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USING DOMAIN DECOMPOSITION METHOD FOR THE STRESS-STRAIN ANALYSIS OF THE ARTIFICIAL JOINT REPLACEMENTS *

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Abstract

The paper presents some results on mathematical simulations of a total knee joint replacement. The finite element method and the non-overlapping decomposition technique for the contact problem in elasticity are applied. Numerical experiments are presented.

1. Introduction

In the present paper we deal with the mathematical simulations of total knee replacements and simulations of mechanical processes taking place during static loadening. The model problem investigated will be formulated as the primal semi-coercive contact problem with a given friction. For the numerical solution of the problem the non-overlapping domain decomposition method is used.

2. The model

The model of the human knee is based on the contact problem in elasticity and on the finite element approximation. Let the investigated part of the knee joint occupy a union Ω of bounded domains Ω^f and Ω^t in $\mathbb{R}^N$ $(N = 2)$. Domains Ω^f and Ω^t denote separate components of the knee joint (the femur - f and the tibia together with the fibula - t) with Lipschitz boundaries. Let the boundary $\partial\Omega = \partial\Omega^f \cup \partial\Omega^t$ consist of three disjoint parts such that $\partial\Omega = \Gamma_\tau \cup \Gamma_u \cup \Gamma_c$. Let $\Gamma_\tau = {}^1\Gamma_\tau \cup {}^2\Gamma_\tau$, where we denote the loaded part of the femur by ${}^1\Gamma_\tau$ and the unloaded part of the boundary $\partial\Omega$ by ${}^2\Gamma_\tau$. By Γ_u, we denote the part of the tibia boundary where we simulate its fixation. A common contact boundary between both joint components Ω^f and Ω^t before deformation we denote by $\Gamma_c = \partial\Omega^f \cap \partial\Omega^t$. Let body forces $\mathbf{F}$, surface tractions $\mathbf{P}$ and slip limits g^{ft} be given.

We have the following problem: find displacements $\mathbf{u}^\iota$ such that

$$\frac{\partial}{\partial x_j}\tau_{ij}(\mathbf{u}^\iota) + F_i^\iota = 0 \quad \text{in } \Omega^\iota,\ \iota = f, t,\ i = 1, \dots, N, \tag{1}$$

where the stress tensor τ_{ij} is defined by

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$$\tau_{ij}(\mathbf{u}^\iota) = c^\iota_{ijkl} e_{kl}(\mathbf{u}^\iota) \qquad \text{in } \Omega^\iota,\ \iota = f, t,\ i = 1, \ldots, N, \tag{2}$$

with boundary conditions

$$\tau_{ij}(\mathbf{u}) n_j = P_i \quad \text{ on } {}^1\Gamma_\tau,\ i = 1, \ldots, N, \tag{3}$$

$$\tau_{ij}(\mathbf{u}) n_j = 0 \quad \text{ on } {}^2\Gamma_\tau,\ i = 1, \ldots, N, \tag{4}$$

$$\mathbf{u} = \mathbf{u}_0\, (= 0) \qquad \text{on } \Gamma_u, \tag{5}$$

$$u^f_n - u^t_n \le 0,\ \tau^f_n \le 0,\ (u^f_n - u^t_n)\tau^f_n = 0 \quad \text{on } \Gamma_c, \tag{6}$$

$$\begin{array}{l} |\tau^{ft}_t| \le g^{ft} \quad \text{on } \Gamma_c, \\ |\tau^{ft}_t| < g^{ft} \implies u^f_t - u^t_t = 0, \\ |\tau^{ft}_t| = g^{ft} \implies \text{there exists } \vartheta \ge 0 \text{ such that } u^f_t - u^t_t = -\vartheta \tau^{ft}_t. \end{array} \tag{7}$$

Here $e_{ij}(\mathbf{u}) = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$ is the small strain tensor, n_i are the components of outward normal to $\partial\Omega$ and $\mathbf{t} = (-n_2, n_1)$ is tangential vector. Normal component of displacement vector $\mathbf{u}$ on the boundary is $u^f_n = u^f_i n^f_i$, resp. $u^t_n = u^t_i n^t_i$. Tangential component of displacement vector $\mathbf{u}$ on the boundary is $u^f_t = u^f_i t^f_i$, resp. $u^t_t = u^t_i t^t_i$. Normal and tangential components of stress vector are given by $\tau^f_n = \tau^f_{ij} n^f_j n^f_i$ and $\tau^f_t = \tau^f_{ij} n^f_j t^f_i$ and $\tau^{ft}_t \equiv \tau^f_t$.

