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ANALYSIS OF INCOMPRESSIBLE FLOW THROUGH A CASCADE OF PROFILES *

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Abstract

The paper deals with analysis of mathematical model of incompressible viscous nonstationary flow through a plane cascade of profiles. We formulate the nonstationary problem and construct a solution by means of semidiscretization in time (Rothe's method).

1. Introduction

The concept "cascade of profiles" represents a 2D model of a 3D blade machine (compressor, pump, turbine). The model is considered in a domain which is bounded in the direction of the x_1–axis and unbounded but periodic in the direction of the x_2–axis. Due to the periodicity we can restrict our considerations only to one period Ω and obtain a solution which can be periodically extended to the whole unbounded domain.

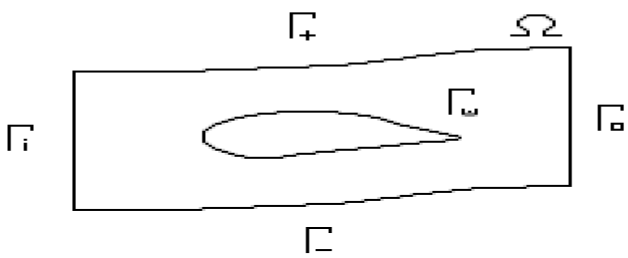

Fig. 1: *Domain* Ω.

2. Formulation of the problem

The classical formulation of the problem consists of the *Navier-Stokes equations*

$$\frac{\partial \mathbf{u}}{\partial t} - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \nabla p = \mathbf{f} \quad \text{in } Q_T = \Omega \times (0, T), \tag{1}$$

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the *continuity equation*

$$\operatorname{div}\mathbf{u} \,=\, 0 \qquad \text{in} \;\; Q_T, \tag{2}$$

the initial condition

$$\mathbf{u}(x,0) = \mathbf{u}_0(x) \qquad \text{in } \Omega, \tag{3}$$

the boundary conditions

$$\mathbf{u}\,|_{\Gamma_i} \,=\, \mathbf{g}, \qquad \mathbf{u}\,|_{\Gamma_w} \,=\, \mathbf{0}, \tag{4}$$

$$-\nu\,\frac{\partial\mathbf{u}}{\partial\mathbf{n}} + p\,\mathbf{n} - \frac{1}{2}\,(\mathbf{u}\cdot\mathbf{n})^{-}\,\mathbf{u} \,=\, \mathbf{h}, \quad [x_1,x_2]\in\Gamma_o\,,\; t\in(0,T), \tag{5}$$

and of the conditions of periodicity in the x_2–direction

$$\mathbf{u}(x_1,x_2+\tau,t) \;=\; \mathbf{u}(x_1,x_2,t), \tag{6}$$

$$\frac{\partial\mathbf{u}}{\partial\mathbf{n}}(x_1,x_2+\tau,t) \;=\; -\frac{\partial\mathbf{u}}{\partial\mathbf{n}}(x_1,x_2,t), \tag{7}$$

$$p(x_1,x_2+\tau,t) \;=\; p(x_1,x_2,t), \tag{8}$$

for $x=(x_1,x_2)\in\Gamma_-$, $t\in(0,T)$. Here $\mathbf{u}=(u_1,u_2)$ denotes the velocity, $\mathbf{f}=(f_1,f_2)$ denotes the external force, p is the pressure, $\nu>0$ is the constant viscosity and $\nabla=\big(\partial/\partial x_1,\,\partial/\partial x_2\big)$.

Theorem. *Let* $\mathbf{u}$, p *be a classical solution of the problem in* $\overline{\Omega}\times[0,T]$. *If we extend* $\mathbf{u}$ *and* p *from* $\overline{\Omega}$ *onto the whole cascade of profiles as functions periodic in* x_2 *with period* τ, *then we obtain a classical solution in the whole unbounded domain.*

The main difficulties: The problem is nonstationary. The Dirichlet boundary condition on $\partial\Omega$ is nonhomogeneous. The outlet boundary condition on Γ_o is nonlinear. The boundary conditions on Γ_- and Γ_+ are periodic.

3. Function spaces

$H^1(\Omega)$ is a classical Sobolev space,
$\mathcal{X}=\{\mathbf{v}\in C^\infty(\overline{\Omega})^2;\; \mathbf{v}=\mathbf{0} \text{ on } \Gamma_i\cup\Gamma_w,\; \mathbf{v}(x_1,x_2+\tau)=\mathbf{v}(x_1,x_2)\;\forall\,(x_1,x_2)\in\Gamma_-\}$,
$\mathcal{V}=\{\mathbf{v}\in\mathcal{X};\; \operatorname{div}\mathbf{v}=0 \text{ in } \Omega\}$, X is the closure of $\mathcal{X}$ in $H^1(\Omega)^2$,
V is the closure of $\mathcal{V}$ in $H^1(\Omega)^2$, H is the closure of $\mathcal{V}$ in $L^2(\Omega)^2$.
The spaces X and V can be characterized as
$X=\{\mathbf{v}\in H^1(\Omega)^2;\; \mathbf{v}=\mathbf{0} \text{ in } \Gamma_i\cup\Gamma_w,\; \mathbf{v}(x_1,x_2+\tau)=\mathbf{v}(x_1,x_2) \text{ for } (x_1,x_2)\in\Gamma_-\}$,
$V=\{\mathbf{v}\in X;\; \operatorname{div}\mathbf{v}=0 \text{ in } \Omega\}$.
Space V is equipped with the norm

