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MATRIX-FREE FORMULATION OF THE STOCHASTIC NEWMARK METHOD*

Cyril Fischer

Abstract

The response of an arbitrary discretized system to the random movement has been solved in probabilistic terms. The excitation has been defined as a combination of the time modulated band limited stationary random processes approximating the evolutionary power spectra of a true seismic record. The solution is based either on the modified version of the stochastic Newmark method or on the spectral differential decomposition of the excitation. Special attention has been paid to the applicability of the methods to the sparse problems, especially to their matrix-free formulation.

1. Introduction

The study of the dynamic behaviour of linear and non-linear systems subjected to random excitation is of great importance in reliability and safety analysis in engineering practice. Recently, there has been an increasing interest in direct stochastic integration schemes due to their simplicity and applicability to multi-degree-of-freedom linear and non-linear systems, see e.g. [2], [5] or [9].

2. Stochastic description of the response of the linear structure

Let us assume the linear system of equations, describing the motion of a structure (given by it's mass $\mathbf{M}$, damping $\mathbf{B}$ and stiffness $\mathbf{C}$) due to the movement of the subsoil

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{B}\dot{\mathbf{u}}(t) + \mathbf{C}\mathbf{u}(t) = -\mathbf{F}\dot{\mathbf{v}}(t) - \mathbf{G}\mathbf{v}(t) \tag{1}$$

where $\mathbf{M}$, $\mathbf{B}$, $\mathbf{C}$ are real (sparse) matrices of dimension $N \times N$, $\mathbf{F}$, $\mathbf{G}$ are real rectangular $N \times N_s$ matrices describing input of the excitation into the structure, $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are the vectors of the unknown response (of length N) and known, but stochastic loading processes (N_s), respectively. The initial conditions are assumed to be zero. The damping matrix $\mathbf{B}$ is supposed to be a general real matrix, so the non-proportional damping model is considered in the following text. This is also the reason why the term $\mathbf{F}\dot{\mathbf{v}}(t)$ in the equation (1) occurs.

The components $v_i(t)$ of the excitation vector process $\mathbf{v}(t)$ are assumed to have the form of a stationary discrete ARMA(p, q) processes $v_{si}(t)$ that are modulated by deterministic functions $m_i(t)$

$$v_i(t) = m_i(t)v_{si}(t). \tag{2}$$

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2.1. Approaches available

Time varying covariance matrix of the response of the structure can be determined using various approaches: Monte Carlo simulations, approximative numerical methods or exact formulae.

Leaving besides simulation techniques, there is only a small selection of numerical methods available for the supposed general case: several variants of central differences or stochastic Newmark methods (proposed in [8], [9]), approximation of the exact formula, introduced by Náprstek in [5] as Correlation method or exact formula (Spectral decomposition method [6]).

2.2. Stochastic Newmark method

2.2.1. Newmark method formulation

The stochastic Newmark method is variant of the famous Newmark method adapted to the stochastic world firstly by To in [9] and further elaborated by Zhang in [10].

The variant presented here originates from Zhang's work, but it is slightly extended for a more general description of input motion; it attempts to introduce the matrix-free method for large sparse systems and, last but not least, it corrects some oversights from the original paper.

The Newmark method can be rewritten in the recurrent form for the unknown vector $\mathbf{U}_n = \begin{pmatrix} \mathbf{u}(t_n) \\ \dot{\mathbf{u}}(t_n) \end{pmatrix}$:

$$\mathbf{U}_{n+1} = \mathbf{T}_A \mathbf{U}_n + \mathbf{T}_B \mathbf{P}_n, \tag{3}$$

where

$$\mathbf{T}_A = \begin{pmatrix} \mathbf{N}_2 & \mathbf{N}_3 \\ \mathbf{N}_5 & \mathbf{N}_6 \end{pmatrix}, \ \mathbf{T}_B = \begin{pmatrix} \mathbf{N}_4 & \beta\Delta t^2\,\mathbf{N}_1 \\ \mathbf{N}_7 & \gamma\Delta t\,\mathbf{N}_1 \end{pmatrix}, \ \mathbf{P}_n = \begin{pmatrix} \mathbf{p}_n \\ \mathbf{p}_{n+1} \end{pmatrix}, \tag{4}$$