Assume that c^ι_{ijkl} are positive definite symmetric matrices such that

$$0 < c^\iota_0 \le c^\iota_{ijkl}\xi_{ij}\xi_{kl} \mid \xi \mid^{-2} \le c^\iota_1 < +\infty \quad \text{for a.a. } \mathbf{x} \in \Omega^\iota, \xi \in \mathbb{R}^{N^2},\ \xi_{ij} = \xi_{ij}\,,$$

where c^ι_0, c^ι_1 are constants independent of $\mathbf{x} \in \Omega^\iota$.

Let us introduce $W = \prod_{\iota=f,t}[H^1(\Omega^\iota)]^N$, $\|\mathbf{v}\|_W = (\sum_{\iota=f,t}\sum_{i\le N}\|v^\iota_i\|^2_{1,\Omega^\iota})^{\frac{1}{2}}$ and the sets of virtual and admissible displacements $V_0 = \{\mathbf{v} \in W \mid \mathbf{v} = 0 \text{ on } \Gamma_u\}$, $V = \mathbf{u}_0 + V_0$, $K = \{\mathbf{v} \in V \mid v^f_n - v^t_n \le 0 \text{ on } \Gamma_c\}$. Assume that $u^f_{0n} - u^t_{0n} = 0$ on Γ_c. Let $c^\iota_{ijkl} \in L^\infty(\Omega^\iota)$, $F^\iota_i \in L^2(\Omega^\iota)$, $P_i \in L^2({}^1\Gamma_\tau)$, $\mathbf{u}^\iota_0 \in [H^1(\Omega^\iota)]^N$.

By reformulating the original problem, we arrive to the following variational problem $(\mathcal{P})$: find a function $\mathbf{u}$, $\mathbf{u} - \mathbf{u}_0 \in K$, such that

$$a(\mathbf{u}, \mathbf{v} - \mathbf{u}) + j(\mathbf{v}) - j(\mathbf{u}) \ge L(\mathbf{v} - \mathbf{u}) \qquad \forall\, \mathbf{v} \in K \tag{8}$$

holds, where

$$\begin{array}{l} a(\mathbf{u}, \mathbf{v}) = \sum_{\iota=f,t}\int_{\Omega^\iota} c^\iota_{ijkl} e_{ij}(\mathbf{u}^\iota) e_{kl}(\mathbf{v}^\iota)\, d\mathbf{x}, \\ j(\mathbf{v}) = \int_{\Gamma_c} g^{ft} \mid \mathbf{v}^f_t - \mathbf{v}^t_t \mid ds, \\ L(\mathbf{v}) = \sum_{\iota=f,t}\int_{\Omega^\iota} F^\iota_i v^\iota_i\, d\mathbf{x} - \sum_{\iota=f,t}\int_{\Gamma^\iota_\tau} P^\iota_i v^\iota_i\, ds. \end{array} \tag{9}$$

Let us define the sets of rigid displacements and rotations
$P = P^f \times P^t$, $P^\iota = \{\mathbf{v}^\iota = (v^\iota_1, v^\iota_2) \mid v^\iota_1 = a^\iota_1 - b^\iota x_2, v^\iota_2 = a^\iota_2 + b^\iota x_1\}$, where $a^\iota_i, i = 1, 2$, and b^ι are arbitrary real constants and $\iota = f, t$.

It can be shown that the problem (8) has a unique solution if:

$$L(\mathbf{v}) < j(\mathbf{v}) \quad \forall \mathbf{v} \in P \cap K - \{0\},$$
$$\{\mathbf{v} \in P \cap V_0 \mid \ v_n^f - v_n^t = 0 \ \text{ on } \Gamma_c\} = \{0\}$$
$$\text{and} \quad |L(\mathbf{v})| > j(\mathbf{v}) \quad \forall \mathbf{v} \in P \cap V_0 - \{0\}.$$

