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ŠINDEL SEQUENCES AND THE PRAGUE HOROLOGE*

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1. Introduction

The mathematical model of the astronomical clock of Prague was developed by the professor of Prague University, Jan Ondřejův, called *Šindel* (see [2]). The clock was realized by Mikuláš from Kadaň around 1410. The ingenuity of clockmakers of that time can be demonstrated by the following construction.

The astronomical clock of Prague contains a large gear with 24 slots at increasing distances along its circumference (see Figure 1). This arrangement allows for a periodic repetition of 1–24 strokes of the bell each day. There is also a small auxiliary gear whose circumference is divided by 6 slots into segments of arc lengths 1, 2, 3, 4, 3, 2 (see Figure 1). These numbers form a period which repeats after each revolution and their sum is $s = 15$. At the beginning of every hour a catch rises, both gears start to revolve and the bell chimes. The gears stop when the catch simultaneously falls back into the slots on both gears. The bell strikes $1 + 2 + \cdots + 24 = 300$ times every day. Since this number is divisible by $s = 15$, the small gear is always at the same position at the beginning of each day.

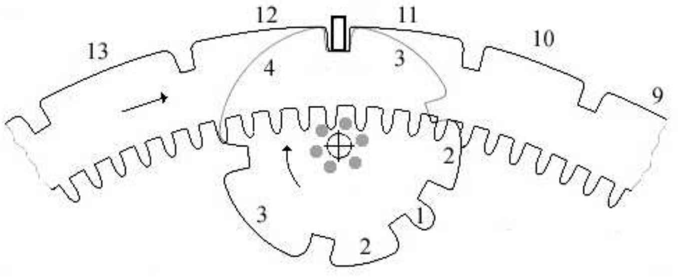

Fig. 1: *The number of bell strokes is denoted by the numbers ..., 9, 10, 11, 12, 13,... along the large gear. The small gear placed behind it is divided by slots into segments of arc lengths 1, 2, 3, 4, 3, 2. The catch is indicated by a small rectangle on the top.*

When the small gear revolves it generates by means of its slots a periodic sequence whose particular sums correspond to the number of strokes of the bell at each hour,

$$1\;2\;3\;4\;\underbrace{3\;2}_{5}\;\underbrace{1\;2\;3}_{6}\;\underbrace{4\;3}_{7}\;\underbrace{2\;1\;2\;3}_{8}\;\underbrace{4\;3\;2}_{9}\;\underbrace{1\;2\;3\;4}_{10}\;\underbrace{3\;2\;1\;2\;3}_{11}\;\underbrace{4\;3\;2\;1\;2}_{12}\ldots \tag{1}$$

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In [4] we showed that we could continue in this way until infinity. However, not all periodic sequences have such a nice summation property. For instance, we immediately find that the period 1, 2, 3, 4, 5, 4, 3, 2 could not be used for such a purpose, since $6 < 4+3$. Also the period 1, 2, 3, 2 could not be used, since $2+1 < 4 < 2+1+2$.

2. Connections with triangular numbers and periodic sequences

In this section we show how the *triangular numbers*

$$T_k = 1 + 2 + \cdots + k = \frac{k(k+1)}{2}, \quad k = 0, 1, 2, \ldots, \tag{2}$$

are related to the astronomical clock. We shall look for all periodic sequences that have a similar property as the sequence 1, 2, 3, 4, 3, 2 in (1), i.e., that could be used in the construction of the small gear. Put $\mathbb{N} = \{1, 2, \ldots\}$.