$$\begin{aligned}
\mathbf{N}_1 &= \left(\mathbf{M} + \gamma\Delta t\,\mathbf{B} + \beta\Delta t^2\,\mathbf{C}\right)^{-1}, \\
\mathbf{N}_2 &= \mathbf{I} - \mathbf{N}_1\left(\frac{1}{2}\Delta t^2\mathbf{C} + \left(\frac{\gamma}{2} - \beta\right)\Delta t^3\mathbf{B}\mathbf{M}^{-1}\mathbf{C}\right), \\
\mathbf{N}_3 &= \mathbf{N}_1\left(\Delta t\mathbf{M} + \left(\gamma - \frac{1}{2}\right)\Delta t^2\mathbf{B} - \left(\frac{\gamma}{2} - \beta\right)\Delta t^3\mathbf{B}\mathbf{M}^{-1}\mathbf{B}\right), \\
\mathbf{N}_4 &= \mathbf{N}_1\left(\left(\frac{1}{2} - \beta\right)\Delta t^2\mathbf{I} + \left(\frac{\gamma}{2} - \beta\right)\Delta t^3\mathbf{B}\mathbf{M}^{-1}\right), \\
\mathbf{N}_5 &= -\mathbf{N}_1\left(\Delta t\mathbf{C} - \left(\frac{\gamma}{2} - \beta\right)\Delta t^3\mathbf{C}\mathbf{M}^{-1}\mathbf{C}\right), \\
\mathbf{N}_6 &= \mathbf{I} - \mathbf{N}_1\left(\gamma\Delta t^2\mathbf{C} + \Delta t\mathbf{B} - \left(\frac{\gamma}{2} - \beta\right)\Delta t^3\mathbf{C}\mathbf{M}^{-1}\mathbf{B}\right), \\
\mathbf{N}_7 &= \mathbf{N}_1\left((1-\gamma)\,\Delta t\mathbf{I} - \left(\frac{\gamma}{2} - \beta\right)\Delta t^3\mathbf{C}\mathbf{M}^{-1}\right),
\end{aligned}$$

and β a γ are the parameters of the Newmark method.

2.2.2. Formulation of ARMA excitation process

The definition of the discrete ARMA(p,q) model represented by a random sequence η_n

$$\eta_n + \sum_{j=1}^{p} a_j \eta_{n-j} = w_n + \sum_{j=1}^{q} b_j w_{n-j} \tag{5}$$

where w_j is the Gaussian white noise process, $\mathsf{E} w_i w_i = \sigma_w^2$, can be written in the matrix form

$$\boldsymbol{\eta}_n = \boldsymbol{\mathcal{A}} \boldsymbol{\eta}_{n-1} + \mathbf{q}_n, \tag{6}$$

where the matrix $\boldsymbol{\mathcal{A}}$ has the form

$$\boldsymbol{\mathcal{A}} = \begin{pmatrix} -a_1 & -a_2 & \cdots & -a_p \\ 1 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}_{p \times p} \tag{7}$$

and

$$\mathbf{q}_n = \begin{pmatrix} w_n + \sum_{j=1}^{q} b_j w_{n-j} \\ \mathbf{0}_{p-1 \times 1} \end{pmatrix} \quad , \quad \boldsymbol{\eta}_n = \begin{pmatrix} \eta_n \\ \vdots \\ \eta_{n-p+1} \end{pmatrix}. \tag{8}$$

2.2.3. Newmark + ARMA combination

The excitation vector $\boldsymbol{\eta}_j$ can be added to the vector of unknowns to set up a new vector $\mathbf{y}_n = \left(\mathbf{u}_{n+1}, \dot{\mathbf{u}}_{n+1}, \boldsymbol{\eta}_{n+2}\right)^{\mathsf{T}}$. The equation (3) then acquires a simpler form

$$\mathbf{y}_{n+1} = \mathbf{T}_n \mathbf{y}_n + \mathbf{z}_{n+2}, \tag{9}$$

where the matrix $\mathbf{T}_n$ (of dimension $(2N+p) \times (2N+p)$) becomes time dependent due to its dependence on the modulation function. It consists of 4 sub-matrices

$$\mathbf{T}_n = \begin{pmatrix} \mathbf{T}_A & \mathbf{T}_{C,n} \\ \mathbf{0} & \boldsymbol{\mathcal{A}} \end{pmatrix} \quad \text{and } \mathbf{z} \text{ is the vector} \quad \mathbf{z}_{n+2} = \begin{pmatrix} \mathbf{0}_{2N} \\ \mathbf{q}_{n+2} \end{pmatrix}.$$

Here $\mathbf{T}_A$ is defined by (4) and

$$\mathbf{T}_{C,n} = \begin{pmatrix} \mathbf{N}_4 & \beta \Delta t^2 \mathbf{N}_1 \\ \mathbf{N}_7 & \gamma \Delta t \mathbf{N}_1 \end{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{s}(t_k) & \mathbf{F} \frac{\mathbf{m}(t_k)}{\Delta t} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{s}(t_{k+1}) & \mathbf{F} \frac{\mathbf{m}(t_{k+1})}{\Delta t} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \end{pmatrix}.$$

The size of $\mathbf{T}_{Cn}$ is $2N \times p$.

