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A SECOND ORDER UNCONDITIONALLY POSITIVE SPACE-TIME RESIDUAL DISTRIBUTION METHOD FOR SOLVING COMPRESSIBLE FLOWS ON MOVING MESHES*

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Abstract

A space-time formulation for unsteady inviscid compressible flow computations in 2D moving geometries is presented. The governing equations in Arbitrary Lagrangian-Eulerian formulation (ALE) are discretized on two layers of space-time finite elements connecting levels n, $n+1/2$ and $n+1$. The solution is approximated with linear variation in space (P1 triangle) combined with linear variation in time. The space-time residual from the lower layer of elements is distributed to the nodes at level $n+1/2$ with a limited variant of a positive first order scheme, ensuring monotonicity and second order of accuracy in smooth flow under a time-step restriction for the timestep of the first layer. The space-time residual from the upper layer of the elements is distributed to both levels $n+1/2$ and $n+1$, with a similar scheme, giving monotonicity without any time-step restriction. The two-layer scheme allows a time marching procedure thanks to initial value condition imposed on the first layer of elements. The scheme is positive and second order accurate in space and time for arbitrary meshes and it satisfies the Geometric Conservation Law condition (GCL) by construction.

Example calculations are shown for the Euler equations of inviscid gas dynamics, including the 1D problem of gas compression under a moving piston and transonic flow around an oscillating NACA0012 airfoil.

1. Introduction

Residual Distribution (RD) schemes have reached a certain level of maturity for the simulation of steady flow problems. The RD approach allows to construct second order methods on a compact stencil, which are positive at the same time. They are used as state of the art methods to solve complex steady problems e.g. 3D turbulent Navier-Stokes equations or Magneto-Hydro-Dynamic equations [5, 2, 1].

In [9] it has been noted that for an unsteady computation a mass matrix coupling space and time discretizations has to be taken into the account, otherwise the *spatial* accuracy is lost. This matrix is not a M-matrix, hence if inverted, the positivity of the spatial discretization is compromised.

In [10, 1, 3] an alternative approach for unsteady problems has been proposed, based on space-time RD schemes for a bilinear space-time element approximation. In particular, in [10, 1] a first order scheme corresponding to the N scheme with Crank-Nicholson time integration has been shown to be positive under a time step restriction. This restriction can be overcome by adding one more time layer [3].

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An extension of the conditionally positive, one layer method for moving meshes was presented in [6]. In this paper we extend the two layer method for computations on moving meshes. Because the underlying scheme can be written as the modification of the spatial N scheme with Crank-Nicholson time integration, we use the Arbitrary Lagrangian-Eulerian formulation of the RD method [11].

2. ALE formulation

We define the ALE mapping which for each $t \in I$ associates a point $\vec{Y}$ of reference configuration Ω_0 to a point $\vec{x}$ on the current domain configuration Ω_t, $\mathcal{A}_t : \Omega_0 \subset \mathbb{R}^d \to \Omega_t \subset \mathbb{R}^d$, $\vec{x}(\vec{Y}, t) = \mathcal{A}_t(\vec{Y})$. The ALE mapping $\mathcal{A}_t$ is chosen sufficiently smooth and invertible with nonzero determinant of Jacobian $J_{\mathcal{A}_t}$. A domain velocity $\vec{w}(\vec{x}, t)$ is defined as the time derivative of $\vec{x}$ for constant $\vec{Y}$. We start from the conservative ALE formulation of the Euler equations in d spatial dimensions

$$\frac{1}{J_{\mathcal{A}_t}} \frac{\partial J_{\mathcal{A}_t} \mathbf{u}}{\partial t}\bigg|_{\vec{Y}} + \nabla_x \cdot [\vec{\mathbf{f}}(\mathbf{u}) - \mathbf{u}\vec{w}] = 0\,, \tag{1}$$

where $\mathbf{u} = (\rho, \rho v_i, E)^T$ is the vector of conserved variables and $\vec{\mathbf{f}}(\mathbf{u})$ the well known vector of flux functions. The system is closed with the equation for a perfect gas. The problem is equipped with an appropriate set of initial and boundary conditions. The following equality, called geometrical conservation law, will be used later

$$\nabla_x \cdot \vec{w} = \frac{1}{J_{\mathcal{A}_t}} \frac{\partial J_{\mathcal{A}_t}}{\partial t}\bigg|_{\vec{Y}}. \tag{2}$$