A sequence $\{a_i\}_{i=1}^{\infty}$ is said to be *periodic,* if there exists $p \in \mathbb{N}$ such that

$$\forall\, i \in \mathbb{N} : \quad a_{i+p} = a_i. \tag{3}$$

The finite sequence $a_1, \ldots, a_p$ is called a *period* and p is called the *period length.* The smallest p satisfying (3) is called the *minimal period length* and the associated sequence $a_1, \ldots, a_p$ is called the *minimal period.*

Definition 1. Let $\{a_i\} \subset \mathbb{N}$ be a periodic sequence. We say that the triangular number T_k for $k \in \mathbb{N}$ is *achievable* by $\{a_i\}$, if there exists a positive integer n such that

$$T_k = \sum_{i=1}^{n} a_i. \tag{4}$$

The periodic sequence $\{a_i\}$ is said to be a *Šindel sequence* if T_k is achievable by $\{a_i\}$ for every $k \in \mathbb{N}$, i.e.,

$$\forall\, k \in \mathbb{N} \quad \exists\, n \in \mathbb{N} : \quad T_k = \sum_{i=1}^{n} a_i. \tag{5}$$

The triangular number T_k on the left-hand side is equal to the sum $1 + \cdots + k$ of hours on the large gear, whereas the sum on the right-hand side expresses the corresponding rotation of the small gear (see Figure 2). For the kth hour, we have

$$k = T_k - T_{k-1} = \sum_{i=m+1}^{n} a_i, \tag{6}$$

where $T_{k-1} = \sum_{i=1}^{m} a_i$. Since $a_i > 0$, the number n depending on k in (5) is unique. From (2) and (4) we also see that $a_1 = 1$ when $\{a_i\}$ is a Šindel sequence.

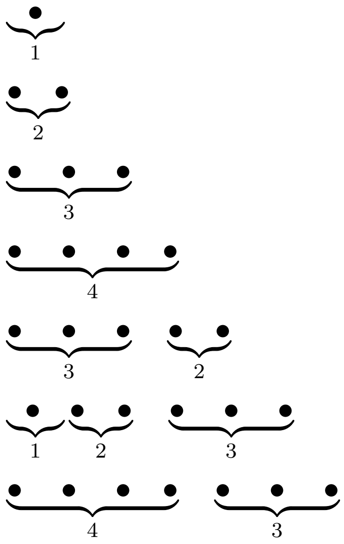

Fig. 2: *The bullets in the kth row indicate the number of strokes at the kth hour (see (6)). The numbers denote lengths of segments on the small gear.*

3. Necessary and sufficient condition for the existence of a Šindel sequence

First we need to define quadratic residues and nonresidues.

Definition 2. Let $n \geq 2$ and a be integers. If the quadratic congruence

$$x^2 \equiv a \pmod{n}$$

has a solution x, then a is called a *quadratic residue modulo* n. Otherwise, a is called a *quadratic nonresidue modulo* n.

Lemma 1. *If* f *and* h *are nonnegative integers, then* $8f+1$ *is a quadratic residue modulo* 2^h.

The proof is a consequence of [5, pp. 105–106]). From now on let

$$s = \sum_{i=1}^{p} a_i \tag{7}$$

denote the sum of the period.

Theorem 1. *A periodic sequence* $\{a_i\}$ *is a Šindel sequence if and only if for any* $n \in \{1, \dots, p\}$ *and any* $j \in \{1, 2, \dots, a_n - 1\}$ *with* $a_n \geq 2$ *the number*

$$w = 8\Big(\sum_{i=1}^{n} a_i - j\Big) + 1$$

is a quadratic nonresidue modulo s.

Proof. $\Longleftarrow$: Let a periodic sequence $\{a_i\}$ not be a Šindel sequence. According to (5), there exist positive integers ℓ, m, and j such that $a_m \geq 2$, $j \leq a_m - 1$, and

$$T_\ell = \sum_{i=1}^{m} a_i - j. \tag{8}$$

Let $n \in \{1, \dots, p\}$ be such that $n \equiv m \pmod{p}$. Then by (2), (8), (7), and (3),

$$(2\ell+1)^2 = 4\ell^2+4\ell+1 = 8T_\ell+1 = 8\Big(\sum_{i=1}^{m} a_i - j\Big)+1 \equiv 8\Big(\sum_{i=1}^{n} a_i - j\Big)+1 \pmod{s},$$

i.e., $8\big(\sum_{i=1}^{n} a_i - j\big) + 1$ is a square modulo s.

