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AN IMPROVEMENT OF EUCLID'S ALGORITHM*

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Abstract

The paper introduces the calculation of a greatest common divisor of two univariate polynomials. Euclid's algorithm can be easily simulated by the reduction of the Sylvester matrix to an upper triangular form. This is performed by using *c*-*s* transformation and QR-factorization methods. Both procedures are described and numerically compared. Computations are performed in the floating point environment.

1 Introduction

Euclid's algorithm and the corresponding manipulations with the Sylvester resultant matrix are two well-known methods for computing the greatest common divisor of two univariate polynomials. See the book [1] or the paper [4].

Theory has been developed in those papers and all practical examples included only low-degree polynomials in which the effect of computing in floating point arithmetic has not shown. That is why we have decided to work on computation of the greatest common divisor of two large-degree polynomials in this article. Many times the numerical experiments have yielded inaccurate or even wrong results caused for instance by the big differences in the absolute value of coefficients of polynomials which are calculated during Euclid's algorithm. Since the problems in real world have demanded the best possible precision on the coefficients of the greatest common divisor some of the ideas on the balancing the coefficients have been introduced in the article [5] and several others. We have developed an improvement of Euclid's algorithm in this paper, called *c*-*s* transformation, which is conducted by the transformation of Sylvester matrix. The above mentioned method, described in [2], has not been published yet and its rigorous analysis has been presented in this article.

Scalars c and s are computed from coefficients of polynomials in every step of Euclid's algorithm and are resembled to scalars used in Givens rotation. Detailed description is given in paragraph 2 where the classic and well known Euclid's algorithm has been compared with *c*-*s* transformation. Let us mention that structure of the Sylvester matrix is preserved by both methods.

We have decided to mention another interesting approach proposed in [7] which does not preserve the structure of the Sylvester matrix. This method is based on QR-factorization of the Sylvester matrix or a part of the Sylvester matrix. Coefficients of the greatest common divisor can be obtained from the last non-vanishing row of

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the upper triangular matrix R obtaining by QR-factorization. Complete description of aforementioned algorithm and numerical experiments are given in paragraph 3.

Reasonable results can be obtained if we know the degree of the greatest common divisor. In that case we know exactly where coefficients of the greatest common divisor can be found in the matrix R. Algorithms for determining the degree of the greatest common divisor have been studied in [3] or [6]. Those methods are not and cannot be included in this article.

All test polynomials have been computed via convolution that is why the degree of the greatest common divisor is known and used in our examples.

In this article, all numerical experiments have been carried out in double precision. We have worked with polynomials having non-trivial greatest common divisor.

2 Euclid's algorithm and transformations of the Sylvester matrix

Let the symbol GCD (f_0, f_1) denotes the greatest common divisor of polynomials f_0 and f_1 and $\deg(f_0)$ the degree of f_0. Let

$$f_0(x) = a_0x^m + a_1x^{m-1} + \cdots + a_{m-1}x + a_m, \qquad (1)$$

$$f_1(x) = b_0x^n + b_1x^{n-1} + \cdots + b_{n-1}x + b_n, \qquad (2)$$

where $m \geq n$, $a_0a_m \neq 0$, $b_0b_n \neq 0$, To illustrate the algorithm, let us consider the polynomials f_0 and f_1 of degrees 5 and 2 respectively:

$$f_0(x) = a_0x^5 + a_1x^4 + a_2x^3 + a_3x^2 + a_4x + a_5,$$
$$f_1(x) = b_0x^2 + b_1x + b_2.$$

The Sylvester resultant matrix $S(f_0, f_1)$ for the polynomials f_0 and f_1 of degrees $m = 5$ and $n = 2$ is

$$S(f_0, f_1) = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & 0 \\ 0 & a_0 & a_1 & a_2 & a_3 & a_4 & a_5 \\ b_0 & b_1 & b_2 & 0 & 0 & 0 & 0 \\ 0 & b_0 & b_1 & b_2 & 0 & 0 & 0 \\ 0 & 0 & b_0 & b_1 & b_2 & 0 & 0 \\ 0 & 0 & 0 & b_0 & b_1 & b_2 & 0 \\ 0 & 0 & 0 & 0 & b_0 & b_1 & b_2 \end{bmatrix}.$$

