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Contribution to construction of global cubic $C^1$ or $C^2$-spline on equidistant knotset

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# CONTRIBUTION TO CONSTRUCTION OF GLOBAL CUBIC $C^1$ OR $C^2$-SPLINE ON EQUIDISTANT KNOTSET *

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## 1. Problem statement

Given the equidistant knotset $\{x_i,\ i = 0 : n+1\}$ with prescribed function values (FV), we can find a quadratic interpolating $C^1$-spline through computing its FV in inserted points of interpolation $t_i = (x_i + x_{i+1})/2$ (see [1], [2]). Cubic $C^1$ splines based on Hermite interpolation with given function and derivative values in knots have a local character (see e.g. [1], [2]). For cubic $C^2$-splines with given function values in knots only we can use the B-spline technique or we have to compute the first or second derivatives in knots to obtain $C^2$-continuity in the local representation. Such splines have a weak localizing property only (the influence of changes in some knot decreases with growing distance).

The aim of this contribution is to discuss the construction of $C^1$-cubic interpolatory splines based on inserting interval midpoints as additional points of interpolation and to compute the unknown spline function values to obtain $C^1$-continuity of spline segments in original knots. For simplicity we will discuss the case of the equidistant knotset only. We will also mention an approach with inserting two points of interpolation to obtain $C^2$-continuity and the solutions which use the B-spline technique (without inserted points).

## 2. Cubic spline knotset with inserted midpoints

On the four-point equidistant knotset $\{x_i,\ i = 0 : 1 : 3\}$ with the stepsize $h$ and given function values $\{y_i\}$ the cubic Lagrange interpolation polynomial and its derivatives we can write with the local parameter $q = (x - x_0)/h$ and well-known Lagrange interpolation coefficients $l_i(q)$ as

$$L_3^{(k)}(x_0 + qh) = \sum_{i=0}^{3} l_i^{(k)}(q) y_i,\ k = 0, 1, 2; \qquad l_i(q) = \prod_{j \neq i} (q - j)/(i - j). \tag{1}$$

Let us extend given equidistant knotset $\{x_i,\ i = 0 : n+1\}$ with inserted midpoints $\{t_i = (x_i + x_{i+1})/2,\ i = 0 : n\}$ and denote $\{s_i = s(x_i),\ i = 0 : n+1\}$ the prescribed FV on the original knotset, $\{u_i = s(t_i),\ i = 0 : n\}$ the unknown FV of cubic interpolants over different intervals containing knot $x_i$ and three neighbours on the extended

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knotset. We shall try to compute the values $u_i$ in such a way to obtain $C^1$-continuity of overlapping segments (one of them have to be defined over the whole interval $[x_i, x_{i+1}]$ or $[x_{i-1}, x_i]$) in the knot $x_i$ . There are several variants of configurations of knots and intervals with common knot $x_i$. Using the explicit expression for $l_i^{(k)}(q)$ with corresponding values of parameter $q$ , we obtain after some simplifications the $C^1, C^2$-continuity conditions (CC) in the knot $x_i$ as the recursions between unknown values $u_i$ and prescribed values $s_i$ written in the following table.

| Overlapping intervals<br>q used | $C^1$-continuity condition<br>$C^2$-continuity condition |
|---|---|
| $[t_{i-1}, x_{i+1}], [x_i, t_{i+1}]$ | $u_{i-1} + 6u_i + u_{i+1} = 4(s_i + s_{i+1})$ |
| 1, 0 | $u_{i-1} + 6u_i + u_{i+1} = 4(s_i + s_{i+1})$ |
| $[x_{i-1}, t_i], [x_i, t_{i+1}]$ | $6u_{i-1} + 16u_i + 2u_{i+1} = s_{i-1} + 14s_i + 9s_{i+1}$ |
| 2, 0 | $u_{i-1} + 6u_i + u_{i+1} = 4(s_i + s_{i+1})$ |
| $[x_{i-1}, t_i], [t_{i-1}, x_{i+1}]$ | $4(u_{i-1} + u_i) = s_{i-1} + 6s_i + s_{i+1}$ |
| 2, 1 | $u_{i-1} - 2s_i + u_i = u_{i-1} - 2s_i + u_{i+1}$ |
| $[t_{i-2}, x_i], [x_i, t_{i+1}]$ | $2u_{i-2} + 18u_{i-1} + 18u_i + 2u_{i+1} = 9s_{i-1} + 22s_i + 9s_{i+1}$ |
| 3, 0 | $-u_{i-2} - 5u_{i-1} + 5u_i + u_{i+1} = -4s_{i-1} + 4s_i$ |
| $[t_{i-2}, x_i], [x_{i-1}, t_i]$ | $2u_{i-2} + 16u_{i-1} + 6u_i = 9s_{i-1} + 14s_i + s_{i+1}$ |
| 3, 1 | $u_{i-2} + 6u_{i-1} + u_i = 4(s_{i-1} + s_i)$ |
| $[t_{i-2}, x_i], [x_{i-1}, t_i]$ | $u_{i-2} + 6u_{i-1} + u_i = 4(s_{i-1} + s_i)$ |
| 3, 2 | $u_{i-2} + 6u_{i-1} + u_i = 4(s_{i-1} + s_i)$ |

