^2(\Omega)$.

3 Discretization

3.1 Time semidiscretization

Let $0 = t_0 < t_1 < ... < t_{\bar{N}} = T$ be a partition of the time interval $[0,T]$ and set $\tau_n = t_n - t_{n-1}$, $\tau = \max\{\tau_n : 1 \le n \le \bar{N}\}$. We use the backward Euler scheme in (2) and get the semi-discrete problem: Find a sequence $\{u^n\}_{1\le n\le\bar{N}}$, $u^n \in H_0^1(\Omega)$ such that

$$\int_\Omega \frac{u^n - u^{n-1}}{\tau_n} v\,dx + \int_\Omega \nabla u^n\cdot\nabla v\,dx = \int_\Omega f^n v\,dx \quad \forall v\in H_0^1(\Omega),$$

where $f^n = f(\cdot, t_n)$.

3.2 Space discretization

We will carry out the space discretization with the aid of the DGFEM. On each time level $t_n,\ n = 1,\dots,\bar{N}$, we consider a family $\{\mathcal{T}_{h,n}\}_{h>0}$ of partitions of Ω into a finite number of closed triangles in 2D and tetrahedra in 3D with mutually disjoint interiors. We assume that the following conditions are satisfied.

$$\text{shape regularity:}\quad \exists C_s > 0 :\ \frac{h_K}{\rho_K} \le C_s\ \forall\, K \in \mathcal{T}_{h,n}, \tag{3}$$

$$\text{local quasi-uniformity:}\quad \exists C_H > 0 :\ h_K \le C_H h_{K'}\ \forall K, K' \in \mathcal{T}_{h,n} \text{ sharing a face,} \tag{4}$$

where $h_K = \mathrm{diam}(K)$ for $K \in \mathcal{T}_{h,n}$, ρ_K denotes the radius of the largest d-dimensional ball inscribed into K, and ∂K denotes the boundary of element K. Moreover, we assume that there exists a triangulation $\widetilde{\mathcal{T}}_{h,n}$ satisfying (3) and (4) which is a refinement of both $\mathcal{T}_{h,n-1}$ and $\mathcal{T}_{h,n}$, $1 \le n \le \bar{N}$ and such that

$$\exists C_{HT} > 0 :\ \forall 1 \le n \le \bar{N}\ \forall K \in \widetilde{\mathcal{T}}_{h,n}\ \forall K' \in \mathcal{T}_{h,n}, K \subset K' :\ \frac{h_{K'}}{h_K} < C_{HT}.$$

By $\mathcal{F}_{h,n}^I$ and $\mathcal{F}_{h,n}^D$ we denote the set of all interior faces (edges for $d = 2$) and faces (edges for $d = 2$) on $\partial\Omega$, respectively. For a simplicity, we put $\mathcal{F}_{h,n} = \mathcal{F}_{h,n}^I \cup \mathcal{F}_{h,n}^D$. Further, we set $h_\Gamma = \mathrm{diam}(\Gamma)$ for $\Gamma \in \mathcal{F}_{h,n}$. For each $\Gamma \in \mathcal{F}_{h,n}^I$ there exist two elements K_Γ^L and K_Γ^R such that $\Gamma \subset \overline{K_\Gamma^L} \cap \overline{K_\Gamma^R}$. We define a unit normal vector $\boldsymbol{n}_\Gamma$

to each $\Gamma \in \mathcal{F}_h^I$ so that it points out of K_Γ^L. Finally, we assume that $\boldsymbol{n}_\Gamma$, $\Gamma \in \mathcal{F}_{h,n}^D$, has the same orientation as the outward normal to $\partial\Omega$.

Over the triangulation $\widetilde{\mathcal{T}}_{h,n}$ we define the so-called broken Sobolev space

$$H^s(\Omega, \widetilde{\mathcal{T}}_{h,n}) = \{v; v|_K \in H^s(K)\, \forall K \in \widetilde{\mathcal{T}}_{h,n}\}$$