We will now formulate modified Euclid's algorithm which can scale down the big differences between the coefficients of f_0 and f_1. Let us define the division f_0/f_1 in the following form:

$$c_0 \underbrace{\left(a_0x^5 + a_1x^4 + a_2x^3 + a_3x^2 + a_4x + a_5\right)}_{f_0(x)} + s_0 \underbrace{\left(b_0x^2 + b_1x + b_2\right)}_{f_1(x)} x^3$$
$$\underbrace{= 0 + \underbrace{(c_0a_1 + s_0b_1)}_{a_1^{(1)}} x^4 + \underbrace{(c_0a_2 + s_0b_2)}_{a_2^{(1)}} x^3 + \underbrace{c_0a_3}_{a_3^{(1)}} x^2 + \underbrace{c_0a_4}_{a_4^{(1)}} x + \underbrace{c_0a_5}_{a_5^{(1)}}}_{h_4(x) := a_1^{(1)}x^4 + a_2^{(1)}x^3 + a_3^{(1)}x^2 + a_4^{(1)}x + a_5^{(1)}}.$$

The numbers c_0 and s_0 are chosen to remove the leading coefficient of f_0. To define the corresponding transformation of the Sylvester matrix, let us define the matrix $G_0^{(1)}(c_0, s_0)$

$$G_0^{(1)}(c_0, s_0) = \begin{bmatrix} c_0 & 0 & s_0 & 0 & 0 & 0 & 0 \\ 0 & c_0 & 0 & s_0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Apparently

$$S^{(1)}(f_0, f_1) := G_0^{(1)}(c_0, s_0) S(f_0, f_1) = \begin{bmatrix} 0 & a_1^{(1)} & a_2^{(1)} & a_3^{(1)} & a_4^{(1)} & a_5^{(1)} & 0 \\ 0 & 0 & a_1^{(1)} & a_2^{(1)} & a_3^{(1)} & a_4^{(1)} & a_5^{(1)} \\ b_0 & b_1 & b_2 & 0 & 0 & 0 & 0 \\ 0 & b_0 & b_1 & b_2 & 0 & 0 & 0 \\ 0 & 0 & b_0 & b_1 & b_2 & 0 & 0 \\ 0 & 0 & 0 & b_0 & b_1 & b_2 & 0 \\ 0 & 0 & 0 & 0 & b_0 & b_1 & b_2 \end{bmatrix},$$

where

$$a_i^{(1)} = \begin{cases} c_0 a_i + s_0 b_i & \text{for} \quad i = 1, 2, \\ c_0 a_i & \text{otherwise.} \end{cases}$$

Let $a_1^{(1)} \neq 0$. Then $\deg(h_4) = 4$. In the opposite case, the process would be performed with the polynomial of degree less than 4. The Euclid's algorithm proceeds according to the following schema:

$$\begin{aligned} & c_1 \underbrace{(a_1^{(1)} x^4 + a_2^{(1)} x^3 + a_3^{(1)} x^2 + a_4^{(1)} x + a_5^{(1)})}_{h_4(x)} + s_1 \underbrace{(b_0 x^2 + b_1 x + b_2)}_{f_1(x)} x^2 \\ & \underbrace{= 0 + \underbrace{\left(c_1 a_2^{(1)} + s_1 b_1\right)}_{a_2^{(2)}} x^3 + \underbrace{\left(c_1 a_3^{(1)} + s_1 b_2\right)}_{a_3^{(2)}} x^2 + \underbrace{c_1 a_4^{(1)}}_{a_4^{(2)}} x + \underbrace{c_1 a_5^{(1)}}_{a_5^{(2)}}.}_{h_3(x) := a_2^{(2)} x^3 + a_3^{(2)} x^2 + a_4^{(2)} x + a_5^{(2)}} \end{aligned}$$

