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In: Jan Chleboun and Petr Přikryl and Karel Segeth (eds.): Programs and Algorithms of Numerical Mathematics, Proceedings of Seminar. Dolní Maxov, June 6-11, 2004. Institute of Mathematics AS CR, Prague, 2004. pp. 242–247.

Persistent URL: http://dml.cz/dmlcz/702801

# DISCONTINUOUS GALERKIN METHOD OF LINES FOR SOLVING NONSTATIONARY SINGULARLY PERTURBED LINEAR PROBLEMS *

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## 1. Introduction

In the numerical solution of a number of complex problems from science and technology it appears that classical finite difference, finite volume or standard finite element methods do not allow to realize numerical approximations of solutions containing shock waves and contact discontinuities or steep gradients in internal or boundary layers. An excellent candidate to overcome the mentioned difficulties is the discontinuous Galerkin finite element method (DGFEM).

The DGFEM uses piecewise polynomial approximations of the sought solution on a finite element mesh without any requirement on the continuity between neighbouring elements. It allows to construct higher order schemes in a natural way and is suitable for the approximation of discontinuous solutions of conservation laws or solutions of singularly perturbed convection-diffusion problems.

Here we discuss some aspects of the DGFEM of lines applied to a linear scalar convection-diffusion-reaction equation with possibly degenerating diffusion.

## 2. Continuous problem

Let $\Omega \subset I\!R^d$ ($d = 2$ or $3$) be a bounded polygonal (for $d = 2$) or polyhedral (for $d = 3$) domain with a Lipschitz boundary $\partial\Omega$ and $T > 0$ (or $\infty$). We consider the following initial-boundary value problem: Find $u : Q_T = \Omega \times (0,T) \to I\!R$ such that

$$
\begin{aligned}
\frac{\partial u}{\partial t} + \boldsymbol{v}\cdot\nabla u - \varepsilon\triangle u + cu &= g \qquad \text{in } Q_T, && (1)\\
u &= u_D \quad \text{on } \partial\Omega^- \times (0,T), && (2)\\
\varepsilon\frac{\partial u}{\partial \boldsymbol{n}} &= u_N \quad \text{on } \partial\Omega^+ \times (0,T), && (3)\\
u(x,0) &= u^0(x), \quad x \in \Omega. && (4)
\end{aligned}
$$

Here we assume that $\partial\Omega = \partial\Omega^- \cup \partial\Omega^+$ and $\boldsymbol{v}(x,t)\cdot\boldsymbol{n}(x) < 0$ on $\partial\Omega^-$ (inflow) and $\boldsymbol{v}(x,t)\cdot\boldsymbol{n}(x) \ge 0$ on $\partial\Omega^+$ (outflow), for all $t \in (0,T)$. By $\boldsymbol{n}(x)$ we denote the unit

*This research was supported by the Grant No. 201/04/1503 of the Czech Grant Agency and the Grant No. MSM 0021620839 of the Ministry of Education of the Czech Republic. The author is very much obliged to Prof. M. Feistauer for fruitful discussions.

outer normal to the boundary of $\Omega$. In the case $\varepsilon = 0$ we put $u_N = 0$ and ignore the Neumann condition (3). We assume that the data are sufficiently regular, $\varepsilon \geq 0$ and that $c - \frac{1}{2}\mathrm{div}\boldsymbol{v} \geq \gamma_0 \geq 0$ in $Q_T$ with a constant $\gamma_0$.

## 3. Discretization of the problem

Let $\mathcal{T}_h = \bigcup_{i\in I} K_i$ ($I \subset \{0, 1, 2, ...\}$ is a suitable index set) be a standard triangulation of the closure of the domain $\Omega$ into a finite number of closed triangles ($d = 2$) or tetrahedra ($d = 3$). We assume that for $K_i, K_j \in \mathcal{T}_h$ either $K_i \cap K_j = \emptyset$ or $K_i \cap K_j$ is a common $(d-1)$-dimensional face which we denote by $\Gamma_{ij} (= \Gamma_{ji})$ or a common vertex (for $d = 3$ it can also be a common edge). In the case when $K_i \cap K_j = \Gamma_{ij}$, we call $K_i$ and $K_j$ neighbours. For $i \in I$ we set $s(i) = \{j \in I;\ K_j \text{ is a neighbour of } K_i\}$. For $K \in \mathcal{T}_h$, by $h_K$ and $\rho_K$ we denote the diameter of $K$ and the diameter of the largest ball inscribed in $K$, respectively. We set $h = \max_{K\in\mathcal{T}_h} h_K$. We assume that the triangulation is shape-regular: there exists a constant $C_{\mathcal{T}}$ independent of $K \in \mathcal{T}_h$ and $h$ such that

