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Hierarchical FEM: Strengthened CBS inequalities, error estimates and iterative solvers

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# HIERARCHICAL FEM: STRENGTHENED CBS INEQUALITIES, ERROR ESTIMATES AND ITERATIVE SOLVERS*

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**Abstract**

This paper describes natural decomposition of hierarchical finite element spaces, discusses a characterization of this decomposition via strengthened CBS inequality and uses this decomposition for development of hierarchical error estimates and iterative solution methods.

## 1. Introduction

A subsequent refinement of a finite element grid provides a sequence of nested grids and hierarchy of nested finite element spaces as well as a natural hierarchical decomposition of these spaces. This decomposition can be characterized by the constant from the corresponding Cauchy–Bunyakowski–Schwarz (CBS) inequality. In Section 2, we summarize some older and recent results concerning this constant. The CBS analysis is exploited in Section 3 for investigation of the so called hierarchical error estimates. We shall show that such estimates are robust with respect to coefficient jumps and anisotropy as well as to the element shape. Hierarchical error estimates can be used for both global and local error assessment. Local estimates can be used for local refinement and construction of hierarchy of locally refined spaces. In Section 4, we outline the hierarchical decomposition in this case. Note that this decomposition can be used for defining various iterative solution methods and preconditioners.

## 2. FE hierarchy and natural decomposition

Let us consider a model boundary value problem in $\Omega \subset R^d$ $(d = 2, 3)$,

$$\text{find} \quad u \in V \ : \ a(u, v) = b(v) \qquad \forall v \in V, \tag{1}$$

where $V = H_0^1(\Omega)$, $b(v) = \int_\Omega f v dx$ for $f \in L_2(\Omega)$ and

$$a(u, v) = \int_\Omega \sum_{ij}^{d} k_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} \, dx \, .$$

Above $K = (k_{ij})$ is a symmetric and positive definite matrix of coefficients.

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We also consider a coarse triangular or tetrahedral finite element grid $\mathcal{T}_H$ of $\Omega$ and a fine grid $\mathcal{T}_h$, which arises by a refinement of the coarse elements. By $\mathcal{N}_H$ and $\mathcal{N}_h$, we denote the set of nodes corresponding to $\mathcal{T}_H$ and $\mathcal{T}_h$, respectively. Naturally, $\mathcal{N}_h = \mathcal{N}_H \cup \mathcal{N}_H^+$, where $\mathcal{N}_H^+$ is the complement of $\mathcal{N}_H$ in $\mathcal{N}_h$.

Now, we can introduce the finite element spaces $U_H$ and $U_h$ ($U_H \subset U_h$) of functions which are continuous and linear on the elements of the triangulation $\mathcal{T}_H$, and $\mathcal{T}_h$, respectively. We shall also speak about a *hierarchy of triangulations and finite element spaces.*

Let $\{\phi_i^H\}$ and $\{\phi_i^h\}$ be the standard nodal finite element bases of $U_H$ and $U_h$, i.e. $\phi_i^H(x_j) = \delta_{ij}$ for all $x_j \in \mathcal{N}_H$, $\phi_i^h(x_j) = \delta_{ij}$ for all $x_j \in \mathcal{N}_h$. Then we can also introduce a *hierarchical basis* $\{\bar{\phi}_i^h\}$ in $U_h$,

$$\bar{\phi}_i^h = \begin{cases} \phi_i^h & \text{if} \quad x_i \in \mathcal{N}_H^+ , \\ \phi_i^H & \text{if} \quad x_i \in \mathcal{N}_H . \end{cases}$$

It gives a *natural hierarchical decomposition* of the space $U_h$,

$$U_h = U_H \oplus U_H^+ , \tag{2}$$

where $U_H^+ = \text{span}\,\{\phi_i^h,\ x_i \in \mathcal{N}_H^+\}$.

The decomposition (2) is characterized by the angle between the subspaces or the strengthened CBS inequality with the constant $\gamma = \cos(U_H, U_H^+)$, which is defined as follows:

$$\begin{aligned} \gamma &= \cos(U_H, U_H^+) \\ &= \sup \left\{ \frac{|\, a(u,v)\,|}{\sqrt{a(u,u)}\sqrt{a(v,v)}} : u \in U_H,\ a(u,u) \neq 0,\ v \in U_H^+,\ a(v,v) \neq 0 \right\} . \end{aligned} \tag{3}$$

If $\mathcal{T}_h$ arises from $\mathcal{T}_H$ by a regular division of the coarse grid triangles into $m^2$ congruent triangles in 2D or a regular division (given by the affine mapping to a reference rectangular tetrahedra) of the coarse grid tetrahedra into $m^3$ tetrahedra (see Fig. 1) and if the coefficients $K = (k_{ij})$ are constant on the coarse grid elements then for general anisotropic coefficients and arbitrary shape of the coarse grid elements, we get

$$\gamma \leq \sqrt{\frac{m^2-1}{m^2}} \qquad \text{and} \qquad \gamma \leq \sqrt{\frac{(m^2-1)(m^2+2)}{m^2(m^2+1)}}$$

for 2D and 3D case, respectively. See [1], [4] and the references given there for more details.