$\Longrightarrow$: Let $\{a_i\}$ be a Šindel sequence with $s = 2^c d$, where $c \geq 0$ and d is odd. Suppose to the contrary that there exist positive integers n, j, and x such that $n \leq p$, $a_n \geq 2$, $j \leq a_n - 1$, $x \leq s$, and

$$w = 8\Big(\sum_{i=1}^{n} a_i - j\Big) + 1 \equiv x^2 \pmod{s}. \tag{9}$$

From Lemma 1 and (9) there exists y such that

$$\begin{aligned} x^2 &\equiv w \pmod{d}, \\ y^2 &\equiv w \pmod{2^{c+3}}. \end{aligned} \tag{10}$$

By the Chinese remainder theorem (see [3, p. 15]) there exists an integer $u \geq 3$ such that $u \equiv x \pmod{d}$ and $u \equiv y \pmod{2^{c+3}}$. Thus, by (10),

$$\begin{aligned} u^2 &\equiv x^2 \equiv w \pmod{d}, \\ u^2 &\equiv y^2 \equiv w \pmod{2^{c+3}}. \end{aligned}$$

Since $\gcd(d, 2^{c+3}) = 1$, we see that

$$u^2 \equiv w \pmod{2^{c+3}d}. \tag{11}$$

Clearly, u is odd, since w is odd. So let $u = 2\ell + 1$, where $\ell \geq 1$. Then, by (11), $u^2 = 4\ell^2 + 4\ell + 1 = w + 2^{c+3}dg$ for some integer g. Hence, since $u \geq 3$, we find by (2), (11), and (9) that

$$T_\ell = \frac{u^2 - 1}{8} = \frac{w-1}{8} + 2^c dg \equiv \sum_{i=1}^{n} a_i - j \pmod{s}.$$

Thus, there exists a positive integer m such that $m \equiv n \pmod{p}$ and

$$T_\ell = \sum_{i=1}^{m} a_i - j,$$

which contradicts the assumption that $\{a_i\}$ is a Šindel sequence. □

As a byproduct of the proof of Theorem 1, we get the well-known result (see also [1, p. 15] and Figure 3):

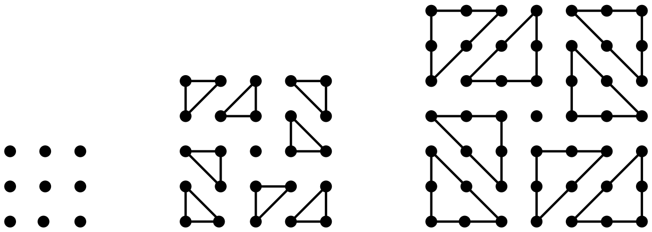

Fig. 3: *The early Pythagoreans knew that if r is a triangular number, then $8r + 1$ is a square. This result is mentioned as early as about 100 A.D. in Platonic Questions by the Greek historian Plutarch, see [6, p. 4].*

Corollary 1. *A positive integer r is a triangular number if and only if $8r + 1$ is a square.*

Remark 1. In Theorem 1, we require that

$$w = 8\Big(\sum_{i=1}^{n} a_i - j\Big) + 1$$

be a quadratic nonresidue modulo s for various values of n and j when $\{a_i\}$ is a Šindel sequence. A sufficient condition for this to occur is that w be a quadratic nonresidue for some odd prime q dividing s. To see that this condition is not necessary, consider the periodic sequence $\{a_i\}$ given in Example 2 below with $p = 11$, $s = 25$, and the period 1, 2, 2, 1, 4, 1, 4, 1, 4, 1, 4. Then

$$8\Big(\sum_{i=1}^{5} a_i - 2\Big) + 1 = 65,$$

which is a quadratic nonresidue modulo 25, but is a quadratic residue modulo 5. Note that 5 is the only odd prime dividing $s = 25$.

Remark 2. Consider the sequence $\{a_i\}$ with period $1, 2, 1, 1, 1, \ldots, 1$. Note that

$$w = 8\Big(\sum_{i=1}^{2} a_i - 1\Big) + 1 = 17.$$

By Theorem 1 and the law of quadratic reciprocity one sees that (cf. [3, pp. 23–25]) if s is an odd prime and $s \equiv 1, 2, 4, 8, 9, 13, 15$ or $16 \pmod{17}$, then w is a quadratic residue modulo s and thus, $\{a_i\}$ is not a Šindel sequence. Other patterns of the period of periodic sequences $\{a_i\}$ can be similarly investigated.

