Pavel Šolín; Karel Segeth
Three ways of interpolation on finite elements

In: Jan Chleboun and Petr Přikryl and Karel Segeth (eds.): Programs and Algorithms of Numerical Mathematics, Proceedings of Seminar. Dolní Maxov, June 6-11, 2004. Institute of Mathematics AS CR, Prague, 2004.
pp. 230–241.

Persistent URL: http://dml.cz/dmlcz/702800

THREE WAYS OF INTERPOLATION ON FINITE ELEMENTS *

Pavel Šolín, Karel Segeth

Abstract

Interpolation on finite elements usually occurs in a Hilbert space setting, which means that interpolation techniques involving orthogonal projection are an alternative for the traditional Lagrange nodal interpolation schemes. In addition to the Lagrange interpolation, this paper discusses the global orthogonal projection and the projection-based interpolation. These techniques are compared from the point of view of quality, efficiency, sensitivity to input parameters and other aspects. Local optimality result for the projection-based interpolation is presented.

1. Introduction

An important and often solved problem in numerical analysis is the approximation (usually interpolation) of functions. It has its place also in the finite element method, in particular when some adaptive approach (e.g. *hp*-adaptivity) is applied. The goal of the *hp*-adaptive computation is to approximate the solution, if it is smooth, on a relatively large element of the partition of the domain by a polynomial of a relatively high degree. Further examples are multigrid methods if we proceed from a coarse to a fine grid, or approximation of the initial condition when we use the finite element method of lines to solve a parabolic equation.

To show only the principles and minimize the necessary mathematical means, we confine ourselves to the 1D case only. A generalization to more dimensions is possible, see, e.g., [2], [5], [6].

We first consider the global orthogonal projection that gives the best results but is, at the same time, the most expensive. The projection-based interpolation is a compromise solution: we interpolate the function at the finite element nodes and then apply the orthogonal projection on each element. The cheapest but not always optimal (in particular, when the approximating polynomial is of a higher degree) is the explicit nodal interpolation. The optimal evaluation algorithm, Horner scheme, is used here. The error depends on the distribution of interpolation nodes.

Several examples to illustrate the above statements are presented.

2. Approximation and interpolation

Assume some restricted set of functions $\mathcal{C}$ (such as, for example, polynomial, piecewise-polynomial or trigonometric polynomial functions) in a linear space V and

*This work was supported by grants No. 201/04/1503 and 102/01/D114 of the Grant Agency of the Czech Republic.

a function $g \in V$ that does not belong to $\mathcal{C}$. The prototype approximation problem is to find a suitable function $g_c \in \mathcal{C}$ (*approximation of* g) such that g_c is in some sense close to g. The measure of the quality of the approximation can be some error estimate, the norm $\|g - g_c\|_V$ if the space V is normed, or other conditions.

Approximation becomes *interpolation* when the sought function $g_c \in \mathcal{C}$ has to satisfy some additional constraints. These conditions are formulated generally as

$$L_i(g_c) = b_i, \quad i = 1, 2, \ldots, N_c,$$

where $L_i : V \to I\!R$ are linearly independent linear forms in V' and $b_1, b_2, \ldots, b_{N_c}$ some given constants.

For example, the traditional Lagrange interpolation is obtained via the choice

$$L_i(g_c) = g_c(x_i)$$

and

$$b_i = g(x_i),$$

where $x_1, x_2, \ldots, x_{N_c}$ is a given set od distinct interpolation points in some domain Ω, $V = V(\Omega)$.

There are many natural questions related to the approximation and interpolation: What assumptions have to be put on V, $\mathcal{C}$ and g to ensure the existence and uniqueness of the best approximation? What conditions the linear forms L_i have to obey to guarantee a unique solution of the interpolation problem? What can be said about the error of the approximation and/or interpolation?

The analysis is highly nontrivial in the general setting of a basic linear or normed space V and a general subset $\mathcal{C} \subset V$. However, the good news is that all important assumptions on the space V, set $\mathcal{C}$ and the function g, developed in the framework of the abstract approximation theory, are fulfilled automatically when V is a Hilbert space and $\mathcal{C}$ its closed subspace.

