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In: Jan Chleboun and Karel Segeth and Tomáš Vejchodský (eds.): Programs and Algorithms of Numerical Mathematics, Proceedings of Seminar. Prague, May 28-31, 2006. Institute of Mathematics AS CR, Prague, 2006. pp. 37–45.

Persistent URL: http://dml.cz/dmlcz/702816

ON A TRAFFIC PROBLEM*

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Abstract

We consider a macroscopic follow-the-leader model of a road traffic. The novelty is that we incorporate the possibility to overtake a slower car. We introduce two ways to simulate overtaking. One is based on swapping initial conditions after the overtaking occurs. Second approach is to formulate the problem as a Filippov system with discontinuous right-hand sides.

1. Introduction

A massive traffic is the phenomenon of our civilization. The mathematical modeling of traffic flows has a long tradition, see e.g. [1] for a recent review. We will consider a class of *macroscopic follow-the-leader models*, see e.g. [2]: Consider the system

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = y_i \,, \qquad \frac{\mathrm{d}y_i}{\mathrm{d}t} = V(x_{i+1} - x_i) - y_i \,, \quad x_{N+1} = x_1 + L, \tag{1}$$

$i = 1, \ldots, N$. It models N cars on a circular road of the length L. The pairs (x_i, y_i) are interpreted as the position $x_i \equiv \mathrm{mod}(x_i, L)$ and the velocity y_i of the car number i. The acceleration $\mathrm{d}y_i/\mathrm{d}t$ of each car depends on the difference between the car velocity y_i and the *optimal velocity function* $V = V(x_{i+1} - x_i)$. In particular, we will consider the hyperbolic optimal velocity function $r \mapsto V(r)$ defined as

$$V(r) = V^{\max} \frac{\tanh\left(a(r-1)\right) + \tanh(a)}{1 + \tanh(a)} \,, \tag{2}$$

where $V^{\max}$ and a are positive constants. The choice of V imposes a driving law and we assume that this law is the same for all N drivers. The difference

$$h_i \equiv x_{i+1} - x_i \,, \quad i = 1, \ldots, N \,, \tag{3}$$

is called *headway* (of the i-th car). Note that we can also formulate the model (1) in the state space of headway and velocity components

$$\frac{\mathrm{d}h_i}{\mathrm{d}t} = y_{i+1} - y_i \,, \qquad \frac{\mathrm{d}y_i}{\mathrm{d}t} = V(h_i) - y_i \,, \qquad i = 1, \ldots, N. \tag{4}$$

*The research of the first author was supported by the grant MSM 6046137306 of the Ministry of Education, Youth and Sports, Czech Republic. The second author was supported by the Grant Agency of the Czech Republic (grant No. 201/06/0356) and also by the research project MSM 0021620839 of The Ministry of Education, Youth and Sports, Czech Republic.

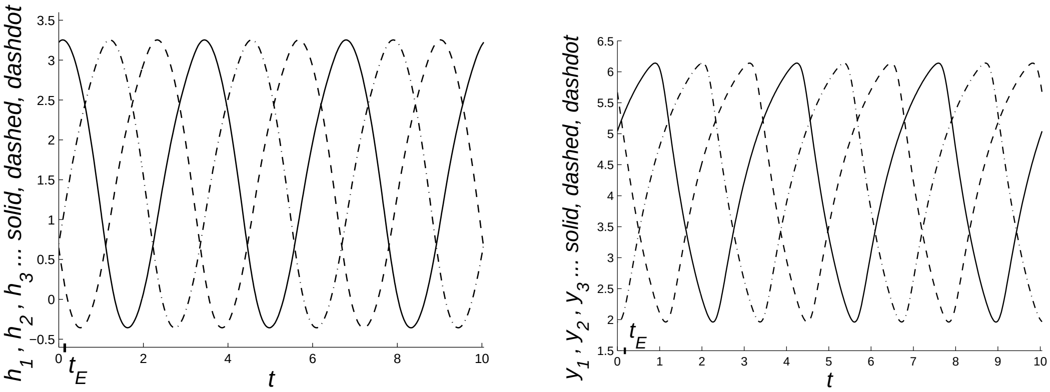


Fig. 1: *Velocity vs. time, headway vs. time: negative headway is non physical.*

Given an initial condition $[x^0, y^0] \in \mathbb{R}^N \times \mathbb{R}^N$, the system (1) defines a flow on $\mathbb{R}^N \times \mathbb{R}^N$

$$[x^0, y^0] \mapsto [x(t), y(t)] \equiv \Psi(t, [x^0, y^0]), \quad t \in \mathbb{R}. \tag{5}$$