The numbers c_1 and s_1 are again chosen to remove the coefficient of x^4. The corresponding matrix operation consists of the construction of the matrix $G_1^{(1)}(c_1, s_1)$, by analogy to the previous case, such that

$$S^{(2)}(f_0, f_1) := G_1^{(1)}(c_1, s_1) S^{(1)}(f_0, f_1) = \begin{bmatrix} 0 & 0 & a_2^{(2)} & a_3^{(2)} & a_4^{(2)} & a_5^{(2)} & 0 \\ 0 & 0 & 0 & a_2^{(2)} & a_3^{(2)} & a_4^{(2)} & a_5^{(2)} \\ b_0 & b_1 & b_2 & 0 & 0 & 0 & 0 \\ 0 & b_0 & b_1 & b_2 & 0 & 0 & 0 \\ 0 & 0 & b_0 & b_1 & b_2 & 0 & 0 \\ 0 & 0 & 0 & b_0 & b_1 & b_2 & 0 \\ 0 & 0 & 0 & 0 & b_0 & b_1 & b_2 \end{bmatrix},$$

where

$$a_i^2 = \begin{cases} c_1 a_i^{(1)} + s_1 b_{i-1} & \text{for} \quad i = 2, 3, \\ c_1 a_i^{(1)} & \text{otherwise.} \end{cases}$$

If $a_2^{(2)} = 0$ and $a_3^{(2)} \neq 0$, then instead of h_3 the polynomial h_2,

$$h_2(x) = a_3^{(2)} x^2 + a_4^{(2)} x + a_5^{(2)} \ ,$$

is considered. Moreover, if $a_3^{(2)} = 0$, then the first stage of Euclid's algorithm terminates. Let us assume that the degrees of all polynomials after division decrease by 1. Hence $a_2^2 \neq 0$. Let the numbers c_2, s_2 and then c_3 and s_3 are chosen to remove the coefficient of dominant power. The last two divisions yield the polynomials

$$\begin{array}{rclcl} h_2(x) & = & a_3^{(3)} x^2 + a_4^{(3)} x + a_5^{(3)} & = & c_2 h_3(x) + s_2 f_1(x) x, \\ h_1(x) & = & a_4^{(4)} x + a_5^{(4)} & = & c_3 h_2(x) + s_3 f_1(x), \end{array}$$

where $a_3^{(3)} \neq 0$ and $a_4^{(4)} \neq 0$. The matrices $G_2^{(1)}(c_2, s_2)$, $G_3^{(1)}(c_3, s_3)$ correspond to the last two divisions. Their construction is omitted because it is the same as in the previous steps. If we define

$$G_1 = G_3^{(1)}(c_3, s_3) G_2^{(1)}(c_2, s_2) G_1^{(1)}(c_1, s_1) G_0^{(1)}(c_0, s_0)$$

and

$$P_1 = [e_3, e_4, e_5, e_6, e_7, e_1, e_2] \ ,$$

then the first stage of Euclid's algorithm can be written in the matrix form as follows

$$P_1 G_1 S(f_0, f_1) = \left[\begin{array}{cccc|ccc} b_0 & b_1 & b_2 & 0 & 0 & 0 & 0 \\ 0 & b_0 & b_1 & b_2 & 0 & 0 & 0 \\ 0 & 0 & b_0 & b_1 & b_2 & 0 & 0 \\ 0 & 0 & 0 & b_0 & b_1 & b_2 & 0 \\ \hline 0 & 0 & 0 & 0 & b_0 & b_1 & b_2 \\ 0 & 0 & 0 & 0 & a_4^{(4)} & a_5^{(4)} & 0 \\ 0 & 0 & 0 & 0 & 0 & a_4^{(4)} & a_5^{(4)} \end{array} \right] =: \left[\begin{array}{cc} F_{1,1} & F_{1,2} \\ F_{2,1} & F_{2,2} \end{array} \right].$$