2.1. The Hilbert space setting

Let $V = V(\Omega)$ be a Hilbert space appropriate for the solved problem (usually the H^1 space), $a(\cdot, \cdot) : V \times V \to I\!R$ a bounded V-elliptic bilinear form, $l \in V'$, and $V_{h,p}$ a finite-dimensional subspace of V determined by the finite element mesh $\mathcal{T}_{h,p}$, where h and p are the discretization parameters, h usually controls the size of subintervals, p the degree of approximating polynomials. Consider the solution $u \in V$ of a model continuous problem

$$a(u, v) = l(v) \quad \text{for all} \quad v \in V$$

and the solution $u_{h,p} \in V_{h,p}$ of the corresponding discrete problem

$$a(u_{h,p}, v) = l(v) \quad \text{for all} \quad v \in V_{h,p}.$$

According to Céa's lemma, the discretization error $\|u - u_{h,p}\|_V$ is bounded by the interpolation properties of the subspace $V_{h,p} \subset V$ and the continuity and V-ellipticity constants C_{c}, C_{e} of the bilinear form $a(\cdot, \cdot)$,

$$\|u - u_{h,p}\|_V \leq \frac{C_{\mathrm{c}}}{C_{\mathrm{e}}} \inf_{v \in V_{h,p}} \|u - v\|_V = \frac{C_{\mathrm{c}}}{C_{\mathrm{e}}} \mathrm{dist}(u, V_{h,p})_V.$$

Hence the interpolation properties of the space $V_{h,p}$ are largely responsible for the final form of the error estimate.

In practice we always have a particular interpolation operator $P : V \to V_{h,p}$ that obviously satisfies

$$\mathrm{dist}(u, V_{h,p})_V \leq \|u - Pu\|_V.$$

Hence, for a sufficiently regular function $u \in V$ it is our aim to estimate the interpolation error $\|u - Pu\|_V$ using some parameters of the mesh $\mathcal{T}_{h,p}$ as well as the regularity properties of the function u. A typical interpolation error estimate has the form

$$\|u - Pu\|_V \leq C(u) h^{\alpha},$$

where $h = \max_i h_i$ is the mesh diameter, $C(u)$ is a constant depending on the regularity properties of the function u, and the exponent α usually depends on the polynomial degrees of basis functions on finite elements in the mesh $\mathcal{T}_{h,p}$.

In addition to its application in error analysis, interpolation also finds practical use in the finite element technology, when a given function $g \in V$ needs to be represented by a sufficiently close function $g_{h,p} \in V_{h,p}$. Problems of this type are encountered in the finite element solution of evolutionary problems, automatic *hp*-adaptivity, multigrid methods, and in many other situations.

3. Global orthogonal projection (best approximation)

In the Hilbert space setting the question of existence and uniqueness of the best approximation is trivial. Since $V_{h,p} \subset V$ is finite-dimensional and therefore closed, the closest approximation of a function $g \in V$ in the norm $\|\cdot\|_V$ is its unique orthogonal projection $g_{h,p} = Pg \in V_{h,p}$. The *orthogonal projection* P is defined via the condition

$$(g - g_{h,p}, v)_V = 0, \quad \text{for all } \ v \in V_{h,p}. \tag{1}$$

With some basis $\{v_1, v_2, \ldots, v_N\} \subset V_{h,p}$, (1) can be equivalently rewritten as

$$(g - g_{h,p}, v_i)_V = 0, \quad i = 1, 2, \ldots, N. \tag{2}$$

Expressing

$$g_{h,p} = \sum_{j=1}^{N} y_j v_j \tag{3}$$

and substituting into (2), one obtains a system of linear algebraic equations

$$\sum_{j=1}^{N} y_j (v_j, v_i)_V = (g, v_i)_V, \quad i = 1, 2, \ldots, N, \tag{4}$$

for the unknown coefficients $y_1, y_2, \ldots, y_N$. Recall that N is the number of all the basis functions on all subintervals and may be very large.