Without loss of generality, we may order x^0 as

$$s \leq x_1^0 \leq x_2^0 \leq \cdots \leq x_{N-1}^0 \leq x_N^0 \leq L + s,$$

where $s \in \mathbb{R}$ is an arbitrary phase shift. It is easy to check that there exists a family of *quasi-stationary solutions*, see e.g. [2]. For example, in case $N = 3$ let $x^0 = [s; s + L/3; s + 2L/3]$, $y^0 = [c; c; c]$, $c \equiv V(L/3)$ where $s \in \mathbb{R}$ is an arbitrary phase shift. Then the flow (5) is given by $x(t) = [s + ct; s + L/3 + ct; s + 2L/3 + ct]$, $y(t) = [c; c; c]$ for all t. Therefore, velocity and headway components are constant.

These solutions were observed both stable and unstable. The stability exchange is due to the Hopf bifurcation, see [3]: In certain parameter regions, quasi-stationary solutions co-exist with periodic solutions to (1).

Fig. 1 shows the periodic solution for $N = 3$ cars and the parameter setting $L = 4.56281$, $V^{\max} = 7$, $a = 2$. The periodicity concerns the velocity and headway components. In [4], the authors noted that the solutions to (1) which yield the negative headway are problematic to interprete physically. They called them *non physical solutions*. For example, the trajectory on Fig. 1 becomes non physical since $t_E = 0.2074$. Observe that

$$h_2(t_E) \equiv x_2(t_E) - x_3(t_E) = 0, \quad y_2(t_E) > y_3(t_E). \tag{6}$$

The natural interpretation is that the car No 2 is about to **overtake** the car No 3.

The authors of [4] tried to generalize the model (1) in such a way that the periodic solutions become physical for a larger parameter regions. We will follow a different idea. We are going to simulate the overtaking. The resulting model is a piecewise smooth dynamical system composed by pieces of (1).

2. Overtaking

The idea is as follows: On the left Fig. 2, three consecutive trajectories due to the flow (5) are sketched. The headway of the k-th car, namely $h_k(t) = x_{k+1}(t) - x_k(t)$, becomes negative for $t > t_E$. Note that we can compute the time t_E for which $h_k(t_E) = 0$ within a prescribed precision in MATLAB environment (see `odeset`, `Event location`). We define a new initial condition at $t = t_E$ by naturally swapping $[x_k(t_E), y_k(t_E)]$ and $[x_{k+1}(t_E), y_{k+1}(t_E)]$. The resulting trajectories, see Fig. 2 on the right, have discontinuous first derivatives (the solid and dashed lines). Note that x_k on the right Fig. 2 corresponds to position of the k-th car only for $t \le t_E$. For $t > t_E$, x_k is position of the $(k+1)$-st car. Overtaking algorithm solves the problem in two runs, simulation with swapping of initial conditions and postprocessing to produce final trajectories of cars. In the postprocessing stage, we assemble pieces of the final trajectories on Fig. 3 from lines obtained on Fig. 2. They have continuous derivatives and discontinuous headway components. The velocities are continuous.