$$h_K/\rho_K \leq C_{\mathcal{T}} \qquad \forall K \in \mathcal{T}_h. \tag{5}$$

We introduce the so-called broken Sobolev space

$$H^k(\Omega, \mathcal{T}_h) = \{\varphi; \varphi|_K \in H^k(K)\ \forall K \in \mathcal{T}_h\} \tag{6}$$

and define the seminorm $|\varphi|^2_{H^k(\Omega,\mathcal{T}_h)} = \sum\limits_{K\in\mathcal{T}_h} |\varphi|^2_{H^k(K)}$, where $|\cdot|_{H^k(K)}$ is the seminorm in the Sobolev space $H^k(K)$. For $\varphi \in H^1(\Omega, \mathcal{T}_h), i \in I$ and $j \in s(i)$ we shall use the following notation: $\varphi|_{\Gamma_{ij}} =$ the trace of $\varphi|_{K_i}$ on $\Gamma_{ij}$, $\varphi|_{\Gamma_{ji}} =$ the trace of $\varphi|_{K_j}$ on $\Gamma_{ji} = \Gamma_{ij}$, $\langle\varphi\rangle_{\Gamma_{ij}} = \frac{1}{2}\left(\varphi|_{\Gamma_{ij}} + \varphi|_{\Gamma_{ji}}\right)$, $[\varphi]_{\Gamma_{ij}} = \varphi|_{\Gamma_{ij}} - \varphi|_{\Gamma_{ji}}$, $\boldsymbol{n}_{ij} =$ the unit outer normal to $\partial K_i$ on the face $\Gamma_{ij}$. Further, for $i \in I$ we set $\partial K_i^-(t) = \{x \in \partial K_i;\ \boldsymbol{v}(x,t)\cdot\boldsymbol{n}(x) < 0\}$ and $\partial K_i^+(t) = \{x \in \partial K_i;\ \boldsymbol{v}(x,t)\cdot\boldsymbol{n}(x) \geq 0\}$.

In the derivation of the discrete problem we start from equation (1), multiply it by any $\varphi \in H^2(\Omega, \mathcal{T}_h)$, integrate over each $K_i$, apply Green's theorem in the diffusion and convective terms, sum over all $i \in I$, add some terms to both sides of the resulting identity or vanishing terms and use the boundary conditions. We find that the exact solution $u$ satisfies the following identity for a.e. $t \in (0, T)$ :

$$\left(\frac{\partial u(t)}{\partial t}, \varphi\right) + a_h(u(t), \varphi) + b_h(u(t), \varphi) + c_h(u(t), \varphi) + \varepsilon J_h^\sigma(u(t), \varphi) = l_h(\varphi)(t). \tag{7}$$

The forms in (7) are defined in the following way:

$$(u, \varphi) = \int_\Omega u\varphi\, dx, \tag{8}$$

$$a_h(u, \varphi) = \varepsilon \sum_{i\in I} \int_{K_i} \nabla u \cdot \nabla\varphi\, dx \tag{9}$$

$$
\begin{aligned}
& -\varepsilon \sum_{i\in I} \sum_{j\in s(i), j<i} \int_{\Gamma_{ij}} \left( \langle \nabla u \rangle \cdot \boldsymbol{n}_{ij} \, [\varphi] - \langle \nabla \varphi \rangle \cdot \boldsymbol{n}_{ij} \, [u] \right) \, dS \\
& -\varepsilon \sum_{i\in I} \int_{\partial K_i^- \cap \partial\Omega} \left( (\nabla u \cdot \boldsymbol{n})\varphi - (\nabla \varphi \cdot \boldsymbol{n}) u \right) \, dS,
\end{aligned}
$$

$$
\begin{aligned}
b_h(u,\varphi) \; = \; & \sum_{i\in I} \int_{K_i} (\boldsymbol{v}\cdot\nabla u)\varphi \, dx - \sum_{i\in I} \int_{\partial K_i^- \cap \partial\Omega} (\boldsymbol{v}\cdot\boldsymbol{n}) u\varphi \, dS \\
& - \sum_{i\in I} \int_{\partial K_i^- \setminus \partial\Omega} (\boldsymbol{v}\cdot\boldsymbol{n}) [u] \varphi \, dS,
\end{aligned} \tag{10}
$$