The RD schemes operate on the quasi-linear form of the equation, which can be obtained with $\nabla_x \cdot (\mathbf{u}\vec{w}) = \vec{w} \cdot \nabla_x \mathbf{u} + \mathbf{u}\nabla_x \cdot \vec{w}$ and identity (2)

$$\frac{1}{J_{\mathcal{A}_t}} \frac{\partial J_{\mathcal{A}_t} \mathbf{u}}{\partial t}\bigg|_{\vec{Y}} + \left(\frac{\partial \vec{\mathbf{f}}}{\partial \mathbf{u}} - I\vec{w}\right) \cdot \nabla_x \mathbf{u} - \frac{\mathbf{u}}{J_{\mathcal{A}_t}} \frac{\partial J_{\mathcal{A}_t}}{\partial t}\bigg|_{\vec{Y}} = 0. \tag{3}$$

3. Numerical scheme

The problem is solved on mesh $\mathcal{T}^h$ consisting of simplex elements $\{E\}$. The unknowns are stored in the vertices of the mesh. A straightforward application of the N scheme [6] with Crank-Nicholson time integrator operating between layers n and $n + 1/2$ (i.e. lower layer of the elements) to the problem (3) gives

$$\frac{S_i^{n+1/2} u_i^{n+1/2} - S_i^n u_i^n}{\Delta t^{\text{lower}}} + \sum_{E \in \mathcal{D}_i} \frac{1}{2} \left[\left(k_i^+ (u_i - u_{\text{in}})\right)^{n+1/2} + \left(k_i^+ (u_i - u_{\text{in}})\right)^n \right]^E -$$
$$-\frac{u_i^{n+1/2} + u_i^n}{2} \frac{S_i^{n+1/2} - S_i^n}{\Delta t^{\text{lower}}} = 0,\ k_i = \overline{\left(\frac{\partial \vec{\mathbf{f}}}{\partial \mathbf{u}} - I\vec{w}\right)} \cdot \frac{\vec{n}_i}{d},\ u_{\text{in}} = -\Big(\sum_{i \in E} k_i^+\Big)^{-1} \sum_{j \in E} k_j^- u_j, \tag{4}$$

where S_i is the area of median dual cell around node i, $\mathcal{D}_i$ denote all the elements sharing node i, u_i^n is the solution in node i at time level n, Δt is the time-step, $\vec{n}_i$ is the normal to the face opposite to the node i scaled by its surface and k_i^+ is the positive part of the upwind matrix k_i in the sense of its eigen-decomposition. Note that the Jacobian includes the mesh velocity. The Jacobian and mesh velocity $\vec{w}$ are taken in an averaged state, such that the resulting method is conservative [11, 4]. Note, that the method presented here is different from [11] in the treatment of the source term, what allows us to show the positivity of the scheme for scalar problems under the time-step restriction

$$\Delta t^{\text{lower}} \leq \frac{\mu(E^{n+1/2}) + \mu(E^n)}{k_i^{+,E}(d+1)}, \qquad \forall i, E \in \mathcal{T}^h, \tag{5}$$

where $\mu(E)$ is the volume of element E. This method can be interpreted as a space-time method, distributing *space-time* nodal contribution $\phi_i^{E^{\text{ST}}}$ from the lower layer of the elements (element between levels n and $n+1/2$) to the nodes at level $n+1/2$

$$u_i^{n+1/2,m+1} = u_i^{n+1/2,m} - \alpha_i \sum_{E \in \mathcal{D}_i} \phi_i^{E^{\text{ST}},\text{lower,n+1/2}}, \tag{6}$$

where α_i is relaxation parameter given by the explicit stability constraint.