Let us illustrate performance of the algorithm. We consider $N = 14$, $V^{\max} = 34$ and $a = 2$; the same data as in [3], Figure 9. The steady state at $L = 15$ is known

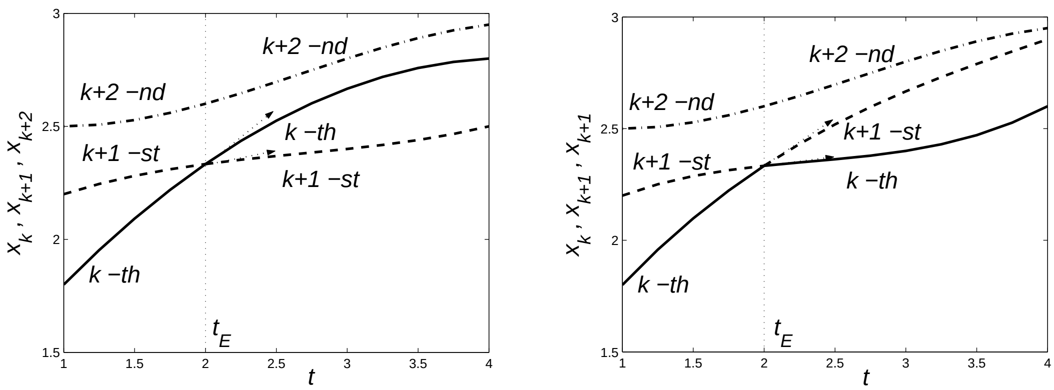


Fig. 2: *On the left: A sketch of three trajectories of the flow (5). On the right: The trajectories after imposing the swap of the initial condition at* $t = t_E$.

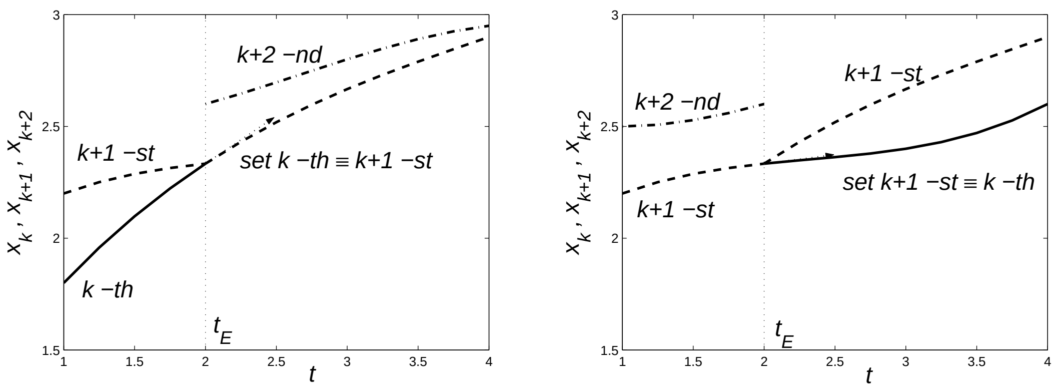


Fig. 3: *On the left: Trajectory of the k-th car. On the right: Trajectory of the* $k+1$*-st car. The relevant headway components are discontinuous at* t_E.