equipped with the norm $\|v\|^2_{H^s(\Omega,\widetilde{\mathcal{T}}_{h,n})} = \sum_{K\in\widetilde{\mathcal{T}}_{h,n}} \|v\|^2_{H^s(K)}$. For $v \in H^1(\Omega, \widetilde{\mathcal{T}}_{h,n})$ we define the broken gradient $\nabla_h v$ of v by $(\nabla_h v)|_K = \nabla(v|_K)$ for $\forall K \in \widetilde{\mathcal{T}}_{h,n}$ and use the following notation: v_Γ^L stands for the trace of $v|_{K_\Gamma^L}$ on Γ, v_Γ^R is the trace of $v|_{K_\Gamma^R}$ on Γ, $\langle v\rangle_\Gamma = \frac{1}{2}(v_\Gamma^L + v_\Gamma^R)$, $[v]_\Gamma = v_\Gamma^L - v_\Gamma^R$, $\Gamma \in \mathcal{F}_{h,n}^I$. Further, for $\Gamma \in \mathcal{F}_{h,n}^D$, we define v_Γ^L as the trace of $v|_{K_\Gamma^L}$ on Γ, and $\langle v\rangle_\Gamma = [v]_\Gamma = v_\Gamma^L$. If $[\cdot]_\Gamma$ and $\langle\cdot\rangle_\Gamma$ appear in an integral of the form $\int_\Gamma \ldots dS$, we will omit the subscript Γ and write $[\cdot]$ and $\langle\cdot\rangle$ instead. Finally, we define the space of discontinuous piecewise polynomial functions

$$S_{hp}^n = \{v; v \in L^2(\Omega), v|_K \in P^p(K)\, \forall\, K \in \widetilde{\mathcal{T}}_{h,n}\},$$

where $P^p(K)$ is the space of all polynomials on K of degree p.

Now, we can state the discrete problem: For a given approximation $u_h^0 \in S_{hp}^0$ of an initial condition u^0 find a sequence $\{u_h^n\}_{1\le n\le \bar{N}}$, $u_h^n \in S_{hp}^n$ such that

$$\int_\Omega \frac{u_h^n - u_h^{n-1}}{\tau_n} v_h\, dx + \sum_{K\in\widetilde{\mathcal{T}}_{h,n}} \int_K \nabla u_h^n \cdot \nabla v_h\, dx - \sum_{\Gamma\in\mathcal{F}_{h,n}} \int_\Gamma \langle\nabla u_h^n \cdot \boldsymbol{n}\rangle [v_h]\, dS$$
$$+\theta \sum_{\Gamma\in\mathcal{F}_{h,n}} \int_\Gamma \langle\nabla v_h \cdot \boldsymbol{n}\rangle [u_h^n]\, dS + \sum_{\Gamma\in\mathcal{F}_{h,n}} \int_\Gamma \sigma[u_h^n][v_h]\, dS = \int_\Omega f^n v_h\, dx$$

for all $v_h \in S_{hp}^n$, where $\theta = -1$, $\theta = 1$, and $\theta = 0$ corresponds to the symmetric, nonsymmetric, and incomplete variants of the DGFEM, respectively.

In this section, we derive a residual-based a posteriori error estimate of the discretization error based on the Helmholtz decomposition of the gradient of the error. This approach was developed in [4], where the heat equation was solved with the aid of the combination of the Crouzeix-Raviart nonconforming finite elements in space and the backward Euler scheme in time.

Since the time error estimation is almost the same as in [4], we focus on the spatial error estimation.

Definition 2. *Let* $\{u^n\}_{1\le n\le\bar{N}}$ *be the semi-discrete solution and* $\{u_h^n\}_{1\le n\le\bar{N}}$ *be the discrete solution of* (1). *Then we set*

$$\{e^n\}_{1\le n\le\bar{N}} = \{u^n - u_h^n\}_{1\le n\le\bar{N}}.$$

We will need an interpolation operator that maps $H^1(\Omega, \widetilde{\mathcal{T}}_{h,n})$ into $S_{hp}^n \cap H_0^1(\Omega)$.

3.3 Oswald interpolation operator

Let $\mathcal{N}^0_{h,n}$ be the set of all Lagrangian vertices of the elements of $\widetilde{\mathcal{T}}_{h,n}$. According to, e.g., [3], we define the Oswald interpolation operator $\mathcal{I}^0_{Os} : S^n_{hp} \to S^n_{hp} \cap H^1_0(\Omega)$ by

$$\begin{aligned} \mathcal{I}^0_{Os}(v_h)(\nu) &= \frac{1}{\mathrm{card}(\omega_\nu)} \sum_{K\in\omega_\nu} v_h|_K(\nu), \quad \nu \in \mathcal{N}^0_{h,n} \backslash \mathcal{N}^B_{h,n} \\ &= 0, \quad \nu \in \mathcal{N}^B_{h,n} \end{aligned}$$

where $\omega_\nu = \{K \in \widetilde{\mathcal{T}}_{h,n};\ \nu \in K\}$, $\mathcal{N}^B_{h,n} = \{\nu \in \mathcal{N}^0_{h,n}; \nu \in \partial\Omega\}$. Moreover, we define the interpolation operator $I^0_{h,n} : H^1(\Omega, \widetilde{\mathcal{T}}_{h,n}) \to S^n_{hp} \cap H^1_0(\Omega)$ by

$$I^0_{h,n}(v) = \mathcal{I}^0_{Os}(\Pi_{hp}(v)) \quad \forall\, v \in H^1(\Omega, \widetilde{\mathcal{T}}_{h,n}),$$

where Π_{hp} denotes the L^2-projection of v on the space S^n_{hp}.