Although the method is implicit, it suffers from the time-step restriction (5). As a cure, we add a second layer of elements with similar scheme, operating between levels $n+1/2$ and $n+1$. This scheme distributes portions of the space-time residual of the upper layer as follows:

- To the nodes at $n+1$:

$$\begin{aligned} \phi_i^{E^{\text{ST}},\text{upper,n+1}} &= \mu(E_i^{n+1})u_i^{n+1} - \mu(E_i^{n+1/2})u_i^{n+1/2} + \\ +\Delta t^{\text{upper}} \sum_{E \in \mathcal{D}_i} \frac{1}{2}\left(k_i^+(u_i - u_{\text{in}})\right)^{n+1,E} &- \frac{u_i^{n+1} + u_i^{n+1/2}}{2}\left(\mu(E_i^{n+1}) - \mu(E_i^{n+1/2})\right). \end{aligned} \tag{7}$$

- To the nodes at $n+1/2$:

$$\phi_i^{E^{\text{ST}},\text{upper,n+1/2}} = \Delta t^{\text{upper}} \sum_{E \in \mathcal{D}_i} \frac{1}{2}\left(k_i^+(u_i - u_{\text{in}})\right)^{n+1/2,E}. \tag{8}$$

Relaxation procedure (6) has then the form

$$u_i^{n+1/2,m+1} = u_i^{n+1/2,m} - \alpha_i \sum_{E \in \mathcal{D}_i} \left(\phi_i^{E^{\text{ST}},\text{lower,n+1/2}} + \phi_i^{E^{\text{ST}},\text{upper,n+1/2}}\right) \tag{9}$$

$$u_i^{n+1,m+1} = u_i^{n+1,m} - \alpha_i \sum_{E \in \mathcal{D}_i} \phi_i^{E^{\text{ST}},\text{upper,n+1}} \tag{10}$$

and the scheme is formally unconditionally stable with arbitrary Δt^{upper}.

The space-time nodal contribution can be seen as a space-time residual distributed with (implicitly defined) distribution coefficient

$$\phi_i^{E^{\mathrm{ST}}} = \beta_i \phi^{E^{\mathrm{ST}}}, \qquad \sum_{i \in E} \beta_i = 1. \tag{11}$$

The scheme described above is at most first order accurate. As it was proven in [1], a condition for second order of accuracy is the uniform boundedness of the distribution coefficients β_i. One of the possibilities to modify the distribution coefficients is [1]

$$\beta_i^{\mathrm{mod}} = \frac{\beta_i^+}{\sum_{j \in E} \beta_j^+}. \tag{12}$$

This modification preserves the sign of the distribution coefficients and ensures its uniform boundedness, hence the method becomes second order accurate, while keeping its positivity. In the case of the Euler equations the modification of the distribution coefficients is performed on *simple waves* given by the projection of the residual to the Jacobian eigenvectors [1].

4. Numerical results

The first test case is motivated by an internal aerodynamics problem, namely flow in a piston engine. A gas at rest is enclosed between two opposite walls in the chamber. One of the walls slowly starts to move, compressing the gas inside the chamber. This problem can be solved by the method of characteristics [12] until the head of the pressure wave reflects from the end wall or a shock is created[1]. We have used a rectangular domain of size 5×1 with initial conditions $u_0 = 0$, $\rho_0 = 1.4$ and $p_0 = 1$. The piston starts to accelerate with derivative of acceleration $\dddot{x} = 0.2$. The numerical solution is plotted at time $t = 4$, when the piston has reached $x = 2.13\bar{3}$. The mesh consist of 372 nodes and 674 triangular elements with 30 nodes along the cylinder wall and 6 nodes along the end wall. Comparison is made with a finite volume method using a linear least square reconstruction, Barth's limiter, three point backward differentiation scheme on moving meshes [7] (Fig. 1). The solution given by the RD scheme perfectly follows the analytical solution, while the FV scheme gives bigger differences.