Example 3.1 Consider a domain $\Omega = (-1, 1)$ covered with a finite element mesh $\mathcal{T}_{h,p} = \{K_1, K_2\}$ consisting of two subintervals $K_1 = (-1, 0)$ and $K_2 = (0, 1)$. Assume the space $V = H_0^1(-1, 1)$ related to some problem with homogeneous Dirichlet boundary conditions. The finite element subspace $V_{h,p}$ is defined as

$$V_{h,p} = \{v \in V;\ v|_{K_i} \in P^1(K_i),\ i = 1, 2\}.$$

Let us construct the best approximation $g_{h,p} \in V_{h,p}$ of the function

$$g(x) = 1 - x^4 \in V.$$

In other words, we are looking for a function $g_{h,p} \in V_{h,p}$ such that

$$\mathrm{dist}(g, g_{h,p}) = \mathrm{dist}(g, V_{h,p}). \tag{5}$$

The linear system (4) reduces to a single equation, which yields the best approximation $g_{h,p}$,

$$g_{h,p}(x) = \begin{cases} \frac{11}{10}(1 + x), & x \in K_1, \\ \\ \frac{11}{10}(1 - x), & x \in K_2, \end{cases}$$

depicted in Fig. 1.

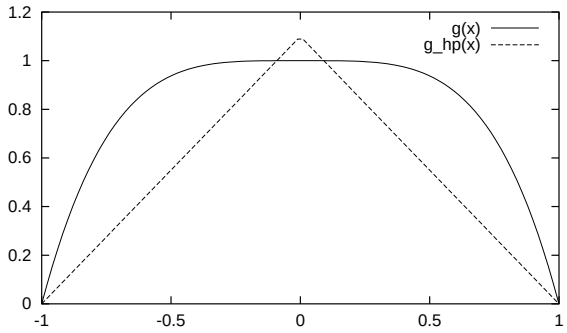

Fig. 1: *Best approximation $g_{h,p} \in V_{h,p}$ of the function $g \in V$.*

Notice that the best approximation $g_{h,p}$ does not coincide with the function g at the grid point $x = 0$. This illustrates that an interpolant, that coincides with the interpolated function at grid vertices, is not likely to be optimal in the norm $\|\cdot\|_V$.

However, in some applications the construction of the best approximation may be too demanding from the practical point of view, since the cost of the calculation of $g_{h,p}$ is similar to the cost of the solution of the global finite element problem. In such cases the only possibility is to abandon the optimality requirement (5) and try to find some less expensive interpolant that still profits from the availability of orthogonal projection. The first natural choice is to combine the Lagrange interpolation of vertex values with the orthogonal projection in element interiors.

4. Projection-based interpolation

A natural way to avoid the large linear system (4) is to split the interval $\Omega = (a, b)$ into several subintervals $\Omega_1, \Omega_2, \ldots, \Omega_k$, and to evaluate the interpolant $g_{h,p}$ locally. From the algorithmic point of view, an obvious choice is $\Omega_i = K_i$, $i = 1, 2, \ldots, M$.

Piecewise-linear case

In the simplest case when basis functions on all the elements $K_1, K_2, \ldots, K_M$ are linear, the continuity requirement implies that the projection-based interpolant $g_{h,p} \in V_{h,p}$ be defined as the usual piecewise-linear *vertex interpolant*,

$$g_{h,p}(x_i) = g^{\mathrm{v}}_{h,p}(x_i) = g(x_i), \quad i = 0, 1, \ldots, M, \tag{6}$$

where $g_{h,p}|_{K_i} \in P^1(K_i)$ for all $K_i \in \mathcal{T}_{h,p}$, as illustrated in Fig. 2.

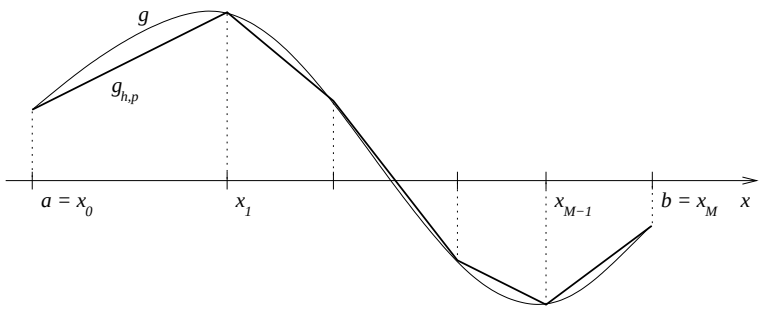

Fig. 2: *Projection-based interpolation reduces to the usual piecewise-linear Lagrange interpolation on elements.*