3. Short description of the domain decomposition algorithm

The principle of domain decomposition method consists in splitting domain on smaller subdomains so that the dimensions of partial problems are smaller than the dimension of the original problem. Let every domain $\overline{\Omega}^\iota = \cup_{i=1}^{J(\iota)} \overline{\Omega}_i^\iota$, where $J(\iota)$ is a number of subdomains of Ω^ι. Suppose that subdomains Ω^ι have Lipschitz boundaries. Let $\Gamma_i^\iota = \partial\Omega_i^\iota \backslash \partial\Omega^\iota$, $\iota = f, t$, $i \in \{1, \ldots, J(\iota)\}$, be a part of interface boundary $\Gamma = \cup_{\iota=f,t} \cup_{i=1}^{J(\iota)} \Gamma_i^\iota$. Let

$$T^\iota = \{j \in \{1, .., J(\iota)\} : \overline{\Gamma}_c \cap \partial\overline{\Omega}_j^\iota = \emptyset\}, \iota = f, t, \tag{10}$$

be the set of all the indices of those subdomains of the domain Ω^ι that are not adjacent to a contact, and let

$$\Omega^{*j} = \cup_{[i,\iota]\in\vartheta} \Omega_i^\iota, \tag{11}$$

where $\vartheta = \{[i, \iota] : \partial\Omega_i^\iota \cap \Gamma_c \neq \emptyset\}$ represents subdomains in unilateral contact. Suppose that $\Gamma \cap \Gamma_c = \emptyset$. Then for the trace operator $\gamma : [H^1(\Omega_i^\iota)]^N \to [L^2(\partial\Omega_i^\iota)]^N$ we have

$$V_\Gamma = \gamma K|_\Gamma = \gamma V|_\Gamma. \tag{12}$$

Let $\gamma^{-1} : V_\Gamma \to V$ be an arbitrary linear inverse mapping satisfying

$$\gamma^{-1}\overline{\mathbf{v}} \ = 0 \quad \text{ on } \Gamma_c \quad \forall \overline{\mathbf{v}} \in V_\Gamma. \tag{13}$$

Let us introduce restrictions $\overline{R}_i^\iota : V_\Gamma \to \Gamma_i^\iota$; $L_i^\iota : L^\iota \to \Omega_i^\iota$; $a_i^\iota(.,.) : a(.,.) \to \Omega_i^\iota$; $V(\Omega_i^\iota) : V \to \Omega_i^\iota$ and let $V^0(\Omega_i^\iota) = \{\mathbf{v} \in V \mid \mathbf{v} = 0 \text{ on } \Omega\backslash\Omega_i^\iota\}$ be the space of functions with zero traces on Γ_i^ι. The algorithm is based on the next theorem and on the use of local and global Schur complements.

Theorem 3.1: A function $\mathbf{u}$ is a solution of a global problem $(\mathcal{P})$, if and only if its trace $\overline{\mathbf{u}} = \gamma\mathbf{u}|_\Gamma$ on the interface Γ satisfies the condition

$$\sum_{\iota=f,t} \sum_{i=1}^{J(\iota)} [a_i^\iota(\mathbf{u}_i^\iota(\overline{\mathbf{u}}), \gamma^{-1}\overline{\mathbf{w}}) - L_i^\iota(\gamma^{-1}\overline{\mathbf{w}})] = 0 \quad \forall \overline{\mathbf{w}} \in V_\Gamma, \overline{\mathbf{u}} \in V_\Gamma \tag{14}$$

and its restrictions $\mathbf{u}_i^\iota(\mathbf{u}) \equiv \mathbf{u}|_{\Omega_i^\iota}$ satisfy:
(i) the condition

$$\begin{aligned} a_i^\iota(\mathbf{u}_i^\iota(\overline{\mathbf{u}}), \varphi_i^\iota) - L_i^\iota(\varphi_i^\iota) \quad \forall \varphi_i^\iota \in V^0(\Omega_i^\iota),\ \mathbf{u}_i^\iota(\overline{\mathbf{u}}) \in V(\Omega_i^\iota), \\ \gamma\mathbf{u}_i^\iota(\overline{\mathbf{u}})|_{\Gamma_i^\iota} = \overline{R}_i^\iota\overline{\mathbf{u}},\ i \in T^\iota, \iota = f, t, \end{aligned} \tag{15}$$