In order to overcome difficulties with the nonconformity of S^n_{hp}, the Helmholtz decomposition of the gradient of the error is carried out as follows (see, e.g., [5]):

$$\nabla_h e^n = \nabla \phi^n + \mathrm{curl}\, \chi^n, \tag{5}$$

where $\phi^n \in H^1_0(\Omega) = \{v \in H^1(\Omega);\ v = 0 \text{ on } \partial\Omega\}$ is the solution of the problem

$$\int_\Omega \nabla \phi^n \cdot \nabla v \, dx = \int_\Omega \nabla_h e^n \cdot \nabla v \, dx \quad \forall v \in H^1_0(\Omega),$$

$\chi^n \in H(\mathrm{curl}, \Omega) = \{v \in (L^2(\Omega))^k;\ \mathrm{curl}\, v \in (L^2(\Omega))^d\}$ ($k = 1$ for $d = 2$ and $k = 3$ for $d = 3$). Moreover, the following holds: $\|\nabla_h e^n\|^2_\Omega = \|\nabla \phi^n\|^2_\Omega + \|\mathrm{curl}\, \chi^n\|^2_\Omega$. The orthogonality of the splitting is crucial because it suffices to estimate each part of the error independently. A proof of the above assertions can be found in [5].

Furthermore, we recall some fundamental properties presented in [6].

Lemma 1. *Let $v_h \in S^n_{hp} \cap H^1_0(\Omega)$, $\phi \in H^1_0(\Omega)$ and $\chi \in (H^1(\Omega))^k$ ($k = 1$ for $d = 2$ and $k = 3$ for $d = 3$) be arbitrary. The error e^n satisfies*

$$\sum_{K\in\widetilde{\mathcal{T}}_{h,n}} \int_K \nabla e^n \cdot \nabla v_h \, dx = \int_\Omega \frac{e^{n-1} - e^n}{\tau_n} v_h \, dx + \theta \sum_{\Gamma\in\mathcal{F}^I_{h,n}} \int_\Gamma \langle \nabla v_h \cdot \boldsymbol{n} \rangle [u^n_h] \, dS, \tag{6}$$

$$\begin{aligned} \sum_{K\in\widetilde{\mathcal{T}}_{h,n}} \int_K \nabla e^n \cdot \nabla \phi \, dx = &\int_\Omega (f^n - \frac{u^n - u^{n-1}}{\tau_n}) \phi \, dx - \sum_{K\in\widetilde{\mathcal{T}}_{h,n}} \int_{\partial K} \nabla u^n_h \cdot \boldsymbol{n} \phi \, dS \\ &+ \sum_{K\in\widetilde{\mathcal{T}}_{h,n}} \int_K \Delta u^n_h \phi \, dx, \end{aligned} \tag{7}$$

$$\sum_{K\in\widetilde{\mathcal{T}}_{h,n}} \int_K \nabla e^n curl \chi \, dx = - \sum_{K\in\widetilde{\mathcal{T}}_{h,n}} \int_{\partial K} u^n_h curl \chi \cdot \boldsymbol{n} \, dS. \tag{8}$$

Definition 3. *Let $n \geq 1$. We define the local spatial error indicator by*

$$\eta_K^n = h_K \left\| f^n + \Delta u_h^n - \frac{u_h^n - u_h^{n-1}}{\tau_n} \right\|_K + h_K^{1/2} \|\nabla u_h^n \cdot \boldsymbol{n}\|_{\partial K} + \|u_h^n\|_{H^{1/2}(\partial K)}$$
$$+ \sum_{\Gamma \in \mathcal{F}_{h,n} \cap \mathcal{F}_K} \left(h_\Gamma^{-1/2} \|[u_h^n]\|_\Gamma + h_\Gamma^{1/2} \|[u_h^n]\|_\Gamma \right),$$

where $\mathcal{F}_K$ denotes the set of all edges or faces of a triangle or of a tetrahedron K, respectively, and $||\cdot||_K$ stands for the $L^2(K)$-norm. The global spatial error estimator is defined by $\eta^n = (\sum_{K \in \widetilde{\mathcal{T}}_{h,n}} (\eta_K^n)^2)^{1/2}$.