Higher-order case

On a general higher-order finite element mesh $\mathcal{T}_{h,p}$, as the reader may guess, the interpolation problem is decoupled by subtracting the piecewise-linear vertex interpolant $g^{\mathrm{v}}_{h,p}$ from the interpolated function g. The function $g - g^{\mathrm{v}}_{h,p}$ vanishes at all grid points and can be projected locally onto the polynomial spaces

$$P_0^{p_i}(K_i) = \{v \in H_0^1(K_i);\ v \in P^{p_i}(K_i)\}, \quad i = 1, \ldots, M.$$

In this way one calculates the *bubble interpolant* $g^{\mathrm{b}}_{h,p}$. The resulting interpolant $g_{h,p}$ is then obtained as a sum of the vertex and bubble parts,

$$g_{h,p} = g^{\mathrm{v}}_{h,p} + g^{\mathrm{b}}_{h,p} \tag{7}$$

Since we are in $H^1_0(K_i)$, either the full $H^1(K_i)$-norm or the equivalent $H^1(K_i)$-seminorm can be used. The fact that the standard vertex interpolation is combined with the orthogonal projection on higher-order subspaces is why one speaks about *projection-based interpolation*.

Let us look at the projection part in more detail. The inner product associated with the H^1-seminorm has the form

$$(\varphi, \psi)_{1,K_i} = \int_{K_i} \varphi'(x)\psi'(x)\,\mathrm{d}x, \quad i = 1, \ldots, M, \tag{8}$$

and $|\varphi|_{1,K_i}$ is the corresponding H^1_0-seminorm. The orthogonality condition that determines $g^{\mathrm{b}}_{h,p}$ is

$$((g - g^{\mathrm{v}}_{h,p}) - g^{\mathrm{b}}_{h,p}, v)_{1,K_i} = 0 \quad \text{for all} \;\; v \in P^{p_i}_0(K_i), \tag{9}$$

which is equivalent to

$$((g - g^{\mathrm{v}}_{h,p}) - g^{\mathrm{b}}_{h,p}, \vartheta^{(i)}_k)_{1,K_i} = 0, \quad k = 2, 3, \ldots, p_i, \tag{10}$$

where $\vartheta^{(i)}_k$, $k = 2, 3, \ldots, p_i$, is a suitable basis of $P^{p_i}_0(K_i)$, $i = 1, \ldots, M$. Denoting by L_j the Legendre polynomial of degree j, and employing the standard higher-order *Lobatto bubble shape functions*

$$l_k(x) \;\; = \;\; \int_{-1}^{x} \mathrm{L}_{k-1}(\xi)\,\mathrm{d}\xi, \quad x \in (-1, 1), \quad 2 \le k,$$

(that vanish at ± 1) and the standard one-dimensional affine reference maps $x_{K_i} : (-1, 1) \to K_i$, this basis can have the form

$$\begin{aligned} \vartheta^{(i)}_2(x) &= l_2(x^{-1}_{K_i}(x)), \\ \vartheta^{(i)}_3(x) &= l_3(x^{-1}_{K_i}(x)), \\ \vdots \qquad & \qquad \vdots \\ \vartheta^{(i)}_{p_i}(x) &= l_{p_i}(x^{-1}_{K_i}(x)). \end{aligned} \tag{11}$$

Equally, instead of the higher-order Lobatto bubble shape functions one can exploit the traditional higher-order Lagrange nodal bubble shape functions $\theta_2, \theta_3, \ldots, \theta_{p_i}$ associated with interior nodal points (and thus also vanishing at the interval end-points).

Expressing now

$$g^{\mathrm{b}}_{h,p}|_{K_i} = \sum_{m=2}^{p_i} \alpha^{(i)}_m \vartheta^{(i)}_m$$

and inserting this linear combination into (10), one obtains on K_i a system of $p_i - 1$ linear algebraic equations,

$$\sum_{m=2}^{p_i} \alpha^{(i)}_m \int_{K_i} \left(\vartheta^{(i)}_m\right)' \left(\vartheta^{(i)}_k\right)' \,\mathrm{d}x = \int_{K_i} \left(g - g^{\mathrm{v}}_{h,p}\right)' \left(\vartheta^{(i)}_k\right)' \,\mathrm{d}x, \quad k = 2, 3, \ldots, p_i, \tag{12}$$

for the unknown coefficients $\alpha^{(i)}_m$. By the substitution theorem, (12) attains on the reference domain $(-1, 1)$ a very simple form

$$\sum_{m=2}^{p_i} \alpha^{(i)}_m \int_{-1}^{1} l'_m(\xi) l'_k(\xi) \,\mathrm{d}\xi = \int_{-1}^{1} \left(\tilde{g}^{(i)} - \tilde{g}^{v(i)}_{h,p}\right)'(\xi)\, l'_k(\xi) \,\mathrm{d}\xi, \quad k = 2, 3, \ldots, p_i. \tag{13}$$