(ii) the condition

$$\begin{aligned}&\textstyle\sum_{[i,\iota]\in\vartheta} a_i^\iota(\mathbf{u}_i^\iota(\overline{\mathbf{u}}),\varphi_i^\iota) + j^\iota(\mathbf{u}_i^\iota(\overline{\mathbf{u}})+\varphi_i^\iota) - j^\iota(\mathbf{u}_i^\iota(\overline{\mathbf{u}})) \geq \\ &\qquad \geq \textstyle\sum_{[i,\iota]\in\vartheta} L_i^\iota(\varphi_i^\iota) \quad \forall\varphi \in (\varphi_i^\iota, [i,\iota]\in\vartheta),\ \varphi_i^\iota \in V^0(\Omega_i^\iota),\end{aligned} \tag{16}$$

and such that

$$\mathbf{u}+\varphi\in K, \quad \gamma\mathbf{u}_i^\iota(\overline{\mathbf{u}})|_{\Gamma_i^\iota} = \overline{R_i^\iota}\overline{\mathbf{u}} \quad \text{for } [i,\iota]\in\vartheta. \tag{17}$$

For the proof see [1].

To analyze the condition (14) the **local and global Schur complements** are introduced. Let

$$V_i^\iota = \{\gamma\mathbf{v}_{|\Gamma_i^\iota} \mid \mathbf{v}\in K\} = \{\gamma\mathbf{v}_{|\Gamma_i^\iota} \mid \mathbf{v}\in V\}$$

and let us define a particular restriction of the inverse mapping $\gamma^{-1}(.)|_{\Omega_i^\iota}$ by

$$\begin{aligned}&Tr_{i\iota}^{-1} : V_i^\iota \to V(\Omega_i^\iota),\ \gamma(Tr_{i\iota}^{-1}\overline{\mathbf{u}})|_{\Gamma_i^\iota} = \mathbf{u}_i^\iota,\ i=1,\ldots,J(\iota),\ \iota=f,t\\ &a_i^\iota(Tr_{i\iota}^{-1}\overline{\mathbf{u}}_i^\iota,\mathbf{v}_i^\iota) = 0 \quad \forall\mathbf{v}_i^\iota\in V_0^0(\Omega_i^\iota),\\ &Tr_{i\iota}^{-1}\overline{\mathbf{u}}_i^\iota \in V(\Omega_i^\iota) \text{ for } i\in T^\iota, \iota=f,t,\end{aligned} \tag{18}$$

where $V_0^0(\Omega_i^\iota) = \{\mathbf{v}\in V_0 \mid \mathbf{v}=0 \ \text{ on } \Omega\backslash\Omega_i^\iota\}$. For $[i,\iota]\in\vartheta$ we complete the definition by the boundary condition (13), i.e.

$$Tr_{i\iota}^{-1}\overline{\mathbf{u}}_i^\iota = 0 \text{ on } \Gamma_c. \tag{19}$$

The local Schur complement for $i\in T^\iota$ is the operator $\mathcal{S}_i^\iota : V_i^\iota \to (V_i^\iota)^*$ defined by

$$\langle\mathcal{S}_i^\iota\overline{\mathbf{u}}_i^\iota,\overline{\mathbf{v}}_i^\iota\rangle = a_i^\iota(Tr_{i\iota}^{-1}\overline{\mathbf{u}}_i^\iota, Tr_{i\iota}^{-1}\overline{\mathbf{v}}_i^\iota) \quad \forall\overline{\mathbf{u}}_i^\iota,\overline{\mathbf{v}}_i^\iota\in V_i^\iota. \tag{20}$$

For subdomains which are in contact we define a **common local Schur complement** for the union $\Omega_i^f\cup\Omega_j^t$ (where $[i,f]\in\vartheta, [j,t]\in\vartheta$) as the operator $\mathcal{S}^{ft}$: $(V_i^f\times V_j^t)\to(V_i^f\times V_j^t)^* = (V_i^f)^*\times(V_j^t)^*$ defined by

$$\begin{aligned}\left\langle\mathcal{S}^{ft}(\overline{\mathbf{y}}_i^f,\overline{\mathbf{y}}_j^t),(\overline{\mathbf{v}}_i^f,\overline{\mathbf{v}}_j^t)\right\rangle &= a_i^f(\mathbf{u}_i^f(\overline{\mathbf{y}}_i^f),Tr_{if}^{-1}\overline{\mathbf{v}}_i^f) + a_j^t(\mathbf{u}_j^t(\overline{\mathbf{y}}_j^t),Tr_{jt}^{-1}\overline{\mathbf{v}}_j^t)\\ &\qquad \forall(\overline{\mathbf{v}}_i^f,\overline{\mathbf{v}}_j^t)\in V_i^f\times V_j^t,\end{aligned} \tag{21}$$

where Tr_{if}^{-1} and Tr_{jt}^{-1} are defined by means of (18) and (19).