The next problem involves a piston instantaneously accelerated to a uniform speed. From the Rankine-Hugoniot jump conditions we can compute the solution analytically.[2] The comparison is shown in Fig. 2 at $t = 2$. Note the perfectly monotone shock capturing. Both the FV and RD schemes give comparable results. Note also the entropy layer in the vicinity of the piston, which is present both for RD and FV methods. Its source has to be still investigated.

[1] The analytical solution is avaliable on an email request. Email: `Jiri.Dobes@fs.cvut.cz`.

[2] Piston velocity is 0.8, flow velocity is $u_L = 0.8$, $u_R = 0$, density is $\rho_L = 2.8191$, $\rho_R = 1.4$ and pressure is $p_L = 2.78$, $p_R = 1$. Shock speed is 0.79461.

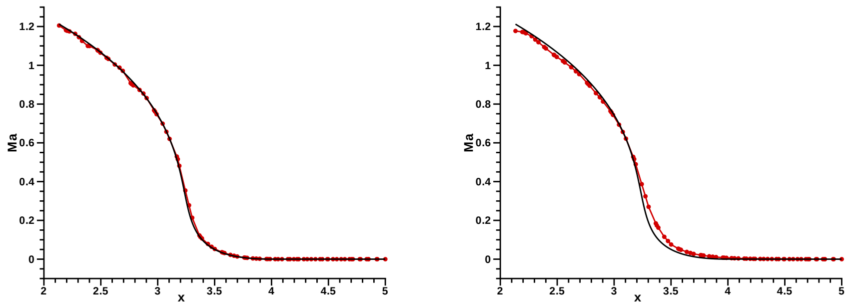

Fig. 1: *Smooth compression of the gas, Mach number cut. Left: present scheme. Right: FV scheme.*

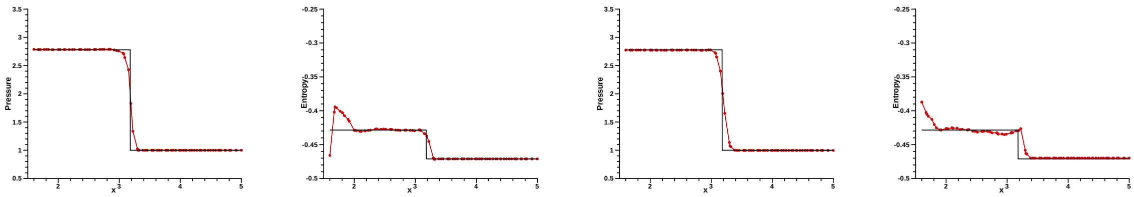

Fig. 2: *Compression of the gas with a shock. Pressure and entropy cut. Left half: present scheme. Right half: FV scheme.*

Finally, a fully 2D test involves a NACA 0012 airfoil which is sinusoidally pitching around its a quarter chord (test case AGARD CT 5[8]). The free stream Mach number is 0.755 and the mean angle of incidence is 0.016° . The airfoil performs a sinusoidal pitching motion with an amplitude of 2.51°

$$\alpha = 2.51 \sin(2kt) + 0.016, \tag{13}$$

where k is the reduced frequency of oscillation with respect to the half chord

$$k = \frac{\omega c}{2u_\infty} = 0.0814, \tag{14}$$

where c is the chord, u_∞ is the free-stream velocity and ω the frequency.