Here, $\tilde{g}^{(i)}(\xi) = g(x_{K_i}(\xi))$ and $\tilde{g}^{v(i)}_{h,p}(\xi) = (g^{v}_{h,p}(x_{K_i}(\xi))$ is nothing else than $l_0(\xi) g(x_{i-1}) + l_1(\xi) g(x_i)$.

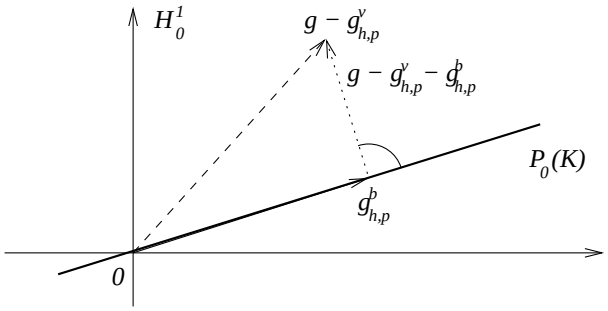

Fig. 3: *Graphical interpretation of the projection problem (13).*

Lemma 4.1 (Local optimality of the projection-based interpolant) *Let $\Omega = (a, b) \subset \mathbb{R}$ be covered with a finite element mesh $\mathcal{T}_{h,p}$ consisting of M finite elements $K_i = (x_{i-1}, x_i)$ equipped with the polynomial degrees $1 \le p_i = p(K_i)$. Let $g \in H^1(\Omega) \cap C(\overline{\Omega})$, $g_{h,p} \in V_{h,p}$ its projection-based interpolant (7) and $\tilde{g}_{h,p} \in V_{h,p}$ an arbitrary other interpolant satisfying*

$$\tilde{g}_{h,p}(x_j) = g(x_j) \qquad \text{for all} \quad j = 0, 1, \ldots, M.$$

Then

$$|g - g_{h,p}|_{1,K_i} \le |g - \tilde{g}_{h,p}|_{1,K_i} \qquad \text{for all} \quad i = 1, 2, \ldots, M \tag{14}$$

and consequently

$$|g - g_{h,p}|_{1,\Omega} \le |g - \tilde{g}_{h,p}|_{1,\Omega}. \tag{15}$$

If the bubble interpolant $g^{\mathrm{b}}_{h,p}$ is calculated using the full H^1-product $(\cdot,\cdot)_1$ instead of (8), the inequalities (14) and (15) hold with the full H^1-norm $\|\cdot\|_1$.

Proof. The fact that the bubble interpolant $g^{\mathrm{b}}_{h,p}$ is defined as the orthogonal projection of $g - g^{\mathrm{v}}_{h,p} \in H^1_0(K_i)$ onto $P^{p_i}_0(K_i)$ implies that

$$\begin{aligned} |g - g_{h,p}|_{1,K_i} &= |(g - g^{\mathrm{v}}_{h,p}) - g^{\mathrm{b}}_{h,p}|_{1,K_i} \\ &= \min_{w \in P^{p_i}_0(K_i)} |(g - g^{\mathrm{v}}_{h,p}) - w|_{1,K_i} \\ &\leq |(g - g^{\mathrm{v}}_{h,p}) - (\tilde{g}_{h,p} - g^{\mathrm{v}}_{h,p})|_{1,K_i} \\ &= |g - \tilde{g}_{h,p}|_{1,K_i}. \end{aligned}$$

The integral $|g - g_{h,p}|^2_{1,\Omega}$ can be written as a sum

$$|g - g_{h,p}|^2_{1,\Omega} = \sum_{i=1}^{M} |g - g_{h,p}|^2_{1,K_i}.$$

The inequality (14) finally yields

$$\sum_{i=1}^{M} |g - g_{h,p}|^2_{1,K_i} \leq \sum_{i=1}^{M} |g - \tilde{g}_{h,p}|^2_{1,K_i} = |g - \tilde{g}_{h,p}|^2_{1,\Omega}.$$

Everything works in the same way when the inner product (8) is replaced by the full H^1-product and the seminorm by the corresponding full H^1-norm. ▫

The projection problem (13) is illustrated in Fig. 3.