Note that in special cases we get smaller values of $\gamma$. For example, $\gamma \leq \sqrt{3/8}$ for isotropic coefficients and equilateral triangles [8], $\gamma \leq \sqrt{1/2}$ for isotropy and rectangular finite elements [8], [1] or $\gamma \leq \sqrt{3/4}$ for orthotropy $k_{ij} = k_i\delta_{ij}$ and rectangular tetrahedra [4] .

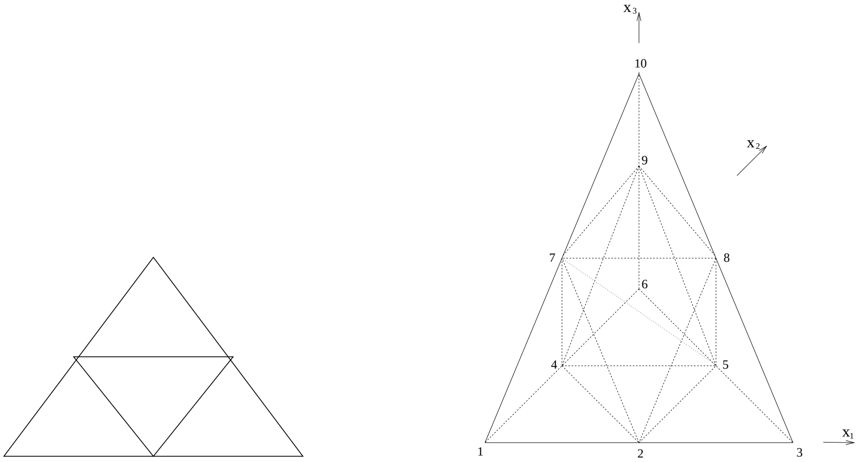


**Fig. 1:** *Decompositions in 2D and 3D with multiplicity $m = 2$.*

## 3. Hierarchical error estimates

Hierarchical error estimates were introduced in papers by R.E. Bank, see [3]. The aim is to estimate the error $e_H = u - u_H$, where $u_H \in V_H$ is the finite element approximation of the exact solution $u \in V$ of the considered boundary value problem (1), $V_H = U_H \cap V$ .

Let us also introduce the spaces $V_h = U_h \cap V$ and $V_H^+ = U_H^+ \cap V,\ V_h = V_H \oplus V_H^+$ and let $u_h$ be the finite element approximation of $u$ in $V_h$, i.e.

$$u_h \in V_h : a(u - u_h, z) = 0 \quad \forall z \in V_h . \tag{4}$$

**Lemma 1** *Let there is a positive constant $\beta < 1$ such that*

$$\| u - u_h \|_a \leq \beta \| u - u_H \|_a, \tag{5}$$

*where $\| v \|_a = \sqrt{a(v, v)}$. Then*

$$\frac{1}{1+\beta} \| u_H - u_h \|_a \leq \| u - u_H \|_a \leq \frac{1}{1-\beta} \| u_H - u_h \|_a . \tag{6}$$

**Proof** see e.g. [3].

The assumption (5) is crucial and need not be fulfilled in any case, see e.g. [6] for a counterexample. If this assumption holds, then

$$\eta = \| u_H - u_h \|_a \tag{7}$$

is the two-level a posteriori error estimate.

For practical use, the computation of $\eta$ is too expensive. The hierarchical decomposition $V_h = V_H \oplus V_H^+$ then suggest to use an approximation $w_h$ to $u_h$,

$$w_h \in V_H^+ : a(u - u_H - w_h, z) = 0 \quad \forall z \in V_H^+ . \tag{8}$$

**Lemma 2** *Let the saturation assumption (5) holds and* $\eta_H = \| w_h \|_a$. *Then*

$$\frac{1}{(1+\beta)(1+\gamma)}\eta_H \leq \| u - u_H \|_a \leq \frac{1}{(1-\beta)(1-\gamma)}\eta_H, \tag{9}$$

*where* $\gamma = \cos(V_H, V_H^+)$.

**Proof** see e.g. [3].

Note that $\eta_H$ is called the hierarchical error estimate.