Now, we state the main result, an upper bound on the error.

Theorem 1. *Let $\{u^n\}_{1 \leq n \leq \bar{N}}$ be the semi-discrete solution and $\{u_h^n\}_{1 \leq n \leq \bar{N}}$ be the discrete solution of (1). Let $1 \leq N \leq \bar{N}$. Then the error e^n satisfies*

$$\sum_{K \in \widetilde{\mathcal{T}}_{h,N}} \|e^N\|_K^2 + \sum_{n=1}^{N} \tau_n \sum_{K \in \widetilde{\mathcal{T}}_{h,n}} \|\nabla e^n\|_K^2 \leq \sum_{K \in \widetilde{\mathcal{T}}_{h,1}} \|e^0\|_K^2 + \sum_{n=1}^{N} C(\eta^n)^2 (1 + \max\{h_n^2, \tau_n\}),$$

where a constant C is independent of the mesh parameter and the time step.

Sketch of the proof: According to (5), we can write

$$\tau_n \sum_{K \in \widetilde{\mathcal{T}}_{h,n}} \|\nabla e^n\|_K^2 = \tau_n \sum_{K \in \widetilde{\mathcal{T}}_{h,n}} \int_K \nabla e^n \cdot \nabla \phi^n \, dx$$
$$+ \tau_n \sum_{K \in \widetilde{\mathcal{T}}_{h,n}} \int_K \nabla e^n \mathrm{curl}\, \chi^n \, dx. \tag{9}$$

Denoting ψ_1 and ψ_2 the two terms on the right-hand side of (9), setting $\phi = \phi^n$ in (7), $\chi = \chi^n$ in (8) and multiplying both inequalities by τ_n yield

$$\psi_1 = \tau_n \int_\Omega (f^n - \frac{u^n - u^{n-1}}{\tau_n}) \phi^n \, dx - \tau_n \sum_{K \in \widetilde{\mathcal{T}}_{h,n}} \int_{\partial K} \nabla u_h^n \cdot \boldsymbol{n} \phi^n \, dS$$
$$+ \tau_n \sum_{K \in \widetilde{\mathcal{T}}_{h,n}} \int_K \Delta u_h^n \phi^n \, dx,$$
$$\psi_2 = -\tau_n \sum_{K \in \widetilde{\mathcal{T}}_{h,n}} \int_{\partial K} u_h^n \mathrm{curl}\, \chi^n \cdot \boldsymbol{n} \, dS.$$

Now, we modify the expression ψ_1. Adding τ_n multiple of (6) with $v_h = I_{h,n}^0 \phi^n$ to ψ_1 and expressing term $-\tau_n \sum_{K \in \widetilde{\mathcal{T}}_{h,n}} \int_K \nabla e^n \cdot \nabla I_{h,n}^0 \phi^n \, dx$ according to identity (7), we obtain

$$\psi_1 = \tau_n \sum_{K\in\widetilde{\mathcal{T}}_{h,n}} \int_K (f^n + \Delta u_h^n - \frac{u^n - u^{n-1}}{\tau_n})(\phi^n - I_{h,n}^0\phi^n)\,dx \qquad (10)$$
$$-\tau_n \int_\Omega \frac{e^{n-1} - e^n}{\tau_n} I_{h,n}^0\phi^n\,dx - \tau_n \sum_{K\in\widetilde{\mathcal{T}}_{h,n}} \int_{\partial K} \nabla u_h^n \cdot \boldsymbol{n}(\phi^n - I_{h,n}^0\phi^n)\,dS$$
$$+\tau_n\theta \sum_{\Gamma\in\mathcal{F}_{h,n}^I} \int_\Gamma \langle \nabla I_{h,n}^0\phi^n \cdot \boldsymbol{n}\rangle [u_h^n]\,dS.$$

By adding and subtracting suitable terms in (10), estimating all terms in ψ_1 and ψ_2 using approximation properties of $I_{h,n}^0$, trace inequalities, inverse inequality, and well known inequalities such as Hölder's, Young's, etc., we finally come to the assertion of Theorem 1.

4 Conclusion

We derived the error upper bound for the heat conduction equation discretized by the high order discontinuous Galerkin finite element method in space and the backward Euler scheme in time. Analogously to [4], the Helmholtz decomposition was used to overcome difficulties arising due to the nonconformity of the DGFEM.

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