The condition (14) can be expressed by means of local Schur complements in the form

$$\begin{aligned}&\textstyle\sum_{\iota=f,t}\sum_{i\in T^\iota}\langle\mathcal{S}_i^\iota\overline{\mathbf{u}}_i^\iota,\overline{\mathbf{v}}_j^\iota\rangle + \sum_{\iota=f,t}\left\langle\mathcal{S}^{ft}(\overline{\mathbf{u}}_i^f,\overline{\mathbf{u}}_j^t),(\overline{\mathbf{v}}_i^f,\overline{\mathbf{v}}_j^t)\right\rangle = \\ &\qquad = \textstyle\sum_{\iota=f,t}\sum_{i=1}^{J(\iota)} L_i^\iota(Tr_{i\iota}^{-1}\overline{\mathbf{v}}_i^\iota) \quad \forall\mathbf{v}\in V_\Gamma, [i,f]\in\vartheta, [j,t]\in\vartheta,\end{aligned} \tag{22}$$

where $\overline{\mathbf{u}} = \gamma\mathbf{u}_{|\Gamma}$, $\overline{\mathbf{v}}_i^\iota = \overline{R}_i^\iota\overline{\mathbf{v}}, \overline{\mathbf{u}}_i^\iota = \overline{R}_i^\iota\overline{\mathbf{u}}$. Then we will solve the equation (22) on the interface Γ in the dual space $(V_\Gamma)^*$. We rewrite (22) into the following form

$$\mathcal{S}_0\overline{\mathbf{U}} + \mathcal{S}_{CON}\overline{\mathbf{U}} = \mathbf{F}, \tag{23}$$

where

$$\begin{array}{c}\mathcal{S}_0 = \sum_{\iota=f,t} \sum_{i\in T^\iota} (\overline{R}^\iota_i)^T \mathcal{S}^\iota_i \overline{R}^\iota_i,\\ \mathcal{S}_{CON} = \sum_{\iota=f,t} \overline{R}^T_{ft} \mathcal{S}^{ft} \overline{R}_{ft},\\ \mathbf{F} = \sum_{\iota=f,t} \sum_{i=1}^{J(\iota)} (\overline{R}^\iota_i)^T (Tr^{-1}_{i\iota})^T \mathcal{S}^\iota_i\end{array} \tag{24}$$

and $\overline{R}_{ft}(\overline{\mathbf{u}}) = (\overline{R}^f_i(\overline{\mathbf{u}}), \overline{R}^t_j(\overline{\mathbf{u}}))^T, \overline{\mathbf{u}} \in V_\Gamma, [i,f] \in \vartheta, [j,t] \in \vartheta$. Equation (23) will be solved by **successive approximations**, because the operators $\mathcal{S}^{ft}$ and therefore $\mathcal{S}_{CON}$ are nonlinear. As an initial approximation $\overline{\mathbf{U}}^0$ we choose the solution of the global primal problem, where the boundary conditions on Γ_c are replaced by the linear bilateral conditions with $g^{ft} \equiv 0$ (i.e. $j(\mathbf{u}) \equiv 0$)

$$u^f_n - u^t_n = 0,\ \tau^{ft}_t = 0 \ \text{ on } \Gamma_c. \tag{25}$$

Then we replace the set K by $K^0 = \{\mathbf{v} \in V \mid v^f_n - v^t_n = 0 \text{ on } \Gamma_c\}$ and solve the following problem

$$\mathbf{u}^0 = \arg\min_{\mathbf{v}\in K^0} \mathcal{L}(\mathbf{v}) \tag{26}$$

where $\mathcal{L}(\mathbf{v}) = \frac{1}{2}a(\mathbf{v},\mathbf{v}) - L(\mathbf{v})$. We set $\overline{\mathbf{U}}^0 = \gamma\mathbf{u}^0|_\Gamma$ to get an initial approximation. The auxiliary problem (26) represents a linear elliptic boundary value problem with bilateral contact and it can be solved by the domain decomposition method again.