Let us now consider algebraic formulation of the fine grid finite element approximation in the hierarchical basis. We get

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

where $u_1$ and $u_2$ correspond to $\mathcal{N}_H^+ \setminus \partial\Omega$ and $\mathcal{N}_H \setminus \partial\Omega$, respectively, and

$$\begin{aligned} A_{11} &= [a(\phi_j^h, \phi_i^h) : x_i, x_j \in \mathcal{N}_H^+ \setminus \partial\Omega], \\ A_{12} &= [a(\phi_j^H, \phi_i^h) : x_i \in \mathcal{N}_H^+ \setminus \partial\Omega, x_j \in \mathcal{N}_H \setminus \partial\Omega], \text{ etc}. \end{aligned}$$

Then

$$\begin{aligned} u_H \text{ is represented by } w_2 : A_{22} w_2 &= b_2 \\ w_h \text{ is represented by } w_1 : A_{11} w_1 &= b_1 - A_{12} u_2 \end{aligned}$$

and $\eta_1 = \| w_h \|_a = \sqrt{\langle A_{11} w_1, w_1 \rangle} = \| w_1 \|_A$ .

The computation of $w_1$ can be still too expensive and we can be interested in a possible simplification, e.g. by approximation $\bar{A}_{11} \sim A_{11}$ such that

$$\bar{w}_1 : \bar{A}_{11} \bar{w}_1 = b_1 - A_{12} u_2$$

can be computed in a number of operations proportional to the number of elements in $\mathcal{N}_H^+$ (i.e. $O(\#\mathcal{N}_H^+)$ operations) and provide a good approximation to $w_1$.

The simplest case is to replace $A_{11}$ by its diagonal, but then the relation between $\| \bar{w}_1 \|_A$ and $\| w_1 \|_A$ depends on anisotropy and/or shape of the elements.

For 2D case, another approximation can be constructed as in the paper [2]. It gives nice bounds independent on the discretization size and both anisotropy and element shape,

$$(1 - \sqrt{\frac{7}{15}}) \| \bar{w}_1 \|_A \leq \| w_1 \|_A \leq (1 + \sqrt{\frac{7}{15}}) \| \bar{w}_1 \|_A .$$

Moreover, $\bar{w}_1$ can be computed in $O(\#\mathcal{N}_H^+)$ operations.

## 4. Locally refined hierarchy

The hierarchical error estimators discussed in the previous section are global, but their value can be computed from contributions of macroelements corresponding to coarse grid elements to $\| w_h \|_a$. These local contributions or another local estimators can be used for determination of these coarse grid elements, which should be refined. After their refining, we can either work with special hanging nodes or make another refinement of the surrounding elements by their bisection, see Fig. 2.

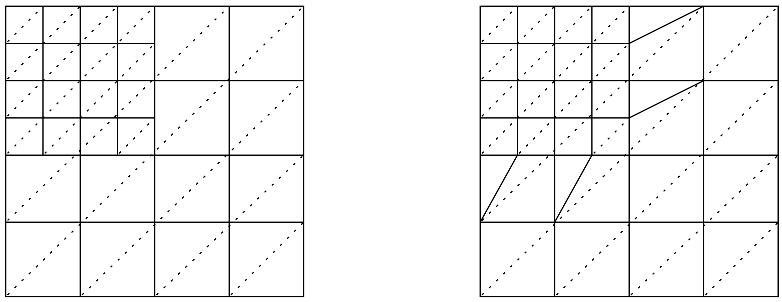

**Fig. 2:** *Local refinement with hanging nodes (left) and bisection (right).*

Again we get spaces $U_H$ and $U_h$ and the natural decomposition $U_h = U_H \oplus U_H^+$. The constant $\gamma = \cos(U_H, U_H^+)$, is then important for special iterative solution methods like FAC or BEPS, see [7], [5] and the references therein.

**Theorem 1**

- *In the case of local refinement with hanging nodes, $\gamma$ remains the same as in the case of global refinement.*
- *In the case of local refinement with bisection, we obtain the same constant $\gamma$ only in special cases (e.g. orthotropic problems $k_{ij} = k_i \delta_{ij}$ and refinement like on Fig.2 right). Generally, $\gamma$ is not further robust with respect to anisotropy or the element shape.*

The proof of the first statement can be found in [7], the second statement will be discussed in a forthcoming paper.

## 5. Conclusions

The paper shows the hierarchical finite element method with hierarchical error estimates, which are robust with respect to coefficients jumps between coarse elements and both physical and numerical anisotropy. The finite element problems on locally refined grids can be solved by iterative methods, see [7] and [5]. The convergence of these methods can be again estimated with the aid of the strengthened CBS constant.

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