Let us close this paragraph by mentioning that the projection-based interpolation is significantly more efficient than the full projection from Paragraph 3. The cost of the local optimality on the elements K_i, $i = 1, 2, \ldots, M$, is the solution of M systems of $p_i - 1$ linear algebraic equations.

However, if we use the Lobatto bubble shape functions l_k, $k = 2, \ldots, p_i$, inner products of their derivatives in (13) vanish for $m \neq k$, the matrix on the left-hand part of (13) is diagonal and the solution of the system costs only p_i arithmetic operations.

5. Nodal interpolation

The last important interpolation technique is the *Lagrange nodal interpolation*, which is based on the evaluation of the interpolated function at a given set of nodal points, and a suitable set of interpolation polynomials. Depending on the selection of the nodal points (such as, e.g., equidistant, Chebyshev, Gauss-Lobatto, Fekete or other points [5], [6]), one obtains various variants of the general Lagrange interpolation method, that produce different interpolants.

By Lemma 4.1, all Lagrange interpolants are equally or less accurate than the projection-based interpolant (7). On the other hand, their explicit nature with no system of linear equations solved makes them extremely efficient. The Lagrange interpolation is a special case of *nodal interpolation* on general nodal elements (see, e.g., [5]).

Remark 5.1 Generally, the discretization of a PDE has nothing in common with the interpolation on the finite elements. Therefore it may be useful to introduce two sets of shape functions – *discretization shape functions* with optimal conditioning properties, and *interpolation shape functions* that minimize the interpolation error.

Although the Lagrange interpolation is natural for Lagrange nodal elements and the projection-based interpolation for Lobatto hierarchic elements, obviously the projection-based interpolation can be performed on Lagrange nodal elements and vice versa.

Interpolation conditions

Consider an interval $K_i = (x_{i-1}, x_i) \subset \Omega \subset I\!R$ and a set of arbitrary Lagrange nodal points $x_{i-1} = \tilde{y}_1^{(i)} < \tilde{y}_2^{(i)} < \ldots < \tilde{y}_{p_i+1}^{(i)} = x_i$. Using the affine reference maps $x_{K_i} : (-1,1) \to K_i$, define the corresponding points in the reference domain $(-1,1)$ as $y_j = x_{K_i}^{-1}(\tilde{y}_j^{(i)})$. On the element K_i, the interpolation conditions

$$g_{h,p}\left(\tilde{y}_j^{(i)}\right) = g\left(\tilde{y}_j^{(i)}\right) \quad \text{for all } \ 1 \le j \le p_i + 1,\ g_{h,p} \in V_{h,p},$$

are equivalent to

$$(g_{h,p} \circ x_{K_i})(y_j) = (g \circ x_{K_i})(y_j) \quad \text{for all } \ 1 \le j \le p_i + 1,\ g_{h,p} \circ x_{K_i} \in P^{p_i}(-1,1).$$

Hence, the interpolation can be performed elementwise on the reference domain $(-1,1)$. In practice a unique set of Lagrange nodal points is defined on the reference domain and used for all elements.

A basic result related to the accuracy of the Lagrange interpolation in the maximum norm is formulated in the following lemma.

Lemma 5.1 (Error of the Lagrange interpolation) *Let* $-1 = y_1 < y_2 < \ldots < y_{p+1} = 1$ *and* $g \in C^{p+1}([-1,1])$. *Define*

$$g_{h,p}(x) = \sum_{i=1}^{p+1} \left(\prod_{j \ne i} \frac{x - y_j}{y_i - y_j} \right) g(y_i). \tag{16}$$

There exists a ξ_y,

$$\min\{-1, x\} \le \xi_y \le \max\{x, 1\},$$

such that

$$g(x) - g_{h,p}(x) = \frac{\prod_{i=1}^{p+1}(x - y_i)}{(p+1)!} g^{(p+1)}(\xi_y). \tag{17}$$