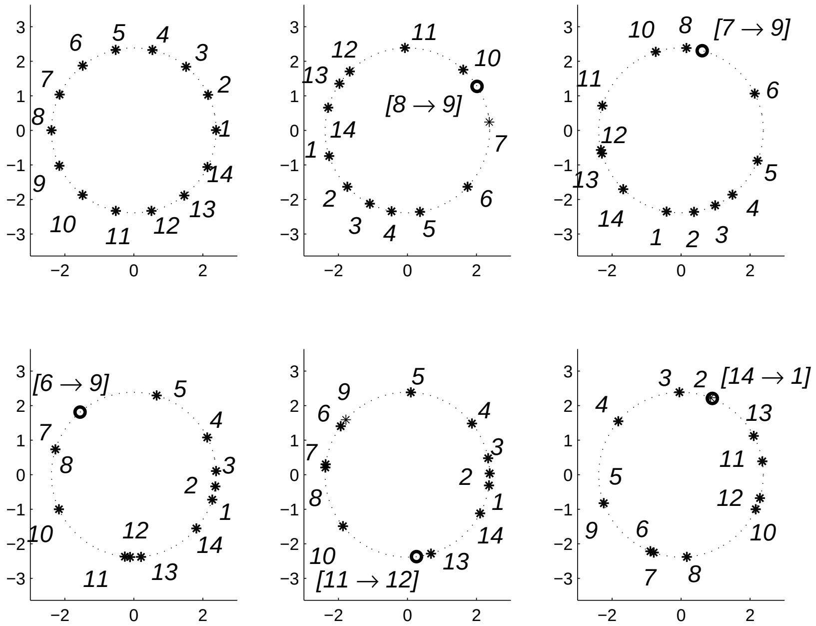


Fig. 4: *A slightly perturbed steady state at* $t = 0$*, top left. Sequence of overtaking (Events): No* 8 *overtakes No* 9*, symbolically* $[8 \to 9]$ *at time* $t_E(1) = 1.9136$*,* $[7 \to 9]$ *at* $t_E(2) = 2.0426$*,* $[6 \to 9]$ *at* $t_E(3) = 2.2294$*,* $[11 \to 12]$ *at* $t_E(4) = 2.2546$*,* $[14 \to 1]$ *at* $t_E(5) = 2.4605$*.*

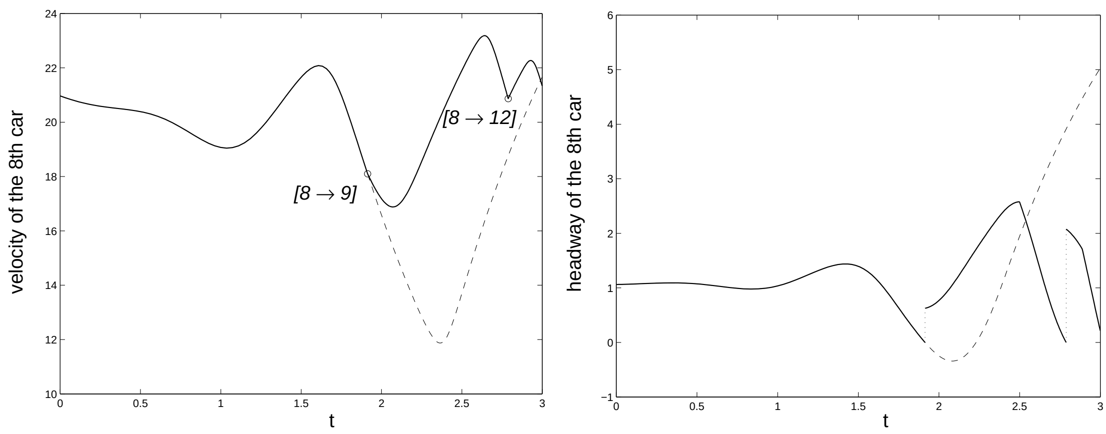


Fig. 5: *Velocity and headway of the* 8*-th car vs time. No* 8 *overtakes No* 9 *and No* 12*. Dashed: The model without overtaking, i.e.* $y_8(t)$*. Since* $t = 1.9136$*, dashed solution becomes non physical.*

to be unstable. Perturbing this steady state slightly, we let the above algorithm work till the time $t = 3$. There were indicated 18 swaps on the track. Five of them are shown on Fig. 4. As an example, we describe the trajectory of the 8-th car for $0 \leq t \leq 3$, see Fig. 5, giving a comparison with the "smooth" model (1).

3. Long time behaviour

Given an initial condition $[x^0, y^0] \in \mathbb{R}^N \times \mathbb{R}^N$ and a time instant $t \geq 0$, let the above algorithm return the actual positions and velocities $[x(t), y(t)] \in \mathbb{R}^N \times \mathbb{R}^N$ of all N cars on the track. We formally define

$$[x^0, y^0] \mapsto [x(t), y(t)] \equiv \Pi(t, [x^0, y^0]), \quad t \geq 0\,. \tag{7}$$

The aim is to investigate asymptotic properties of the overtaking model as $t \to \infty$. We report on invariant objects we observed. For instance, in the case $N = 3$, one can observe phase-shifted reflectionally symmetric oscillations similar to those predicted for a ring of coupled oscillators, see [5], Chapter XVIII, § 4: Let $N = 3$, $V^{\max} = 7$, $a = 2$ and $L = 3.6998$. Let us set $x^0 = [0.1504; 2.6756; 3.5599]$, $y^0 = [4.2668; 5.1647; 2.9087]$. Due to (7), the velocity $y(t)$ is periodic, see Fig. 6. Its period T can be computed numerically. The cars No 1 and No 2 oscillate out-of-phase with the period $T = 4.8525$. The 3-rd car oscillates twice as rapidly as the other two. The corresponding headway components $h(t)$ oscillate similarly, see Fig. 7. Consider $N = 3$ for simplicity. We will show that Overtaking Model, in the state space of headway and velocity components, can be formulated as a *Filippov system*, see [6].