Note that we have obtained the coefficients of the polynomial h_1 in the last two rows. The formula

$$\underbrace{c_3 c_2 c_1 c_0 f_0(x)}_{\widetilde{f}_0(x)} = \underbrace{-(c_3 c_2 c_1 s_0 x^3 + c_3 c_2 s_1 x^2 + c_3 s_2 x + s_3)}_{\widetilde{q}_0(x)} \underbrace{f_1(x)}_{\widetilde{f}_1(x)} + \underbrace{h_1(x)}_{\widetilde{f}_2(x)}.$$

summarises the first stage of Euclid's algorithm. Hence we have

$$\widetilde{f}_0(x) = \widetilde{q}_0(x) \widetilde{f}_1(x) + \widetilde{f}_2(x).$$

The block $F_{2,2}$ is again the Sylvester matrix $S(\widetilde{f}_1, \widetilde{f}_2)$. We suppose that $\widetilde{f}_2(x) \neq 0$. If $\widetilde{f}_2(x) = 0$, then $\widetilde{f}_1(x) = GCD(f_0, f_1)$. The transformation of the Sylvester resultant matrix to an upper triangular matrix requires that the same procedure is applied to the matrix $S(\widetilde{f}_1, \widetilde{f}_2)$, and this corresponds to the second stage of Euclid's algorithm, that is, the division $\widetilde{f}_1/\widetilde{f}_2$. Analogously there exist matrices G_2 and P_2 such that

$$P_2G_2P_1G_1S(f_0,f_1) = \begin{bmatrix} b_0 & b_1 & b_2 & 0 & 0 & 0 & 0 \\ 0 & b_0 & b_1 & b_2 & 0 & 0 & 0 \\ 0 & 0 & b_0 & b_1 & b_2 & 0 & 0 \\ 0 & 0 & 0 & b_0 & b_1 & b_2 & 0 \\ 0 & 0 & 0 & 0 & a_4^{(4)} & a_5^{(4)} & 0 \\ 0 & 0 & 0 & 0 & 0 & a_4^{(4)} & a_5^{(4)} \\ 0 & 0 & 0 & 0 & 0 & 0 & b_2^{(2)} \end{bmatrix}.$$

It is $\widetilde{f}_3(x) = b_2^{(2)}$. Let us remark that $\widetilde{f}_2(x) = GCD(f_0, f_1)$ if $\widetilde{f}_3(x) = 0$. Otherwise f_0 and f_1 are coprime.

We will now demonstrate how to pick the numbers c and s. If we take

$$c_0 = 1 \qquad \text{and} \qquad s_0 = -\frac{a_0}{b_0}\ ,$$

then the division in Euclid's algorithm has the following form

$$\underbrace{(a_0x^5 + a_1x^4 + a_2x^3 + a_3x^2 + a_4x + a_5)}_{f_0(x)} - \underbrace{(b_0x^2 + b_1x + b_2)}_{f_1(x)}(\frac{a_0}{b_0})x^3$$
$$= 0 + \underbrace{\left(a_1 - \frac{a_0b_1}{b_0}\right)}_{a_1^{(1)}} x^4 + \underbrace{\left(a_2 - \frac{a_0b_2}{b_0}\right)}_{a_2^{(1)}} x^3 + \underbrace{a_3}_{a_3^{(1)}} x^2 + \underbrace{a_4}_{a_4^{(1)}} x + \underbrace{a_5}_{a_5^{(1)}}.$$

In the next step we have considered c_1 and s_1 in the form

$$c_1 = 1 \qquad \text{and} \qquad s_1 = -\frac{a_1^{(1)}}{b_0}\ ,$$

and analogously are defined the numbers c_i a s_i in the following steps. This choice forms Euclid's algorithm in the well known form.