The problem was solved on an unstructured mesh consisting of 5711 nodes and 11153 elements with 206 nodes around the airfoil. The free stream boundary was located 20 chords away from the airfoil. The solution at time $t = 115$ is plotted in Fig. 3. The FV solution is plotted by a dotted line, while the RD solution is plotted as the continuous lines. The FV solution is more dissipative, as one can notice above the profile, where the RD isolines are more crisp and running straight into the shock. Interesting is a comparison of the lift coefficient depending on the angle of incidence. On the zoom, one can notice a higher peak of the lift given by the RD method than by the FV method, which points to the higher accuracy.

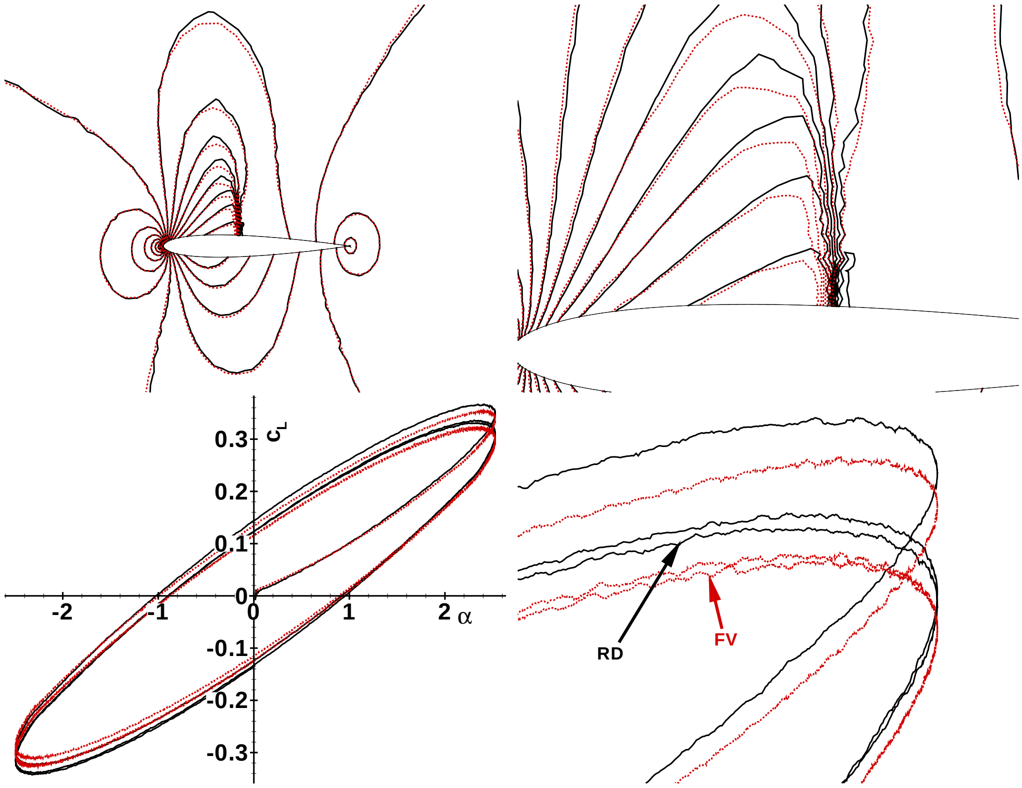

Fig. 3: *Flow past oscillating NACA 0012 airfoil. Top: isolines of the pressure. Bottom: dependence of the lift on the angle. RD method – continuous line, FV method dotted line.* $CFL = 5$.

5. Conclusions

The two layer N-modified space-time multidimensional upwind residual distribution scheme of [1] was extended for computations on moving meshes. The scheme is unconditionally positive and second order accurate on moving meshes. The method was tested on a 1D piston problem (solved in 2D settings), where we have shown excellent agreement with the analytical solution. The method was then applied to the problem of a transonic flow around an oscillating NACA 0012 airfoil, showing the more accurate and less dissipative behavior of RD scheme with respect to the state of the art FV scheme.

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