4. Construction of the primitive Šindel sequence

Definition 3. A Šindel sequence $\{a'_i\}$ with the minimal period length $p+1$ is said to be *composite,* if there exists a Šindel sequence $\{a_i\}$ and $\ell \in \mathbb{N}$ such that

$$\begin{aligned} &a_i = a'_i, \quad i = 1, \dots, \ell - 1, \\ &a_\ell = a'_\ell + a'_{\ell+1}, \\ &a_i = a'_{i+1}, \quad i = \ell + 1, \dots, p. \end{aligned}$$

The period 1, 2, 3, 2, 2, 3, 2 derived from the period 1, 2, 3, 4, 3, 2 of sequence (1) produces a composite Šindel sequence. In other words, the astronomical clock would also work with the small gear corresponding to this composite Šindel sequence.

Definition 4. A Šindel sequence $\{a_i\}$ is called *primitive* if it is not composite. The sequence 1, 1, 1, ... is called a *trivial* Šindel sequence.

The proof of the next theorem contains an explicit algorithm for finding a primitive Šindel sequence for a given s.

Theorem 2. *Let s be a positive integer. Then there exists a unique primitive Šindel sequence $\{a_i\}$ such that* (7) *holds for one of its not necessarily minimal period lengths p. The primitive Šindel sequence $\{a_i\}$ is trivial if and only if $s = 2^h$ for $h \geq 0$.*

Proof. Let $1 \leq b_1 < b_2 < \cdots < b_t \leq s$ be all the integers such that each $8b_n + 1$ is a square modulo s for $n = 1, \dots, t$. We observe that $b_1 = 1$ and $b_t = s$. Now choose the period as follows: $a_1 = b_1$ and $a_n = b_n - b_{n-1}$ for $n = 2, 3, \dots, t$. Then

$$\forall n \in \{1, 2, \dots, t\} : \quad b_n = \sum_{i=1}^{n} a_i.$$

We claim that $\{a_i\}$ is a Šindel sequence. Note that if $n \in \{1, \dots, t\}$, $a_n \geq 2$, and $j \in \{1, 2, \dots, a_n - 1\}$, then $b_{n-1} < \sum_{i=1}^{n} a_i - j < b_n$. Then $8(\sum_{i=1}^{n} a_i - j) + 1$ is a quadratic nonresidue modulo s, since $8b_1 + 1, \dots, 8b_t + 1$ are all the quadratic residues modulo s. It now follows from Theorem 1 that $\{a_i\}$ is a Šindel sequence.

Moreover, one sees that $\{a_i\}$ is a primitive Šindel sequence having a period length $p = t$ and satisfying (7). It is also clear by construction that $\{a_i\}$ is the unique primitive Šindel sequence satisfying (7) for some period length p.

$\Longleftarrow$: By the above construction of the period, the primitive Šindel sequence corresponding to s is nontrivial if and only if there exists a positive integer $f \leq s$ such that $8f + 1$ is a quadratic nonresidue modulo s. By Lemma 1, $8f + 1$ is always a quadratic residue modulo $s = 2^h$ for $h \geq 0$. Hence, the primitive Šindel sequence corresponding to $s = 2^h$ is the trivial Šindel sequence.

$\Longrightarrow$: Conversely, assume that s has an odd prime divisor q. Let d be a quadratic nonresidue modulo q. Since 8 is invertible modulo q, one sees that if z is the inverse of 8 modulo q and $f \equiv z(d-1) \pmod q$, then $8f+1 \equiv d \pmod q$. It now follows that the primitive Šindel sequence corresponding to s is nontrivial. $\square$

We have the following immediate corollaries to Theorems 2 and 1:

Corollary 2. *Let $\{a_i\}$ be a periodic sequence with the minimal length p of the period and $s = 2^m$, where m is a nonnegative integer. Then $\{a_i\}$ is a Šindel sequence if and only if $\{a_i\}$ is the trivial Šindel sequence.*

Corollary 3. *A periodic sequence $\{a_i\}$ is a primitive Šindel sequence if and only if for any $n \in \{1, \dots, p\}$ and any $j \in \{1, 2, \dots, a_n - 1\}$ with $a_n \geq 2$ the number*

$$w = 8\Big(\sum_{i=1}^{n} a_i - j\Big) + 1$$

is a quadratic nonresidue modulo s and

$$v = 8\sum_{i=1}^{n} a_i + 1$$

is a quadratic residue modulo s.