The second possible choice of c and s is based on the idea of balance of the coefficients of f_0 and f_1. In the first step these numbers are defined as

$$c_0 = \frac{b_0}{\sqrt{a_0^2 + b_0^2}} \qquad \text{and} \qquad s_0 = -\frac{a_0}{\sqrt{a_0^2 + b_0^2}}\ ,$$

and analogously in the next steps. This form of Euclid's algorithm will be called c-s transformation. Let us denote for the polynomials (1) and (2)

$$d_1 = \max_{i,j\in\{1,\dots,m\}} \Big||a_i| - |a_j|\Big|, \quad d_2 = \max_{i,j\in\{1,\dots,n\}} \Big||b_i| - |b_j|\Big|, \quad \mathrm{diff}(f_0, f_1) = (d_1, d_2).$$

Example 1. Let f_0 and f_1 be two polynomials such that

$$\begin{aligned} f_0(x) &= (x-4.1)(x-3)(x-\sqrt{2})^2(x+1)(x+\sqrt{2})(x+5)^2, \\ f_1(x) &= (x-3)(x-\sqrt{3})(x-1)(x+1)(x+\sqrt{3})(x+8). \end{aligned}$$

Their greatest common divisor u has the form

$$u(x) = (x-3)(x+1) = x^2 - 2x - 3.$$

Let u_{Euc} and $u_{c\text{-}s}$ denotes the greatest common divisor computed by Euclid's algorithm and c-s transformation, respectively. We have obtained

$$\begin{aligned} u_{Euc}(x) &= x^2 - 1.99999999999994x - 2.99999999999993, \\ u_{c\text{-}s}(x) &= x^2 - 1.99999999999998x - 2.99999999999998, \\ \|u_{Euc} - u\|_2 = 9.&26402450510883e{-14}, \quad \|u_{c\text{-}s} - u\|_2 = 3.14269606124535e{-14} \ . \end{aligned}$$

Both procedures yielded practically the exact greatest common divisor. Let us remark, that for the first division f_0/f_1 in Euclid's algorithm we have obtained $\mathrm{diff}(f_0, f_1) = (1189.33794283234, 98)$.

Example 2. Let

$$\begin{aligned} f_0(x) &= (x-4)^2(x-\sqrt{5})^2(x-\sqrt{3})^2(x-\sqrt{2})^2(x+0.5)^2(x+1)^2, \\ f_1(x) &= (x-6.51)^2(x-5)^2(x-4)(x+0.5)^2(x+0.9)(x+1)^2. \end{aligned}$$

Their greatest common divisor u has the form

$$u(x) = (x-4)(x+0.5)^2(x+1)^2 = x^5 - x^4 - 8.75x^3 - 11.5x^2 - 5.75x - 1 \ .$$

We have obtained

$$\begin{aligned} &u_{Euc}(x) = x^5 - 0.9999999x^4 - 8.75000000x^3 - 11.5000000x^2 - 5.75000000x - 1.00000000, \\ &u_{c\text{-}s}(x) = x^5 - 0.9999999x^4 - 8.74999999x^3 - 11.49999999x^2 - 5.74999999x - 0.99999999, \\ &\|u_{Euc} - u\|_2 = 1.70205143034978e{-11}, \quad \|u_{c\text{-}s} - u\|_2 = 6.36839947245598e{-12} \ . \end{aligned}$$

We have obtained again a good result. Let us remark, that for the first division f_0/f_1 in Euclid's algorithm we have obtained $\mathrm{diff}(f_0, f_1) = (1116.93467622, 12607.8786650)$.

Example 3. Let f_0 and f_1 be the following polynomials:

$$\begin{aligned} f_0(x) &= (x-11)^2(x-8)^2(x-6)^2(x-1)^2(x+2)^2(x+3)^2, \\ f_1(x) &= (x-15)^2(x-8)(x-6)(x+5)^2(x+11)^2. \end{aligned}$$

Their greatest common divisor u has the form $u(x) = (x-8)(x-6) = x^2 - 14x + 48$.

Let us compare the result which yields the modification of Euclid's algorithm by using c-s transformation with the result which yields the standard implementation represented by the m-file `poly_gcd.m`[1] denoted by u_{Euc}. We have obtained

$$\begin{aligned}
&u_{Euc}(x) = x^2 - 13.99999999946275x + 47.99999999538410,\\
&u_{c\text{-}s}(x) = x^2 - 14.00000000045505x + 48.00000000093540,\\
&\|u_{Euc} - u\|_2 = 4.64705900662480e{-}09,\\
&\|u_{c\text{-}s} - u\|_2 = 1.04021318800892e{-}09,\\
&\mathrm{diff}(f_0, f_1) = (11024639, 32669999)
\end{aligned}$$

More examples have been calculated and we have found out that Euclid's algorithm in matrix form and c-s transformation yield almost the same results for low-degree polynomials, in some cases Euclid's algorithm gives better results. If the degree of both polynomials gets larger, then the c-s transformation yields more accurate results.