$$\mathbf{p}_k = \mathbf{p}(t_k) = \mathbf{s}(t_k)\eta_k + \mathbf{F} \frac{\mathbf{m}(t_k)}{\Delta t} \eta_{k-1}\,, \quad \text{where} \quad \mathbf{s}(t) = -\mathbf{F} \left(\dot{\mathbf{m}}(t) + \frac{\mathbf{m}(t)}{\Delta t} \right) + \mathbf{G}\mathbf{m}(t).$$

The derivation of the loading process $\dot{\mathbf{v}}(t)$ is approximated by the backward difference

$$\mathbf{v}(t_k) = \mathbf{m}(t_k)\eta_k \quad \text{and} \quad \dot{\mathbf{v}}(t_k) = \dot{\mathbf{m}}(t_k)\eta_k + \mathbf{m}(t_k)\frac{\eta_k - \eta_{k-1}}{\Delta t}.$$

It should be noted here that it is necessary to use at least 3 autoregressive coefficients a_1, a_2, a_3 of the considered ARMA model (5) even if some of them would be zero.

2.2.4. Formula for the autocovariance matrix

The formula for the autocovariance matrix follows from the equation (9):

$$\mathbf{R}(t_n) = \mathsf{E}\mathbf{y}_{n+1}\mathbf{y}_{n+1}^{\mathsf{T}} = \mathbf{T}_n\mathsf{E}\mathbf{y}_n\mathbf{y}_n^{\mathsf{T}}\mathbf{T}_n^{\mathsf{T}} + \mathbf{K}_0 + \mathbf{K}_n + \mathbf{K}_n^{\mathsf{T}} = \mathbf{T}_n\mathsf{E}\mathbf{y}_n\mathbf{y}_n^{\mathsf{T}}\mathbf{T}_n^{\mathsf{T}} + \mathbf{Z}_n, \quad (10)$$

where $\mathbf{Z}_n = \mathbf{K}_0 + \mathbf{K}_n + \mathbf{K}_n^{\mathsf{T}}$, $\mathbf{K}_0 = (\mathsf{E}\mathbf{z}_n\mathbf{z}_n^{\mathsf{T}})$ and $\mathbf{K}_n = \mathbf{T}_n\mathsf{E}\mathbf{y}_n\mathbf{z}_{n+2}^{\mathsf{T}}$. Matrix $\mathbf{K}_0$ has the only nonzero element c_0 at the position $(2N+1, 2N+1)$

$$c_0 = \left(1 + \sum_{i=1}^{q} b_i^2\right)\sigma_w^2,$$

while

$$\mathbf{K}_n = \mathbf{T}_n\mathsf{E}\mathbf{y}_n\mathbf{z}_{n+2}^{\mathsf{T}} = \sum_{j=1}^{q}\left(\prod_{k=n}^{n-q+1}\mathbf{T}_k\right)\boldsymbol{\xi}_j. \quad (11)$$

In the equation (11), the matrices $\boldsymbol{\xi}_j,\ j = 1, \ldots, q$, are the square $(2N+p)\times(2N+p)$ matrices, each of them has one nonzero element c_j again at the position (2*N*+1,2*N*+1);

$$c_j = \begin{cases} \left(b_j + \sum_{i=1}^{q-j} b_i b_{i+j}\right)\sigma_w^2 & \text{for } j \le q\ , \\ \\ 0 & \text{otherwise.} \end{cases}$$

As the matrices $\boldsymbol{\xi}_j$ have only one nonzero entry, the matrices $\mathbf{K}_n$ have nonzero the $(2N+1)$-th column only.

The formula (10) is not applicable in the case of a large sparse system because the covariance matrix is dense, even when its rank is 1. Moreover, only some entries of the covariance matrix are significant for the assessment of the structure in the most cases. At some additional expense, the algorithm can be reformulated also for the sparse systems.

Taking into account the deterministic (zero) initial condition $\mathbf{y}_0$, it can be derived from the equation (9) that

$$\mathsf{E}\mathbf{y}_{n+1}\mathbf{y}_{n+1}^{\mathsf{T}} = \sum_{i=0}^{n}\left[\prod_{k=i}^{1}\mathbf{T}_k\right]\mathbf{Z}\left[\prod_{k=i}^{1}\mathbf{T}_k\right]^{\mathsf{T}} = \sum_{i=0}^{n}\left[\prod_{k=i}^{1}\mathbf{T}_k\right]\left(c_1\mathbf{e}\mathbf{e}^{\mathsf{T}} + \mathbf{k}\mathbf{e}^{\mathsf{T}} + \mathbf{e}\mathbf{k}^{\mathsf{T}}\right)\left[\prod_{k=i}^{1}\mathbf{T}_k\right]^{\mathsf{T}}, \quad (12)$$

where $\mathbf{e} = (0, \ldots, 0, 1, 0, \ldots, 0)^{\mathsf{T}}$ with the 1 at the position $2N + 1$ and $\mathbf{k}$ is the only nonzero column of the matrix $\mathbf{K}_0$, i.e., that one with index $(2N + 1)$. Instead of the matrix $\mathbf{K}_0$ another suitable matrix can be used, which fulfills the initial conditions for the ARMA(p, q) process.