4. Formulation via a Filippov system

Let us consider $N = 3$. In this case we have only two possible configurations of the cars on the road, see Fig. 8. In the first configuration, cars are running ordered "123" along the circuit in the anticlockwise direction, whereas in the second configuration, the cars are ordered "132". It should be noted that the car numbering is fixed during the computation. The configuration of the cars changes when any car overtakes the other one.

Let us define new variables

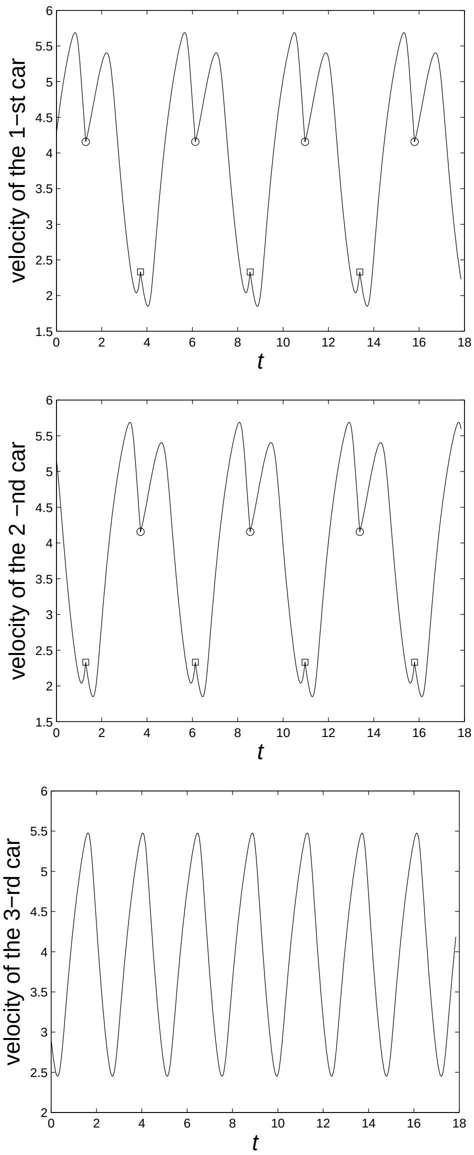
$$h_{ij} = x_j - x_i\,, \quad i \neq j\,, \tag{8}$$

which describe a gap between the car No i and the car No j. It is clear that h_{ji} can be computed from the relation

$$h_{ji} = L - h_{ij}\,, \quad i \neq j\,, \tag{9}$$

which reflects the fact that we consider a closed road. Therefore we can use h_{12}, h_{23} and h_{31} as state variables, only. Remaining gaps h_{13}, h_{21} and h_{32} can be computed from the equation (9).

We will redefine the optimal velocity function as follows. We use the function (2) on the interval $[0, L]$ only, and repeat function values with period L, see Fig. 9 for example. Driving law is independent on whether the car ahead is lap down or lap forward. We denote this new periodic discontinuous optimal velocity function as $\tilde{V}$.