3 QR factorization method for computing the greatest common divisor

The following idea described in [7] will be illustrated for the polynomials of degree $m = 4$ and $n = 3$. Let

$$\begin{aligned}
f_0(x) &= x^4 + a_1x^3 + a_2x^2 + a_3x + a_4,\\
f_1(x) &= b_0x^3 + b_1x^2 + b_2x + b_3.
\end{aligned}$$

A companion matrix C_4 associated with the polynomial f has the form

$$C_4 = \begin{bmatrix} 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -a_4 & -a_3 & -a_2 & -a_1 \end{bmatrix}.$$

It is assumed that the coefficient $a_0 = 1$. The matrix $f_1(C_4)$ is very important. The $\mathrm{GCD}(f_0, f_1)$ can be obtained very easily from the matrix $f_1(C_4)$. See the book [1]. Let the Sylvester matrix be split into the four blocks

$$S(f_0, f_1) = \begin{bmatrix} 1 & a_1 & a_2 & | & a_3 & a_4 & 0 & 0\\ 0 & 1 & a_1 & | & a_2 & a_3 & a_4 & 0\\ 0 & 0 & 1 & | & a_1 & a_2 & a_3 & a_4\\ - & - & - & + & - & - & - & -\\ b_0 & b_1 & b_2 & | & b_3 & 0 & 0 & 0\\ 0 & b_0 & b_1 & | & b_2 & b_3 & 0 & 0\\ 0 & 0 & b_0 & | & b_1 & b_2 & b_3 & 0\\ 0 & 0 & 0 & | & b_0 & b_1 & b_2 & b_3 \end{bmatrix} =: \begin{bmatrix} S_{1,1} & S_{1,2}\\ S_{2,1} & S_{2,2} \end{bmatrix}.$$

[1] http://www.mathworks.com/matlabcentral/fileexchange/20859-gcd-of-polynomials

It is clear that all blocks are Toeplitz matrices. It is easy to calculate the Schur complement $S_{2,2}^{(*)} = S_{2,2} - S_{2,1}S_{1,1}^{-1}S_{1,2}$ and according to the well known theory $S_{2,2}^{(*)} = J_4 f_1(C_4) J_4$. Moreover, there exists an orthogonal matrix Q such that

$$QJ_4 f_1(C_4)J_4 = R,$$

where J_4 is a matrix with ones on the counter diagonal and R is an upper-triangular matrix, the last nonzero row of which contains the coefficients of the GCD of f_0 and f_1. Let $\mathrm{GCD}(f_0, f_1) = d_0x^2 + d_1x + d_2$ in our special case. Then the matrix R has the form

$$R = \begin{bmatrix} x & x & x & x \\ 0 & d_0 & d_1 & d_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where the x's indicates elements whose values are unimportant.

In the following examples let u_{Schu} and u_{Hor} denote the greatest common divisor which was obtained using QR-factorization of $S_{2,2}^{(*)} = S_{2,2} - S_{2,1}S_{1,1}^{-1}S_{1,2}$ and from the QR-factorization of $J_m f_1(C_m) J_m$, where $f_1(C_m)$ was constructed by Horner's scheme, respectively. Let us remark that m is the degree of the polynomial f_0.