$$
c_h(u,\varphi) = \int_\Omega c u \varphi \, dx, \tag{11}
$$

$$
J_h^\sigma(u,\varphi) = \sum_{i\in I} \sum_{j\in s(i)} \int_{\Gamma_{ij}} \sigma \, [u] \, [\varphi] \, dS + \sum_{i\in I} \int_{\partial K_i^- \cap \partial\Omega} \sigma u \varphi \, dS, \tag{12}
$$

$$
\begin{aligned}
l_h(\varphi)(t) \; = \; & \int_\Omega g(t)\varphi \, dx + \sum_{i\in I} \int_{\partial K_i^+ \cap \partial\Omega} u_N(t)\varphi \, dS \\
& + \varepsilon \sum_{i\in I} \int_{\partial K_i^- \cap \partial\Omega} \sigma u_D(t)\varphi \, dS + \varepsilon \sum_{i\in I} \int_{\partial K_i^- \cap \partial\Omega} u_D(t) (\nabla\varphi\cdot\boldsymbol{n}) \, dS \\
& - \sum_{i\in I} \int_{\partial K_i^- \cap \partial\Omega} (\boldsymbol{v}\cdot\boldsymbol{n}) u_D(t)\varphi \, dS,
\end{aligned} \tag{13}
$$

where $\sigma|_{\Gamma_{ij}} = 1/\operatorname{diam}(\Gamma_{ij})$.

In the form $a_h(u,\varphi)$ representing the discretization of the diffusion term we use the nonsymmetric formulation. In the discretization of the convective terms the idea of upwinding is used. We apply Green's theorem and get

$$
\int_{K_i} (\boldsymbol{v}\cdot\nabla u)\varphi \, dx = \int_{\partial K_i^-} (\boldsymbol{v}\cdot\boldsymbol{n}) u\varphi \, dS + \int_{\partial K_i^+} (\boldsymbol{v}\cdot\boldsymbol{n}) u\varphi \, dS - \int_{K_i} u \operatorname{div}(\varphi \boldsymbol{v}) \, dx. \tag{14}
$$

On the inflow part of the boundary of $K_i$ (i.e., $\partial K_i^-$) we use the information from outside the element $K_i$. Therefore, we write $u^-$ instead of $u$. Here $u^-$ is a simplified notation for $u|_{\Gamma_{ji}}$ where $j$ is the index of the corresponding neighbour $K_j$ to $K_i$. On $\Gamma_{ij} = \Gamma_{ji} \subset \partial\Omega^-$ we set $u^- := u_D$. We further rearrange the resulting terms and obtain the form $b_h$. The form $J_h^\sigma$ represents the interior and boundary penalty replacing the continuity of conforming finite elements.

The approximate solution will be sought for each $t \in (0,T)$ in the FE space

$$
S_h = S^{p,-1}(\Omega, \mathcal{T}_h) = \left\{ \varphi \in L^2(\Omega); \varphi|_K \in P^p(K) \;\; \forall K \in \mathcal{T}_h \right\}, \tag{15}
$$

where $p \geq 1$ is an integer and $P^p(K)$ is the space of polynomials on $K$ of degree at most $p$. Now the *DGFE discrete problem* reads: Find an approximate solution $u_h$ of problem (1)-(4) such that

$$\text{a) } u_h \in C^1([0,T); S_h), \tag{16}$$
$$\text{b) } \left(\frac{\partial u_h(t)}{\partial t}, \varphi_h\right) + a_h(u_h(t),\varphi_h) + b_h(u_h(t),\varphi_h) + c_h(u_h(t),\varphi_h) + \varepsilon J_h^\sigma(u_h(t),\varphi_h)$$
$$= l_h(\varphi_h)(t) \quad \forall \varphi_h \in S_h,\ \forall t \in (0,T),$$
$$\text{c) } (u_h(0),\varphi_h) = (u^0,\varphi_h) \qquad \forall \varphi_h \in S_h.$$

If $\varepsilon = 0$, we can also choose $p = 0$. In this case we get the finite volume method.

## 4. Error estimates

In [2] we analyzed error estimates of the above method and obtained the following result.