The non-linear equation (23) is solved by successive approximations. We assume that the approximation $\overline{\mathbf{U}}^{k-1}$ is known, and we find the next approximation $\overline{\mathbf{U}}^k$as the solution of the following linear problem

$$\mathcal{S}_0\overline{\mathbf{U}}^k = \mathbf{F} - \mathcal{S}_{CON}\overline{\mathbf{U}}^{k-1}, k = 1, 2, \ldots \tag{27}$$

In [1] the convergence of the successive approximation (27) to the solution of the original problem (23) is proved in the space $(V_\Gamma)^*$.

Problems (26) and (27) are solved by the finite element method.

4. Discussion of numerical results

The model of the knee joint replacement was derived from the X-ray image after application the total knee prosthesis. The paper presents three models. Differences are given by varied angle between the vertical axis and axis of the femur. The values are 3 degree in model (a), 5 degree in model (b) and 7 degree in model (c). All models are presented in Fig. 1.

In the model the material parameters are as follows: Bone: Young's modulus $E = 1.71 \times 10^{10}$ $[Pa]$, Poisson's ratio $\nu = 0.25$, (M1) $Ti6Al4V$: $E = 1.15 \times 10^{11}$ $[Pa]$, $\nu = 0.3$, (M2) Chirulen: $E = 3.4 \times 10^8$ $[Pa]$, $\nu = 0.4$, (M3) $CoCrMo$: $E = 2.08 \times 10^{11}$ $[Pa]$, $\nu = 0.3$. The femur is loaded between points 5 and 6 by $0.215 \times 10^7 [Pa]$. The tibia is fixed between points 1 and 2, the fibula is fixed between points 3 and 4. The unilateral contact boundary is supposed between points 7 and 8 as well as between 9 and 10. The domain is decomposed into 13 subdomains. The discretization