**Tab. 1.**

We can see that with $q = [2, 1]$ the $C^2$- CC is identically fulfilled and for $q = [1, 0], [3, 2]$ the conditions for $C^1, C^2$ -continuity are identical. We can also consider the relevant problem with given values $u_i$ and free parameters $s_i$.

## 3. $C^1$- continuity conditions for cubic splines

We can find various systems of overlapping intervals with four points of the extended mesh such that each knot $x_i, i = 1 : n$ belongs just to two such intervals. To obtain the cubic spline on the original knotset we need to have for each interval $[x_j, x_{j+1}]$ some cubic interpolant defined over the whole such interval. When we then write down the corresponding $C^1$-CC in all internal knots $x_i$, we obtain system of linear equations for unknown parameters $u_i$. Then we choose for each interval $[x_i, x_{i+1}]$ the corresponding cubic polynomial and we obtain the interpolating cubic $C^1$-spline on the original knotset.

### 3.1. The intervals with q=[2,0] or q=[3,1] connected

When we use for the cubic interpolants the knots $[x_i, t_i, x_{i+1}, t_{i+1}],\ i = 0 : n - 1$, then these interpolants have just the common knots $x_i,\ i = 1 : n$ with the $C^1$- CC from the case with $q = [2, 0]$ ($q = [2, 1]$ for the last couple). Using Table 1, we obtain so the system of n linear relations between parameters $\mathbf{u} = [u_0, \cdots, u_n]$ and $\mathbf{s} = [s_0, \cdots, s_{n+1}]$

$$\mathbf{A_u u} = \mathbf{A_s s} \tag{2}$$

with the tridiagonal matrices $\mathbf{A_u}, \mathbf{A_s}$ of the sizes $(n, n+1)$, $(n, n+2)$ and coefficients

$$\begin{bmatrix} 6 & 16 & 2 & & & \\ & 6 & 16 & 2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 6 & 16 & 2 \\ & & & & 4 & 4 \end{bmatrix}, \quad \begin{bmatrix} 1 & 14 & 9 & & & & \\ & 1 & 14 & 9 & & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & 1 & 14 & 9 \\ & & & & & 1 & 6 & 1 \end{bmatrix}. \tag{3}$$

Both these matrices are of the full rank and with given values $\{s_i, i = 0 : n+1\}$ we have one free parameter $u_i$ – our problem has the solution depending on one parameter, which can be chosen or used for some optimization purposes. The cubic $C^1$-spline then consists of corresponding parts of the cubic interpolants. Let us mention some most interesting cases in these variants.

1. We can choose arbitrary free parameter $u_0$. The tridiagonal matrix of the reduced system is then regular and diagonally dominant (exception - the last row) and we obtain the unique solution for arbitrary values $\{u_0; s_i, i = 0 : n\}$. The corresponding spline will have the mentioned weak localising property.
2. For similar proper system of overlapping intervals for cubic interpolation corresponding to $q = [2,1], [3,1], [3,1], ...$ we obtain the $C^1$-CC with slightly modified matrices $\mathbf{A_u}, \mathbf{A_s}$. With the choice of the free parameter $u_n$ we obtain now uniquely solvable system with weak localising property.
3. In both mentioned cases we can obtain the unique solution of the relevant problem with given parameters $s_0, s_{n+1}; u_i, i = 0 : n$ with diagonally dominant reduced (n,n)-matrix $\mathbf{A_u}$.
4. To obtain the solution with minimal 2-norm we can use pseudoinverse for the solution of the original system - but we obtain usually the solution with some oscillations.

*Example 1*: For the data $x = 0 : 1 : 9$, $y(x) = 1 - x * cos(x)/2$ we can see plotted in Fig. 1 the original function and the values $s_i$ of the corresponding $C^1$-cubic spline in midpoints, computed with pseudoinverse (circles) and with $u_0 = y(0.5)$ from (2). The 2-norms of the vector $\mathbf{u}$ are here [5.69, 5.73]. When we use as free parameter $y(8.5)$, we obtain the oscillating solution with norm equal to 127.25 !

### 3.2. Another configurations

We can find also another possible variants with still worse properties.