Example 4. Let f_0, f_1 and u be polynomials from Example 1. We have calculated

$$\begin{aligned} u_{Schu}(x) &= x^2 - 2.00000000001121x - 3.00000000001110, \\ u_{Hor}(x) &= x^2 - 2.00000000000175x - 3.00000000000208, \end{aligned}$$
$$\|u_{Schu} - u\|_2 = 1.577426943291 52e{-11}, \quad \|u_{Hor} - u\|_2 = 2.71252314056005e{-12}\ .$$

Example 5. Let f_0, f_1 and u be polynomials from Example 2. We have obtained

$$\begin{aligned} u_{Schu}(x) &= x^5 - 0.999999985809256x^4 - 8.749999957416437x^3 \\ &\quad -11.499999953695204x^2 - 5.749999978552083x - 0.999999996447228, \\ u_{Hor}(x) &= x^5 - 1.00000000005345x^4 - 8.75000000015926x^3 \\ &\quad -11.50000000017163x^2 - 5.75000000007888x - 1.00000000001314, \end{aligned}$$
$$\|u_{Schu} - u\|_2 = 6.80551732665222e{-08}, \quad \|u_{Hor} - u\|_2 = 2.53131658107744e{-10}\ .$$

Example 6. Let f_0, f_1 and u be the same polynomials as in Example 3. Then the following results have been calculated.

$$\begin{aligned} u_{Schu}(x) &= x^2 - 14.00000000217399x + 48.00000001232704, \\ u_{Hor}(x) &= x^2 - 14.00000000138107x + 48.00000000802783, \end{aligned}$$
$$\|u_{Schu} - u\|_2 = 1.25172701840642e{-08}, \quad \|u_{Hor} - u\|_2 = 8.14575566893229e{-09}\ .$$

The coefficients of the greatest common divisor are obtained from the 10th row of matrix R. Let R be a matrix from QR-factorization of the matrix $J_{12}f_1(C_{12})J_{12}$

where $f_1(C_{12})$ was constructed by Horner's scheme. The coefficients of the greatest common divisor can be obtained from the 10th row. Let us present the elements of R in 9th–11th row.

row 9: $-8.32446551450895e{+}05$, $1.13733574278167e{+}07$, $-3.60249143760594e{+}07$, $-1.34829260320804e{+}07$

row 10: $-7.99170726641915e{+}05$, $1.11883901740905e{+}07$, $-3.83601948852275e{+}07$ (u_{Hor} is obtained after transformation to monic form)

row 11: $5.13598466751748e{-}03$, $-2.90332235929244e{-}02$.

If the degree of greatest common divisor is not known, the following problem appears: Are the numbers in the 11th row zero? This difficult question is behind the topic of this short paper. For more details see [7].

4 Summary

Euclid's algorithm is composed from sequence of steps and division of two polynomials, whose degree is decreasing to zero, is represented in each particular step. New modification of Euclid's algorithm called *c-s* transformation has been introduced in this article. This modification produces better numerical results in comparison with classical Euclid's algorithm and it is conducted by transformation of the Sylvester matrix and structure of the Sylvester matrix is preserved. The algorithm based on QR-decomposition was mentioned and its numerical results were compared with *c-s* transformation. The second algorithm does not preserve the structure of Sylvester matrix.

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References

[1] Barnett, S.: *Polynomial and linear control system.* Marcel Dekker, New York, USA, 1983.

[2] Eliaš, J.: *Problems connected with the calculation of the GCD.* Bachelor thesis, Charles University, Faculty of Mathematics and Physics at Prague, 2009.

[3] Kaltofen, E., Yang, Z., and Zhi. L.: Structured low rank approximation of a Sylvester matrix. Preprint, 2005.

[4] Laidacker, M. A.: Another Theorem Relating Sylvester's Matrix and the Greatest Common Divisors. Mathematics Magazine **42** (1969), 126–128.

[5] Li, B., Yang, Z., and Zhi, L.: Fast low rank approximation of a Sylvester matrix by structured total least norm. J. Japan Soc. Symbolic and Algebraic Comp. **11** (2005), 165–174.

[6] Winkler, J. R. and Allan, J. D.: Structured total least norm and approximate GCDs of inexact polynomials. J. Comput. Appl. Math. **215** (2008), 1–13

[7] Zarowski, C. J., Ma, X., and Fairman, F. W.: QR- factorization method for computing the greatest common divisor of two polynomials with inexact coeffixients. IEEE Trans. Signal Processing **48**, No. 11 (2000), 3042–3051.