**Theorem** *Let us assume that $\{\mathcal{T}_h\}_{h\in(0,h_0)}$ is a system of triangulations of $\Omega$ with property (5) and that conditions on the data are satisfied. Let the exact solution $u$ of problem (1)-(4) be regular enough and let $u_h$ be the approximate solution obtained by the method of lines (16), a) - c). Then the error $e_h = u_h - u$ satisfies the estimate*

$$\max_{t\in(0,T)} \|e_h(t)\|_{L^2(\Omega)} + \sqrt{\varepsilon}\sqrt{\int_0^T |e_h(\vartheta)|^2_{H^1(\Omega,\mathcal{T}_h)}\,d\vartheta + \int_0^T J_h^\sigma(e_h(\vartheta),e_h(\vartheta))\,d\vartheta}$$
$$+\sqrt{2\gamma_0}\|e_h\|_{L^2(Q_T)} + \sqrt{\frac{1}{2}\sum_{i\in I}\int_0^T \left(\|e_h(t)\|^2_{\boldsymbol{v}(t),\partial K_i\cap\partial\Omega} + \|[e_h(t)]\|^2_{\boldsymbol{v}(t),\partial K_i^-(t)\setminus\partial\Omega}\right)dt}$$
$$\le C(T)h^p(\sqrt{\varepsilon}+\sqrt{h}). \tag{17}$$

Comparing this estimate with results from [1] we see that our estimate is better due to the linearity of our problem.

## 5. Numerical experiments

In order to test the theoretical error estimate we present numerical experiments with the DGFEM of lines introduced in (16).We deal with the hyperbolic equation

$$\frac{\partial u}{\partial t} + v_1\frac{\partial u}{\partial x_1} + v_2\frac{\partial u}{\partial x_2} + cu = g \quad \text{ in } \Omega\times(0,T), \tag{18}$$

with $\Omega = (0,1)^2, v_1 = 0.3, v_2 = 0.4$ and $c = 0.5$, equipped with initial condition (4) and boundary condition (2), where $u_D$ is prescribed on the whole $\partial\Omega$.

Let us define the function $g$ and the initial and boundary conditions in such a way that the exact solution has the form

$$u(x_1,x_2,t) = \left(1-e^{-t}\right)\left(x_1x_2^2 - x_2^2 e^{2\frac{x_1-1}{\nu}} - x_1 e^{3\frac{x_2-1}{\nu}} + e^{\frac{2x_1+3x_2-5}{\nu}}\right), \tag{19}$$

where $\nu > 0$ is a given number. We seek the steady-state solutin of (19) by the time stabilization method. The steady-state solution has the form

$$\lim_{t\to\infty} u(x_1,x_2,t) = u^{\mathrm{st}}(x_1,x_2) = x_1x_2^2 - x_2^2e^{2\frac{x_1-1}{\nu}} - x_1e^{3\frac{x_2-1}{\nu}} + e^{\frac{2x_1+3x_2-5}{\nu}}. \qquad (20)$$

Function (20) has two steep “boundary layers”, the steepness of which is given by the parameter $\nu$. The computation was performed for $\nu = 0.1$ and 0.01, see Fig. 1.