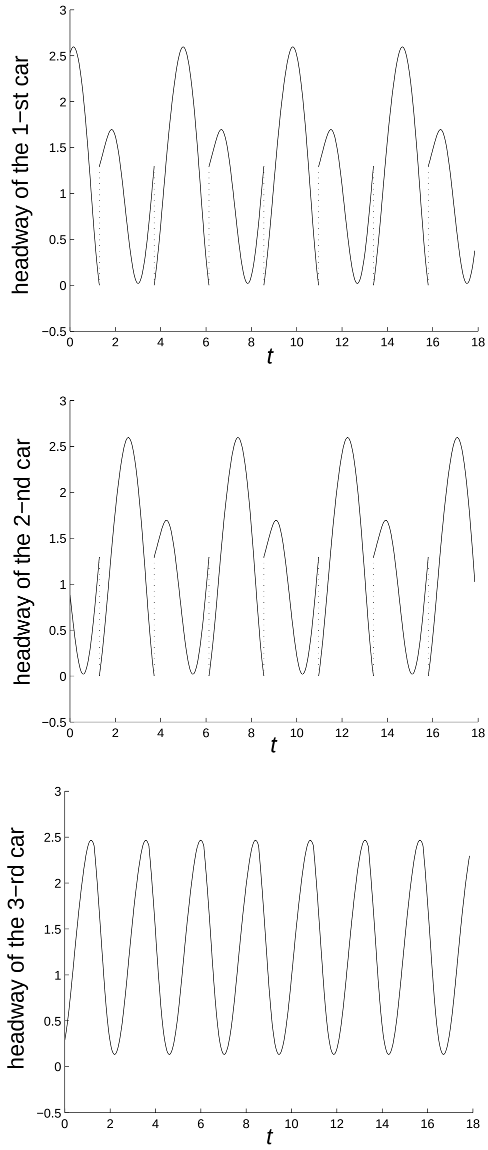


Fig. 6: *The velocity waveforms of period* $T = 4.8363$.

Fig. 7: *Discontinuous headway components.*

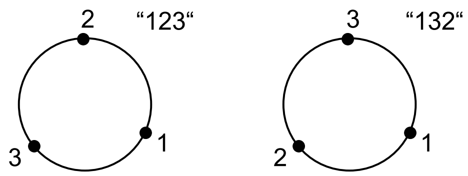


Fig. 8: *Two possible configurations of the cars on the road.*

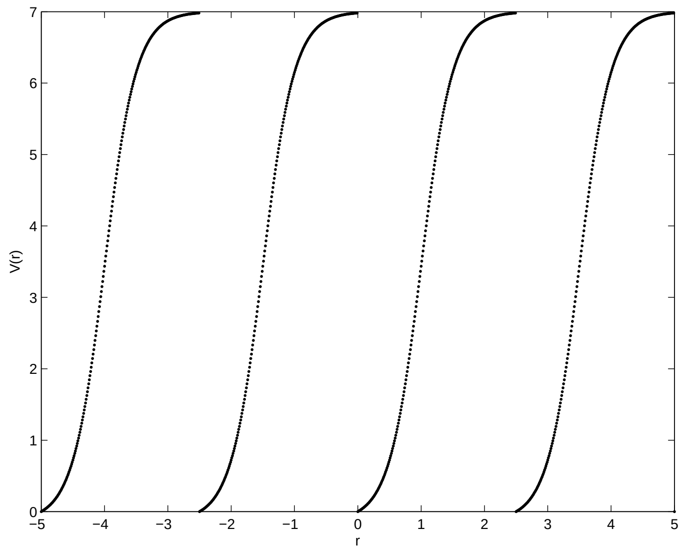


Fig. 9: *Discontinuous optimal velocity function* $\tilde{V}$*;* $V^{\max} = 7$*,* $a = 2$*,* $L = 2.5$*.*

If the system is in the configuration "123" it is described by the following system of differential equations

$$\begin{aligned} \frac{\mathrm{d}h_{12}}{\mathrm{d}t} &= y_2 - y_1\,, \qquad & \frac{\mathrm{d}y_1}{\mathrm{d}t} &= \tilde{V}(h_{12}) - y_1\,, \\ \frac{\mathrm{d}h_{23}}{\mathrm{d}t} &= y_3 - y_2\,, \qquad & \frac{\mathrm{d}y_2}{\mathrm{d}t} &= \tilde{V}(h_{23}) - y_2\,, \\ \frac{\mathrm{d}h_{31}}{\mathrm{d}t} &= y_1 - y_3\,, \qquad & \frac{\mathrm{d}y_3}{\mathrm{d}t} &= \tilde{V}(h_{31}) - y_3\,. \end{aligned} \tag{10}$$