1. In the configuration of two families of local intervals with repeating values of $q = [2,1], [3,1], [3,1], [3,0], ...$ we recognize from Table 1 the $C^2$-continuity of overlapping segments in knots $x_{4k+1}$. The system of CC consists of (4,6)-blocks, some proper choice of free parameter is $u_n$.

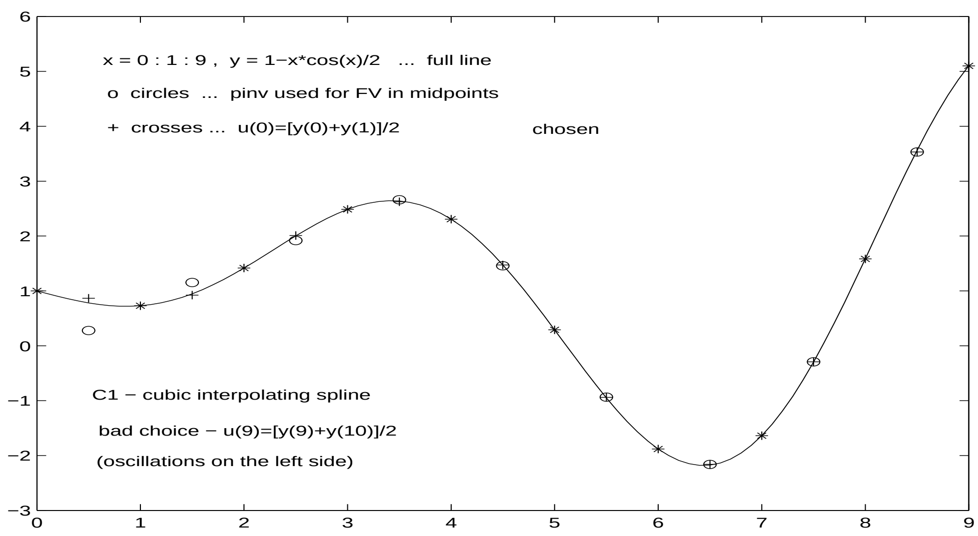


**Fig. 1.**

Similarly we find another union of two families of intervals - the first starting with the knots $x_{3k}$ only, the second family of connected intervals starting with the inserted knot $t_0$ where the repeated triples of corresponding values of the local parameter $q$ for CC in internal knots $x_i$ are $[2,1],[3,0],[2,0]$ for $i = 3k+1, 3k+2, 3k+3$ (with possible changes in the last CC). The whole system of $C^1$-CC we can write now again using Table 1 as $\mathbf{A_u u} = \mathbf{A_s s}$ with block diagonal matrices with (3,4),(3,5)-blocks

$$\begin{bmatrix} 4 & 4 & & \\ 2 & 18 & 18 & 2 \\ & 6 & 16 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 6 & 1 & & \\ & 9 & 22 & 9 & \\ & & 1 & 14 & 9 \end{bmatrix} \tag{4}$$

and to choose $u_n$ as free parameter. We find now also the $C^2$-continuity of the resulting cubic spline in knots $x_{3k+1}$.

We have found in both such cases as the most proper free parameter $u_n$, the unique solution and diagonal dominancy in some rows. But in rows with coefficients [4,4] and [2,18,18,2] the diagonal dominancy is lost and we can find some oscillations in the solution. The structure of the matrix $\mathbf{A_u}$ shows that the choice of the free parameter $u_0$ will result in more oscillating solution. When we are interested to use $u_0$ as the free parameter, then we can use some more appropriate variant of two families of covering intervals with q=[2,0],[2,1],3,0] and changes in row orderings in blocks mentioned in (4).

2. When we try to use another possible variants with corresponding repeating values q=[2,0],[2,1],[3,1],[3,0] or q=[2,1],[3,0], then the structure of the full rank matrix $\mathbf{A_u}$ shows the danger of strong oscillations of computed values $u_i$ with the choice of any free parameter $u_i$ - as we can justify it in computations.

3. In all cases discussed till now we find for the relevant problem with given values $u_i,\ i = 0 : n$ and parameters $s_0, s_n$ the reduced matrix $\mathbf{A_s}$ to be diagonally dominant – *all such problems have the unique solution with weak localising property!*

4. The configurations with q=[1,0],[2,1],[3,2] or q=[1,0],[3,2] seem to result in $C^2$-cubic spline - but we have not unique cubic polynomials for each interval $[x_i, x_{i+1}]$ in such cases.