Thus, in every step there are only two vectors to be dealt with.

Algorithm 1.

1. prepare two empty matrices $\mathbf{E}$ and $\mathbf{K}$,
2. for $i = 1, 2, \ldots$ set up the matrix $\mathbf{T}_{Ci}$ for step i,
3. if $i > 1$ set $\mathbf{E} = \mathbf{T}_i\mathbf{E}$ and $\mathbf{K} = \mathbf{T}_i\mathbf{K}$,
4. append the vectors $\mathbf{e}$ and $\mathbf{k}$ to the matrices $\mathbf{E}$ and $\mathbf{K}$ respectively (as a new column),
5. calculate requested entries $R_{p,q}(t)$ of the autocovariance function $R(t)$,

$$R_{p,q}(t) = c_1 \sum_{j=1}^{i} \mathbf{E}_{p,j}\mathbf{E}_{q,j} + \sum_{j=1}^{i} (\mathbf{E}_{p,j}\mathbf{K}_{q,j} + \mathbf{K}_{p,j}\mathbf{E}_{q,j}).$$

There are some limitations that prevent the really general usage of this procedure. First, the single matrix-vector multiplication $\mathbf{T}_n\mathbf{x}$ requires 2 solutions of the matrix equation of the dimension N, each for several right hand sides. Next, the single n-th step requires $2n$ matrix-vector multiplications. And finally, it is necessary to store intermediate results during the whole cycle, it means 2 dense matrices of dimension $((2N + p) \times i)$ for i steps of the stochastic Newmark method.

3. Numerical example

Let us assume a model of a single span concrete bridge. The model was chosen sufficiently small to provide possibility of computation of the exact solution. The structure is subjected to the seismic excitation modeled as the stationary ARMA process modulated by double exponential modulation function. The excitation enters the structure on both ends. Only the transverse direction of the motion is taken into account. The figure below shows some properties of the proposed method, namely the influence of length of the integration step Δt (Fig. 1). The curves depict the expected dispersion of the vibration of the bridge in its center (higher curves) and in a quarter of its length.

4. Conclusions

Numerical solution of the stochastic differential equations is of the great importance. The significant complexity of the task demands special approach, notably in the case of large systems. It is clear that there is no cheap or easy solution and careful choice of the numerical procedure is necessary.

The stochastic version of the Newmark method can be formulated using the matrix-free approach utilizing only the matrix – vector multiplication. However, additional workload is considerable and it even increases as the integration path elongates.

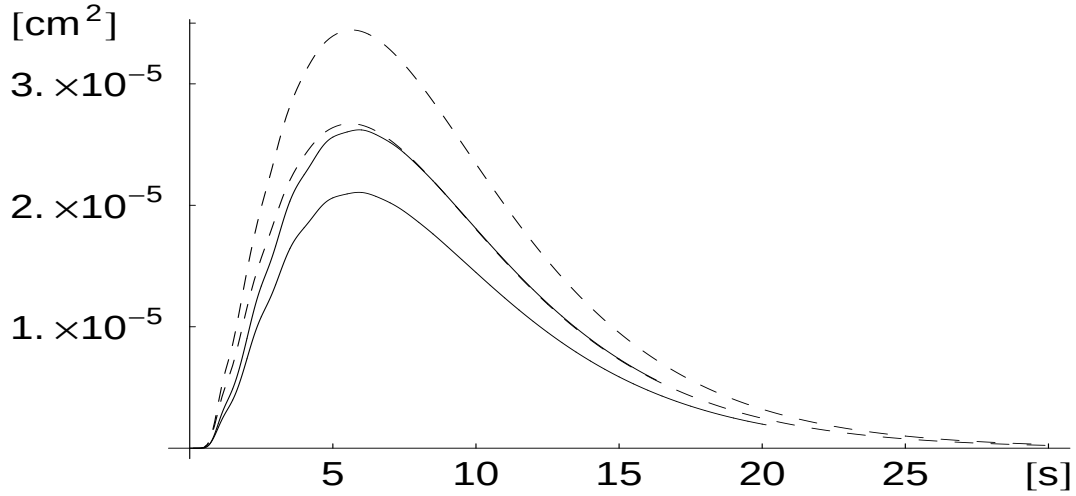

Fig. 1: *Result of the stochastic Newmark method for* $\Delta t = 0.125s$ *(dashed line) and* $\Delta t = 0.01s$ *(solid line).*

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