After overtaking occurs, the configuration of the cars changes to "132" and then the system (10) changes to the following one

$$\begin{aligned} \frac{\mathrm{d}h_{12}}{\mathrm{d}t} &= y_2 - y_1\,, \qquad & \frac{\mathrm{d}y_1}{\mathrm{d}t} &= \tilde{V}(h_{13}) - y_1 = \tilde{V}(L - h_{31}) - y_1\,, \\ \frac{\mathrm{d}h_{23}}{\mathrm{d}t} &= y_3 - y_2\,, \qquad & \frac{\mathrm{d}y_2}{\mathrm{d}t} &= \tilde{V}(h_{21}) - y_2 = \tilde{V}(L - h_{12}) - y_1\,, \\ \frac{\mathrm{d}h_{31}}{\mathrm{d}t} &= y_1 - y_3\,, \qquad & \frac{\mathrm{d}y_3}{\mathrm{d}t} &= \tilde{V}(h_{32}) - y_3 = \tilde{V}(L - h_{23}) - y_1\,. \end{aligned} \tag{11}$$

Finally, if

$$h_{ij} = kL\,, \quad k \in \mathbb{Z}\,, \tag{12}$$

for some i, j then the i-th car and the j-th car are involved in overtaking. More precisely, if h_{ij} increases when it crosses the boundary (12), then the i-th is overtaken by the j-th one. On the other hand, if h_{ij} decreases when it crosses the boundary (12), then the i-th car overtakes the j-th one. During the computation, we swap systems (10) and (11) after each overtaking. Since the function $\tilde{V}$ is discontinuous and right hand sides of systems (10) and (11) are different, the system given by equations (10), (11) and (12) is a Filippov system, see [6].

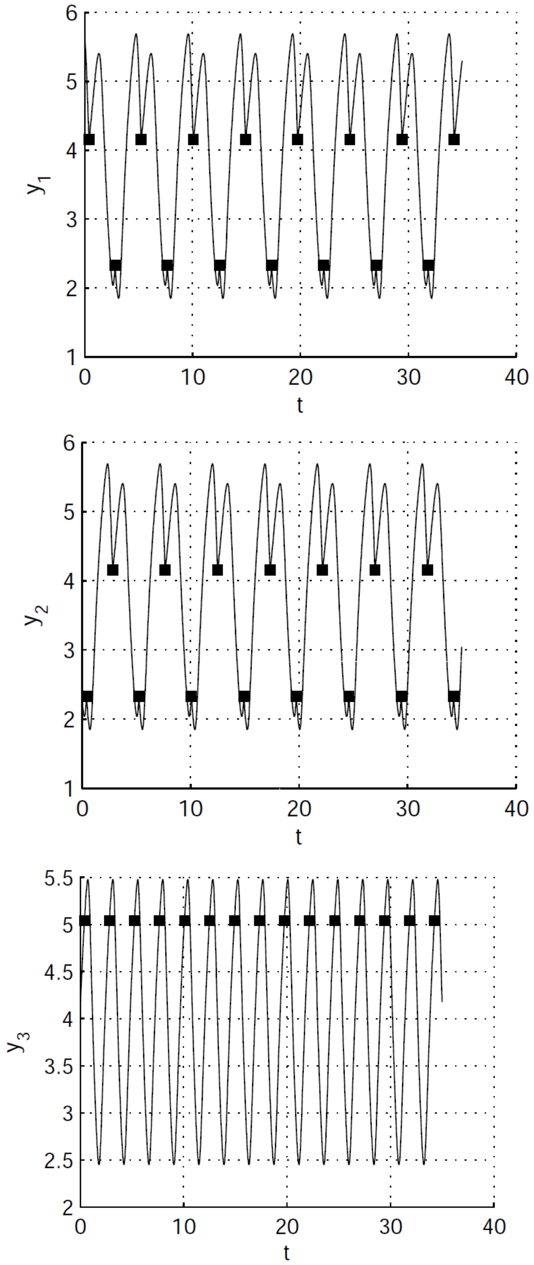


Fig. 10: *The velocity components of the solution of Filippov system* (10),(11),(12).