**Statement 1**: *On the equidistant knot mesh $\{x_i\}$ with given function values $\{s_i\}$ we can find (with one free parameter) function values $u_i$ in each of interval midpoints $t_i = (x_i + x_{i+1})/2$ such that the corresponding parts of the local cubic interpolants on intervals $[x_i, x_{i+1}]$ will form the cubic interpolating $C^1$- spline with knots $\{x_i\}$ for the original data $\{x_i, s_i\}$. One free parameter we can use for optimization purposes, or to choose $u_0, u_n$ (according to the structure of the matrix $\mathbf{A_u}$) to obtain weak localising property of the spline.*
*The relevant problem with given values $s_0, s_n; \{u_i, i = 0 : n\}$ has in all cases discussed above the unique solution with weak localising property.*

## 4. Two inserted knots per interval for $C^2$-continuity

Let us insert into each interval of the original equidistant knotset $\{x_i,\ i = 0 : n + 1\}$ two uniformly situated knots $x_i < t_i^1 < t_i^2 < x_{i+1},\ i = 0 : n$. We want try to find function values in knots $t_i^1, t_i^2$ to obtain $C^2$-cubic spline on the original knotset. We can write the $C^1, C^2$- CC in knots $x_i$ ($q = [3, 0]$ used from Table 1) with the notation $u_i = s(t_i^1), v_i = s(t_i^2), s_i = s(x_i)$ as

$$\begin{aligned} -9u_{i-1} + 18v_{i-1} + 18u_i - 9v_i &= -2s_{i-1} + 22s_i - 2s_{i+1}, \\ -4u_{i-1} + 5v_{i-1} - 5u_i + 4v_i &= -s_{i-1} + s_{i+1}, \quad i = 1 : n. \end{aligned} \tag{5}$$

With $n + 1$ intervals $[x_i, x_{i+1}],\ 2(n + 1)$ inserted knots $t_i^1, t_i^2,\ i = 0 : n$ and given values $s_i = s(x_i)$ we obtain so the system of $2n$ $C^1, C^2$- CC with the block structure and full rank matrix. So we have two free parameters, similarly as for classical cubic $C^2$-splines. We again can use free parameters to various purposes.
1. We can solve the system of CC (5) with the pseudoinverse, to obtain the spline with minimal norm of parameters $[u_i, v_i]$.
2. We can choose the values $u_0, v_n$ and compute the remaining parameters from the regular reduced system (5).
3. We can prescribe boundary conditions for the spline computed - the first or second derivatives as with classical cubic splines (details in [3]).
4. There are also solutions with free parameters $u_0, v_0$ or $u_n, v_n$ which can be computed recursively. But in both such cases we obtain strongly oscillating solutions.

**Statement 2**: *On the equidistant mesh with given function values in knots and two inserted points of interpolation we can compute the function values in inserted points to obtain cubic $C^2$-spline. Two free parameters can be prescribed, used for boundary conditions with the first (second) derivative or for optimization purposes.*

In Fig. 2 we can see the FV in inserted knots computed with pseudoinverse (circles) and from the system of CC extended with boundary conditions (natural spline - zero values of the second derivative at boundaries).

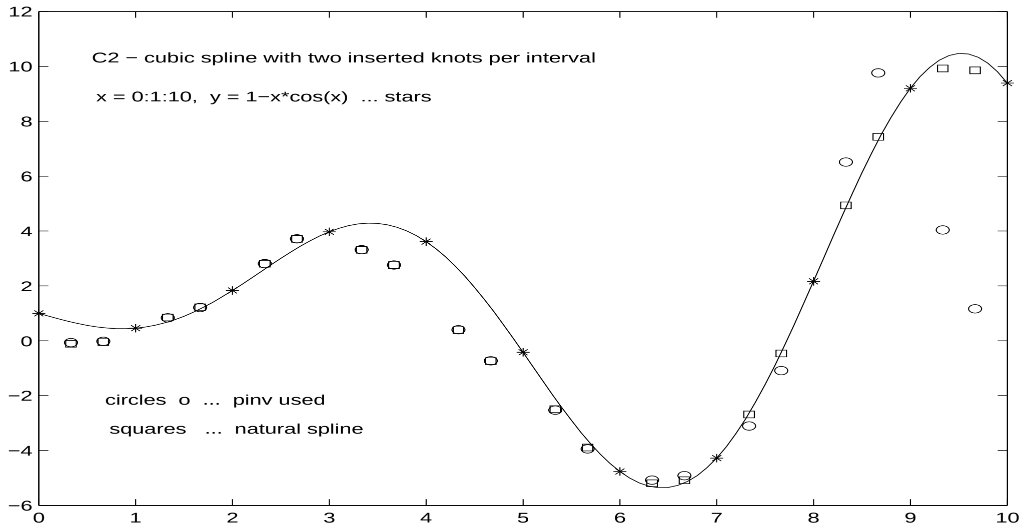


**Fig. 2.**

*Remarks:*
1) We can use two inserted points per interval to find cubic $C^2$-spline interpolating the given mean values (see [3]).
2) We have to use B-splines with coinciding knots to find cubic $C^1$-spline interpolating in knots $x_i$ (several possible variants).

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