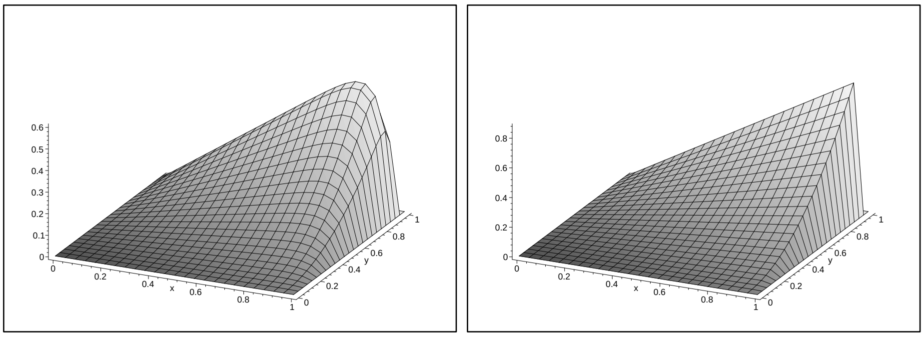

**Fig. 1:** *The steady-state solution (20) for $\nu = 0.1$ (left) and $\nu = 0.01$ (right).*

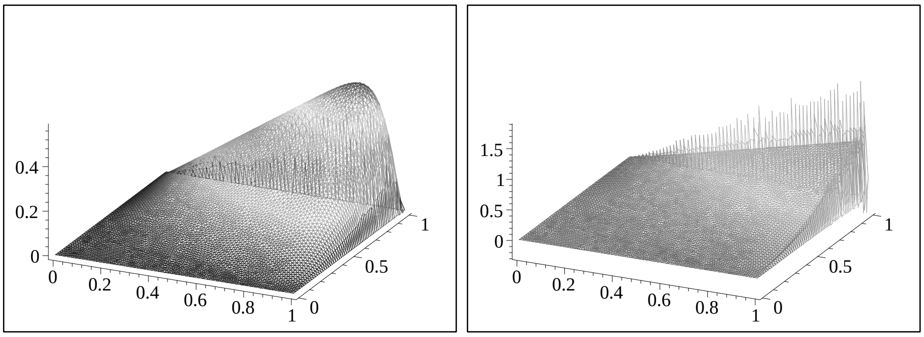


**Fig. 2:** *Numerical solution computed on $\mathcal{T}_{h_7}$ for $\nu = 0.1$ (left) and $\nu = 0.01$ (right).*

We solved the initial-boundary value problem by the presented numerical method (16) with $p = 1$, i.e. piecewise linear elements. The resulting system of ODE’s was solved by the forward Euler method with a small time step $\tau = 10^{-4}$, which guarantees stability and sufficiently precise resolution with respect to time. The computational error of the steady state solution is evaluated in $L^2(\Omega)$-norm $e_h \equiv \|u_h^{\mathrm{st}} - u^{\mathrm{st}}\|_{L^2(\Omega)}$. We define the *local experimental order of convergence* by

$$\alpha_l = \frac{\log\left(e_{h_l}/e_{h_{l-1}}\right)}{\log\left(h_l/h_{l-1}\right)}, \quad l = 2,\ldots,7. \qquad (21)$$

Moreover, we compute the *global experimental order of convergence* $\bar{\alpha}$ by the least squares method. Table 1 shows the $L^2$-error, the values of $\alpha_l$, $l = 2, \ldots, 7$, and $\bar{\alpha}$. Fig. 2 shows the computed numerical results on the mesh $\mathcal{T}_{h_7}$. We observe a continuous numerical solution although the discontinuous approximation is used.

| | | | $\nu = 0.1$ | | $\nu = 0.01$ | |
|---|---|---|---|---|---|---|
| $l$ | $\#\mathcal{T}_{h_l}$ | $h_l$ | $e_{h_l}$ | $\alpha_l$ | $e_{h_l}$ | $\alpha_l$ |
| 1 | 125 | 0.173 | 0.0257 | – | 0.400 | – |
| 2 | 250 | 0.128 | 0.0158 | 1.61 | 0.272 | 1.28 |
| 3 | 500 | 0.090 | 0.0068 | 2.40 | 0.136 | 1.97 |
| 4 | 1000 | 0.064 | 0.0048 | 1.01 | 0.098 | 0.96 |
| 5 | 2000 | 0.045 | 0.0020 | 2.53 | 0.044 | 2.27 |
| 6 | 4000 | 0.032 | 0.0014 | 1.00 | 0.033 | 0.84 |
| 7 | 8000 | 0.023 | 0.0006 | 2.67 | 0.014 | 2.47 |
| global order of accuracy $\bar{\alpha}$ | | | | 1.85 | | 1.66 |

**Tab. 1:** *Errors in $L^2$-norm, $\alpha_l$, $l = 2, \ldots, 7$ and $\bar{\alpha}$ for $\nu = 0.1$ and $\nu = 0.01$.*

From Fig. 2 we see that for $\nu = 0.01$ the approximate solution suffers from spurious overshoots and undershoots manifesting the so-called Gibbs phenomenon. They can be avoided by a suitable limiting of the order of accuracy of the space discretization in the vicinity of a steep gradient. Nevertheless, from numerical experiments we see that the Gibbs phenomenon does not effect the theoretical order of convergence in a negative way.

## 6. Conclusion

We derived $L^\infty(L^2), L^2(L^2)$ and $\sqrt{\varepsilon}L^2(H^1)$ estimates for the error of the approximate solution which are of order $h^p(\sqrt{h} + \sqrt{\varepsilon})$. This is the optimal estimate. We cannot get reasonable estimates when upwinding is not used in the scheme. The estimates hold true even if $\varepsilon = 0$ and are of order $h^{p+1/2}$.

## References


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