$$|||\mathbf{v}||| \;=\; \Big(\int_\Omega \sum_{i,j=1}^{2} \frac{\partial v_i}{\partial x_j}\,\frac{\partial v_i}{\partial x_j}\,dx\Big)^{1/2}, \tag{9}$$

which is equivalent with the norm $\|\cdot\|_{H^1(\Omega)}$.

4. Weak formulation

Using Green's theorem and the classical formulation of the problem, we can formally derive the following integral identity:

$$(\frac{\partial \mathbf{u}}{\partial t}, \mathbf{v}) \,+\, a(\mathbf{u}, \mathbf{v}) \,=\, (\mathbf{f}, \mathbf{v}) \,+\, b(\mathbf{h}, \mathbf{v}), \tag{10}$$

where

$$\begin{aligned}
(\frac{\partial \mathbf{u}}{\partial t}, \mathbf{v}) &= \int_\Omega \frac{\partial \mathbf{u}}{\partial t} \mathbf{v}\, dx, \\
a_1(\mathbf{u}, \mathbf{v}) &= \nu \int_\Omega \sum_{i,j=1}^{2} \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j}\, dx, \\
a_2(\mathbf{u}, \mathbf{v}, \mathbf{w}) &= \int_\Omega \sum_{i,j=1}^{2} u_j \frac{\partial v_i}{\partial x_j} w_i\, dx, \\
a_3(\mathbf{u}, \mathbf{v}, \mathbf{w}) &= \int_{\Gamma_o} \frac{1}{2} (\mathbf{u} \cdot \mathbf{n})^- \, \mathbf{v} \cdot \mathbf{w}\, dS, \\
a(\mathbf{u}, \mathbf{v}) &= a_1(\mathbf{u}, \mathbf{v}) + a_2(\mathbf{u}, \mathbf{u}, \mathbf{v}) \,+\, a_3(\mathbf{u}, \mathbf{u}, \mathbf{v}), \\
(\mathbf{f}, \mathbf{v}) &= \int_\Omega \mathbf{f} \cdot \mathbf{v}\, dx, \\
b(\mathbf{h}, \mathbf{v}) &= -\int_{\Gamma_o} \mathbf{h} \cdot \mathbf{v}\, dS.
\end{aligned}$$

We look for a function $\mathbf{u} \in L^2(0, T;\, V) \cap L^\infty(0, T;\, H)$ satisfying (10) with $f \in L^2(0, T;\, V^*)$ for each function $\mathbf{v} \in V$, the initial condition, the boundary conditions on Γ_i, Γ_w and the condition of periodicity on Γ_-. This $\mathbf{u}$ is called the *weak solution*.

5. Existence of a weak solution

The weak solution is constructed by means of the *semidiscretization in time* (the Rothe method, see [2]). This method transforms the nonstationary problem into a sequence of stationary problems.

For arbitrary $n \in \mathbb{N}$ we put $\theta = \theta_n = T/n$ and we consider the partition of the interval $[0, T]$ defined by the points $t_k = k\theta,\ k = 0, 1, \ldots, n$. We search for a sequence of stationary solutions of the modified stationary problems $\mathbf{u}^0$, $\mathbf{u}^1$, …, $\mathbf{u}^n$ on the time levels t_k $(k = 1, \ldots, n)$ in the form

$$\mathbf{u}^k = \mathbf{g}^* \,+\, \mathbf{z}^k, \tag{11}$$

where $\mathbf{g}^*$ is the extension of the function $\mathbf{g}$ from Γ_i onto domain Ω fulfilling

$$\|\mathbf{g}^*\|_{H^1(\Omega)^2} \le c_3\, \|\mathbf{g}\|_{H^{1/2}(\partial\Omega)^2} \,\le\, c_4\, \|\mathbf{g}\|_{H^s(\Gamma_i)^2},$$

more details can be found in [1]. We assume that $\mathbf{g}^* \in W^{1,\infty}(\Omega)$ and put $\mathbf{u}^0 = \mathbf{u}_0$ $(\in H)$. The solutions of the stationary problems satisfy: $\mathbf{u}^k \in H^1(\Omega)^2$ have the form (11) with, $\mathbf{z}^k \in V$ and

$$\left(\frac{\mathbf{u}^k - \mathbf{u}^{k-1}}{\theta}, \mathbf{v}\right) + a(\mathbf{u}^k, \mathbf{v}) = <\mathbf{f}^k, \mathbf{v}> \quad \forall\, \mathbf{v} \in V \tag{12}$$

where

$$\mathbf{f}^k = \frac{1}{\theta} \int_{t_{k-1}}^{t_k} \mathbf{f}(t)\, dt \quad \in V^*, \quad k = 1, \ldots, n.$$

It can be proven that there exist solutions of this modified stationary problems by the technique similar to [1].