Proof. The result obviously holds if $x = y_i$. Hence suppose $x \ne y_i$ for all $1 \le i \le p+1$, and denote

$$e(x) = g(x) - g_{h,p}(x).$$

The function

$$\sigma(t) = e(t) - \frac{\prod_{i=1}^{p+1}(t - y_i)}{\prod_{i=1}^{p+1}(x - y_i)} e(x)$$

has $p + 2$ distinct roots $t = x$ and $t = y_i$, $1 \le i \le p + 1$. The mean value theorem implies that $\sigma'(t)$ has $p+1$ distinct roots. Applying the mean value theorem to higher derivatives of σ, we find that $\sigma^{(p+1)}(t)$ has a single root $\xi_y \in (\min\{-1, x\}, \max\{x, 1\})$, satisfying

$$0 = \sigma^{(p+1)}(\xi_y) = g^{(p+1)}(\xi_y) - \frac{(p+1)!}{\prod_{i=1}^{p+1}(x - y_i)} e(x),$$

and (17) follows. □

The function $\beta_p(x) = \prod_{i=1}^{p+1}(x - y_i)$ in (17) is the only way the distribution of the nodal points influences the distribution of the interpolation error. Compare with the projection-based interpolation from Paragraph 4. where the interpolation error was independent of the concrete representation of the polynomial space.

Let us look at $\beta_p(x)$ for equidistributed nodal points in Fig. 4. Notice the different scales.

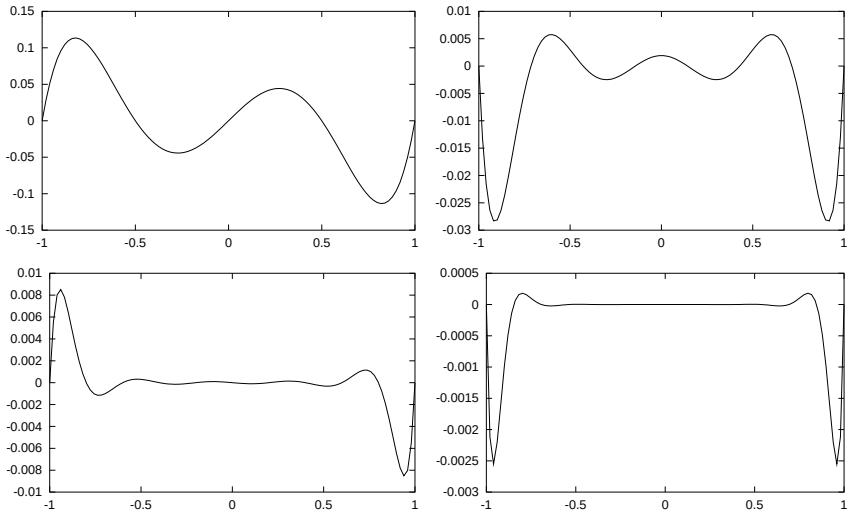

Fig. 4: *Error factor $\beta_p(x)$ for equidistributed nodal points, $p = 4, 7, 10$ and 13.*

From these plots it is clear that the behaviour of the error $e(x) = g(x) - g_{h,p}(x)$ is significantly worse near the endpoints than in the interior.

The Lagrange interpolation with equidistributed nodal points is known to be notoriously bad. In his famous example from 1901, Carl Runge showed that the sequence of Lagrange interpolants $g_{h,p}$ with equidistributed nodal points diverges for otherwise a very nice function $g(x) = 1/(1 + 25x^2)$ in the interval $(-1, 1)$ as $p \to \infty$ (for details see, e.g., [3]).

Important special case – the Chebyshev interpolation

The Lagrange interpolant (16) based on the nodal points

$$y_j = \cos\left(\frac{\pi(j-1)}{p}\right), \qquad j = 1, 2, \ldots, p+1, \tag{18}$$

is called the *Chebyshev interpolant*. The error factors β_p for the Chebyshev interpolation with $p = 4, 7, 10$ and 13 are shown in Fig. 5. Compare with Fig. 4 and notice the different scales.