Theorem 3. *For any $k \in \mathbb{N}$ there exist $\ell \in \mathbb{N}$ and a Šindel sequence $\{a_i\}$ such that $a_\ell = k$.*

Proof. It was stated in Corollary 1 that for $r \in \mathbb{N}$, $8r+1$ is a square if and only if r is a triangular number. Let $k = T_k - T_{k-1}$ be given (see (6)). Thus it suffices by the proof of Theorem 2 to find a positive integer $s \geq T_k$ such that $8(T_{k-1}+j)+1$ is a quadratic nonresidue modulo s for $j = 1, 2, \dots, k-1$.

For a fixed $j \in \{1, \dots, k-1\}$ let

$$8(T_{k-1}+j)+1 = \prod_{i=1}^{v} p_i^{\alpha_i}$$

be the prime power factorization. Since $8(T_{k-1}+j)+1$ is not a square, some α_i is odd. Without loss of generality, we can assume that α_1 is odd. Let c_1 be a quadratic nonresidue modulo p_1. By the Chinese remainder theorem and Dirichlet's theorem on the infinitude of primes in arithmetic progressions, one can find a prime $q_j \geq T_k$ such that $q_j \equiv 1 \pmod 4$, $q_j = c_1 \pmod{p_1}$, and $q_j \equiv 1 \pmod{p_i}$ for $i \in \{2, \dots, v\}$. By the law of quadratic reciprocity and the properties of the Jacobi symbol (see [3, p. 24–25]), $8(T_{k-1}+j)+1$ is a quadratic nonresidue modulo q_j. Now simply let s be the product of the distinct q_j's for $j \in \{1, \dots, k-1\}$. $\square$.

5. Numerical examples

We developed a program that generates the primitive Šindel sequence for a given s. It is based on the numerical algorithm presented in the proof of Theorem 2. By this theorem we know that the primitive primitive Šindel sequence is uniquely determined for each positive integer s.

s	Primitive Šindel sequences
1	1
2	1 1
3	1 2
4	1 1 1 1
5	1 2 2
6	1 2 1 2
7	1 2 3 1
8	1 1 1 1 1 1 1 1
9	1 2 3 3
10	1 2 2 1 2 2
11	1 2 1 2 4 1
12	1 2 1 2 1 2 1 2
13	1 1 1 3 2 2 3
14	1 2 3 1 1 2 3 1
15	1 2 3 4 3 2
16	1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
17	1 1 1 1 2 4 1 4 2
18	1 2 3 3 1 2 3 3
19	1 1 1 3 1 2 1 5 2 2
20	1 2 2 1 2 2 1 2 2 1 2 2
21	1 2 3 1 3 3 2 6
22	1 2 1 2 4 1 1 2 1 2 4 1
23	1 2 2 1 3 1 3 2 5 1 1 1
24	1 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2
25	1 2 2 1 4 1 4 1 4 1 4

Example 1. The period 1, 2, 3, 4, 5, 3, 3, 7, 2, 3, 3, 9 with minimal period length $p = 12$ and $s = 45$ yields a primitive Šindel sequence $\{a_i\}$ with a large value of $a_{12} = 9$ relative to s (see Theorem 3).

Example 2. The next table shows values of all primitive Šindel sequences for $s = 1, \ldots, 25$. Anyway, we verified that no primitive Šindel sequence up to $s = 1000$ has such a nice symmetry property as that in (1). From the table we also observe that trivial primitive Šindel sequences appear when $s = 2^h$ for some $h \geq 0$ (see Theorem 2).

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