6. Construction of a weak solution of the nonstationary problem

Let $\mathbf{u}^0$, $\mathbf{u}^1$, ..., $\mathbf{u}^n$ be a sequence of solutions of the modified stationary problems on the time levels t_0, t_1, ..., t_n. Using this sequence, we construct the time–dependent functions:

$$\begin{aligned} \mathbf{u}_\theta : & \quad [0,T] \longrightarrow V \\ \mathbf{w}_\theta : & \quad [0,T] \longrightarrow H \\ \mathbf{u}_\theta(0) = \mathbf{u}^1,\ \mathbf{u}_\theta(t) & = \mathbf{u}^k \ \text{ for } t \in (t_{k-1}, t_k]\ k = 1, \ldots, n, \end{aligned} \tag{13}$$

$\mathbf{w}_\theta$ is continuous on $[0,T]$, linear on each $[t_{k-1}, t_k]$ $(k = 1, \ldots, n)$ and $\mathbf{w}_\theta(t_k) = \mathbf{u}^k$ for $k = 0, 1, \ldots, n$, $\mathbf{w}_\theta : [0,T] \longrightarrow H$ because $\mathbf{u}^0 = \mathbf{u}_0 \in H$. However $\mathbf{w}_\theta : [\theta, T] \longrightarrow V$. We denote by $\tilde{\mathbf{w}}_\theta$ the function $\mathbf{w}_\theta$ extended from $[\theta, T]$ onto the whole time interval $[0,T]$ by the equality $\tilde{\mathbf{w}}_\theta = \mathbf{u}^1$ on $[0, \theta]$.

We can deduce from the form of the modified stationary problem and from the properties of the bilinear form $a(\mathbf{u}, \mathbf{v})$ that

$$\begin{aligned} \mathbf{u}_\theta,\ \mathbf{w}_\theta & \quad \text{is bounded in} \quad L^\infty(0,T;\ H) \\ \mathbf{u}_\theta,\ \tilde{\mathbf{w}}_\theta & \quad \text{is bounded in} \quad L^2(0,T;\ V) \\ \frac{d\mathbf{w}_\theta}{dt} & \quad \text{is bounded in} \quad L^1(0,T;\ V^*) \\ (\mathbf{u}_\theta - \mathbf{w}_\theta) & \quad \longrightarrow 0 \quad \text{in } L^2(0,T;\ H) \text{ for } \theta \to 0+. \end{aligned} \tag{14}$$

It is possible to prove that $d\mathbf{w}_\theta/dt$ is bounded in the space $L^1(0,T;\ V^*)$. (The boundedness in $L^2(0,T;\ V^*)$ is open.) Nevertheless, we can derive the strong convergence in $L^2(0,T;\ H)$ by means of the generalization of the Lions–Temam theorem on the compact imbedding (see [3]).

Since the sequences are bounded, we can choose subsequences (which we denote in the same way) such that

$$\mathbf{u}_\theta \quad \longrightarrow \quad \mathbf{u} \text{ weakly in } L^2(0,T;\ V)$$

$$
\begin{aligned}
\mathbf{u}_\theta &\longrightarrow \mathbf{u} \text{ weakly} - * \text{ in } L^\infty(0,T;\, H)\\
\mathbf{w}_\theta &\longrightarrow \mathbf{u} \text{ weakly} - * \text{ in } L^\infty(0,T;\, H)\\
\frac{d\mathbf{w}_\theta}{dt} &\longrightarrow \frac{d\mathbf{u}}{dt} \text{ weakly in } L^1(0,T;\, V^*)\\
\mathbf{w}_\theta &\longrightarrow \mathbf{u} \text{ weakly in } L^2(\epsilon,T;\, V), \quad \forall \epsilon > 0
\end{aligned}
$$

The weak–* convergence of a sequence in $L^\infty(0,T;\, H)$ means its convergence as a sequence of continuous linear functionals on $L^1(0,T;\, H)$.

Now it is already possible to prove that

$$\mathbf{w}_\tau \longrightarrow \mathbf{u} \text{ strongly in } L^2(0,T;\, H),$$

and that $\mathbf{u}$ is a sought weak solution.

This theory will be the basis for the derivation of a numerical solution of our problem. This detailed analysis will be a subject of a paper in preparation.

References

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