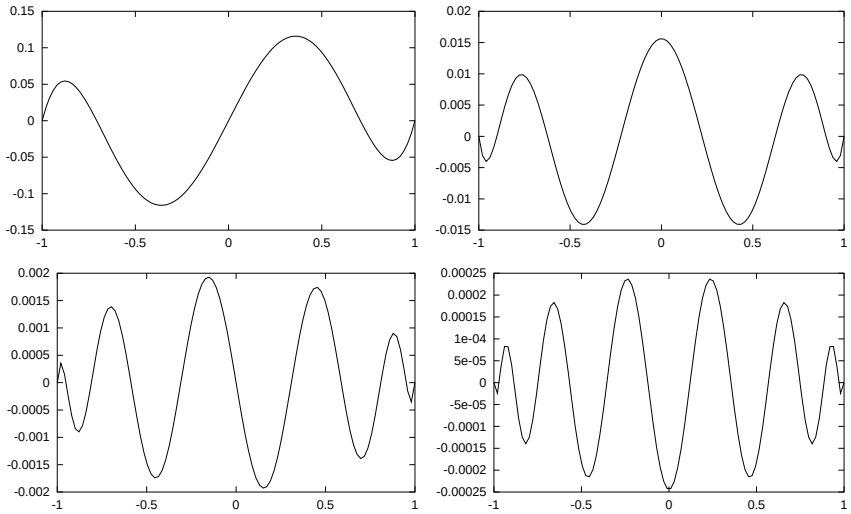

Fig. 5: *Error factor $\beta_p(x)$ for Chebyshev nodal points, $p = 4, 7, 10$ and 13.*

Before introducing the basic Chebyshev interpolation error estimate, we need the weighted L^2-space

$$L^2_w(-1,1) = \{v \in L^2(-1,1);\ v \text{ is measurable and } \|v\|_{0,w} < \infty\}$$

with

$$\|v\|^2_{0,w} = \int_{-1}^{1} |v(x)|^2 w(x)\,\mathrm{d}x, \tag{19}$$

where

$$w(x) = \frac{1}{\sqrt{1-x^2}}, \qquad x \in (-1,1),$$

is the *Chebyshev weight function*. The norm (19) induces an inner product

$$(u,v)_w = \int_{-1}^{1} u(x)v(x)w(x)\,\mathrm{d}x$$

on $L^2_w \times L^2_w$. Further define a weighted Sobolev space

$$H^s_w(-1,1) = \left\{ v \in L^2_w(-1,1);\ v^{(k)} \in L^2_w \text{ for all } k = 1,2,\ldots,s \right\}$$

with the norm

$$\|v\|_{s,w} = \left(\sum_{k=0}^{s} \|v^{(k)}\|^2_{0,w} \right)^{1/2}.$$

Here $v^{(k)}$ denotes the kth weak derivative of v.

Theorem 5.1 (Chebyshev interpolation error estimate) *Let $u \in H^s_w(-1,1)$ for some $s \geq 1$. Let $P_p u$ be the Chebyshev interpolant of degree p based on the $p+1$ nodal points* (18)*. Then there exists a constant C independent of u such that*

$$\|u - P_p u\|_{0,w} \leq C p^{-s} \|u\|_{s,w}.$$

Proof can be found, e.g., in [4]. The theorem shows that the Chebyshev interpolation converges. Among nodal interpolation schemes, Chebyshev interpolation is very popular due to its accuracy. More details can be found, e.g., in [1], [3].

References

[1] Atkinson K.: *An introduction to numerical analysis.* 2nd ed. New York, John Wiley & Sons 1989.

[2] Demkowicz L., Babuška I.: *Optimal p interpolation error estimates for edge finite elements of variable order in 2D.* TICAM Report 01-11. Austin, University of Texas 2001.

[3] Heath M.T.: *Scientific computing. An introductory survey.* 2nd ed. New York, McGraw-Hill 2002.

[4] Pasciak J.E.: *Spectral and pseudospectral methods for advection equations.* Math. Comp. **35**, 1980, 1081–1092.

[5] Šolín P., Segeth K., Doležel I.: *Higher-order finite element methods.* Boca Raton, FL, Chapman & Hall/CRC Press 2004.

[6] Taylor M.A., Wingate B.A., Vincent R.E.: *An algorithm for computing Fekete points in the triangle.* SIAM J. Numer. Anal. **38**, 2000, 1707–1720.