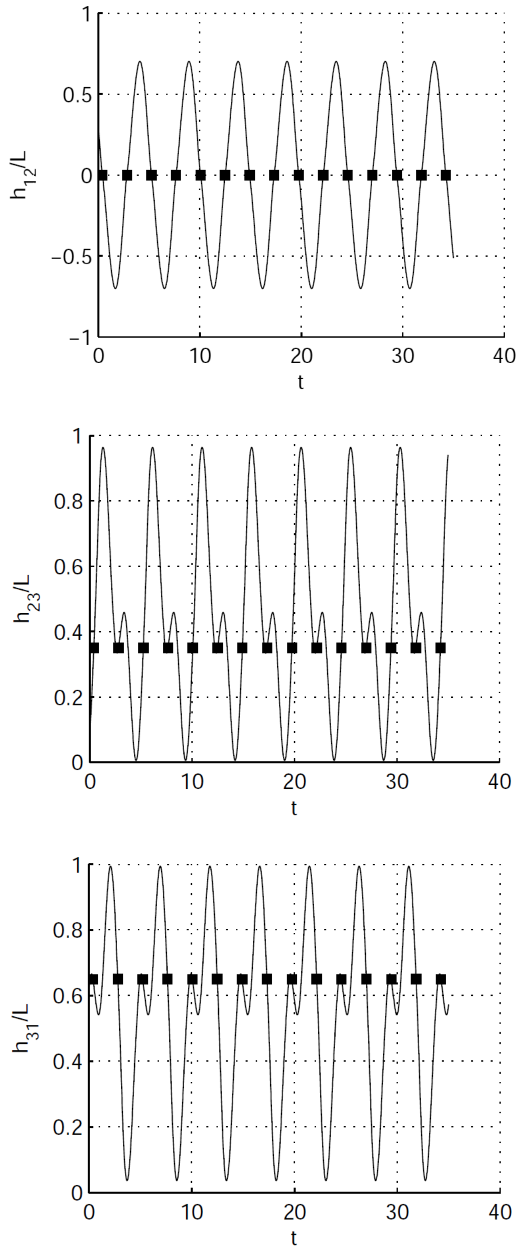


Fig. 11: *The gap components of the solution of Filippov system* (10),(11),(12).

5. Comparison and comments

In this section, we provide a numerical solution of the discontinuous model. The problem was solved in MATLAB by `ode15s` procedure with the event location to detect overtaking. Special attention is paid to the comparison of the results given by the overtaking algorithm described in the section 2 and 3 and results obtained from the discontinuous model. We have fixed values of parameters $V^{\max} = 7$ and $a = 2$. Experiments were started from the "123" configuration.

The numerical solution was obtained for the length of the track $L = 3.6998$, with the initial condition

$$[h_{12}(0), h_{23}(0), h_{31}(0)] = [1.1396, 0.3138, 2.2464]\,, \tag{13a}$$

$$[y_1(0), y_2(0), y_3(0)] = [5.6485, 2.2919, 4.0906]\,. \tag{13b}$$

Results are plotted on the Fig. 10 and 11. On each figure, the overtaking events are marked by the full square.

The velocity components of the solution of both models are similar, compare Fig. 10 and Fig. 6. This shows that both approaches results in the same behaviour of the cars on the track.

Headway components of the solutions are not similar. The model (4) is identical to the "123" configuration of the discontinuous model, i.e. the system (10). Thus, $h_{12} = h_1$, $h_{23} = h_2$ and $h_{31} = h_3$ until no overtaking occurs in the system. After overtaking, since $h_1 = h_{13}$, h_{12} is not equal to h_1 (until next overtaking occurs), etc. Therefore, the solution curves on Fig. 11 correspond to that ones on Fig. 7 only partially. Since the function $h_{12}(t)$ is the only one crossing the boundary represented by the equation (12), the cars No 1 and No 2 overtake each other alternatively and the car No 3 is not involved in any overtaking.

Let us note that gaps $h_{ij}(t)$ are continuous functions, but headway components $h_i(t)$ are discontinuous, see Fig. 11 and 7. The velocities y_i are continuous but they do not have continuous derivatives, see Fig. 10 and 6.

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