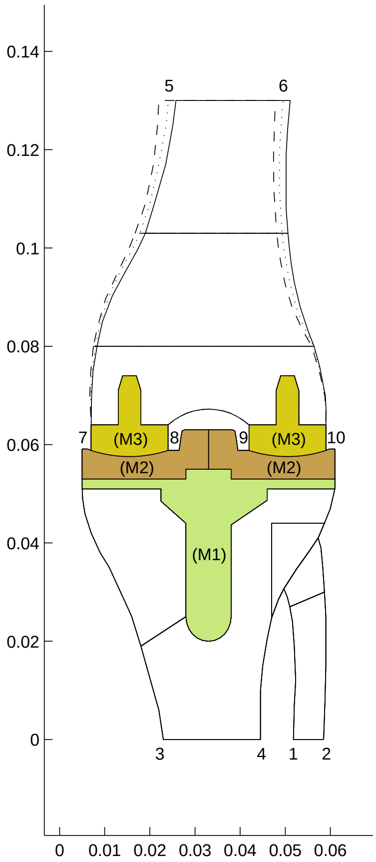

Fig. 1: *The models.*

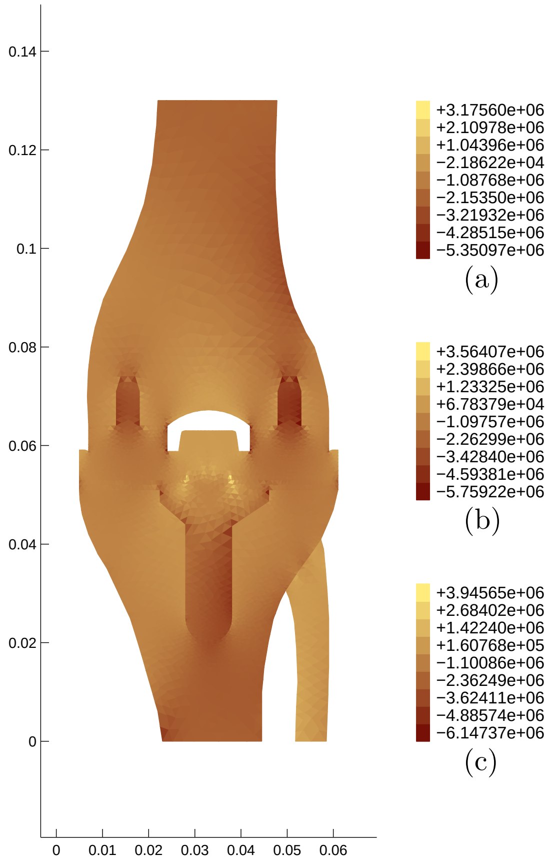

Fig. 2: *The vertical stress tensor component.*

comprises 3800 nodes, 7200 elements, 62 unilateral contact nodes and 350 interface elements. The loadings evoked by muscular forces were neglected. For each model, we obtain solution (with accuracy 10^{-6}) after 19 iterations of the PCG algorithm for the auxiliary problem and 14 iterations of the successive approximations method with total 36 iterations of the PCG algorithm for the original problem. In Fig. 2 the vertical stress tensor components for the frontal cross-section are presented, while in Figs 3 a,b,c the principal stresses are presented. The presented graphical results represent distribution of stresses in the femur, in the total prosthesis and in the tibia as well as in the fibula.

The obtained numerical results correspond to the stress fields in the bones and in the knee prosthesis observed in orthopaedic practice. The presented models facilitate to comparison of the prostheses made from different materials as the $CoCrMo$ alloy, the Al_2O_3 and ZrO_2 ceramics. The aim of the mathematical modelling of the knee prosthesis is to determine the best version of the knee prosthesis.

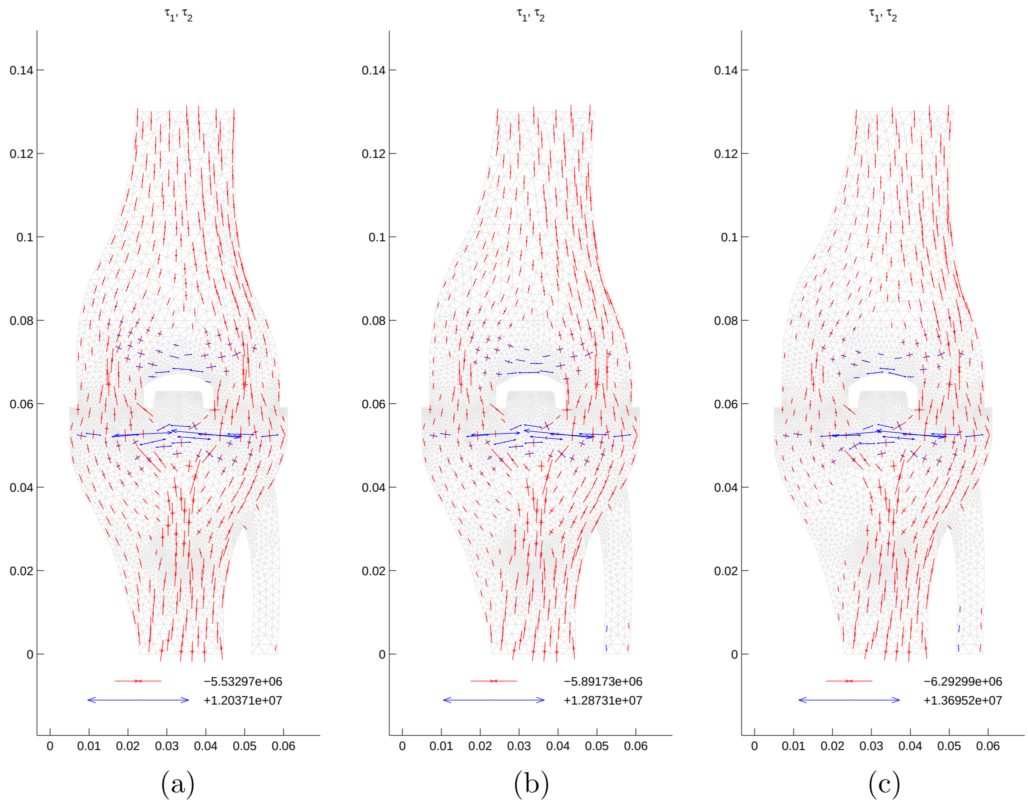

Fig. 3: *The principal stresses (→← represents compression and ←→ extension).*

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