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NUMERICAL INTEGRATION IN THE DISCONTINUOUS GALERKIN METHOD FOR ELLIPTIC PROBLEMS*

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1. Introduction

The use of numerical integration is considered as one of *variational crimes* often committed in practical applications of the finite element method. In the theoretical study of the Discontinuous Galerkin method exact integration is almost exclusively considered. We refer to one of exceptions, [5], where the effect of numerical integration applied to the evaluation of nonlinear convective terms is studied while the diffusion term is set in such a way that application of appropriate quadrature formulae yields exact integration.

The aim of this paper is to study various aspects of the use of numerical integration for the evaluation of integrals appearing in Discontinuous Galerkin formulations of a linear elliptic (diffusion) problem. Our aim is to obtain sufficient conditions on quadrature formulae which ensure that there exists a unique solution of the corresponding discrete problem. Moreover, we shall study how the use of numerical integration impacts error estimates.

Let us consider simple model problem

$$
\begin{aligned}
-\nabla \cdot (A(x)\nabla u) &= f \quad \text{in } \Omega, &(1)\\
u &= g_D \quad \text{on } \Gamma_D, &(2)\\
(A(x)\nabla u)\cdot \boldsymbol{n} &= g_N \quad \text{on } \Gamma_N. &(3)
\end{aligned}
$$

We assume that $\Omega \subset \mathbb{R}^2$ is a bounded polygonal domain with a Lipschitz-continuous boundary $\partial\Omega$ divided into two disjoint parts Γ_D and Γ_N such that $\partial\Omega = \overline{\Gamma}_D \cup \overline{\Gamma}_N$, where $\mathrm{meas}_1(\Gamma_D) \neq 0$.

We assume that functions f, g_D and g_N are sufficiently regular. Further, let there exists a constant $K > 0$ such that the matrix $A \in [W^{1,\infty}(\Omega)]^{2\times 2}$ satisfies

$$\boldsymbol{\xi}^T A(x)\boldsymbol{\xi} \geq K\ \boldsymbol{\xi}^T \cdot \boldsymbol{\xi} \quad \forall\ \boldsymbol{\xi} \in \mathbb{R}^2,\ \text{a. e. on } \Omega. \tag{4}$$

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2. Discontinuous Galerkin formulation

Let $\mathcal{T}_h$ be a conforming triangulation of $\overline{\Omega}$. We shall denote individual triangles of $\mathcal{T}_h$ by T and put $h_T = \mathrm{diam}(T)$. For the theoretical study it is convenient to consider that a family of triangulations $\{\mathcal{T}_h\}_{h>0}$ of a domain Ω is *regular*, see [4].

Let $\mathcal{E}_h$ stand for the set of all *edges* of $\mathcal{T}_h$. These edges represent the interfaces between pairs of adjacent elements, or sides of triangles lying on the boundary of the domain Ω. Let us distinguish sets of *internal edges* ($\mathcal{E}_h^I$), *Dirichlet edges* ($\mathcal{E}_h^D$) and *Neumann edges* ($\mathcal{E}_h^N$). The length of the edge $S \in \mathcal{E}_h$ will be denoted by $|S|$.

Let us define the space $V_h = \{v \in L^2(\Omega)\ ;\ v|_T \in P_p(T)\ \forall T \in \mathcal{T}_h\}$, where $P_p(T)$ is the space of polynomials of degree at most $p \geq 1$ on T.

For $S \in \mathcal{E}_h^I$ let us denote by T_1 and T_2 the two triangles sharing the edge S. Then we define the *average* on the side S by $\{u\} = \frac{1}{2}((u|_{T_1})|_S + (u|_{T_2})|_S)$ and $\{u\} = u|_S$ for $S \in \mathcal{E}_h^D$. The *jump* on $S \in \mathcal{E}_h^I$ is defined by $[\![u]\!] = (u|_{T_1})|_S - (u|_{T_2})|_S$ and again $[\![u]\!] = u|_S$ for $S \in \mathcal{E}_h^D$. Orientation of the vector $\boldsymbol{n}$ is in accord with the orientation of the *jump*.

For the Discontinuous Galerkin formulation let us introduce bilinear forms $a^+, a^- : V_h \times V_h \to \mathbb{R}$,

$$\begin{aligned} a^{\pm}(u,v) = \sum_{T\in\mathcal{T}_h} \int_T (A\nabla u)\cdot\nabla v\,dx - \sum_{S\in\mathcal{E}_h^I\cup\mathcal{E}_h^D} \int_S \{A\nabla u\}\cdot\boldsymbol{n}[\![v]\!]\,ds \\ \pm \sum_{S\in\mathcal{E}_h^I\cup\mathcal{E}_h^D} \int_S \{A\nabla v\}\cdot\boldsymbol{n}[\![u]\!]\,ds + \sum_{S\in\mathcal{E}_h^I\cup\mathcal{E}_h^D} \frac{\sigma_S}{|S|}\int_S [\![u]\!][\![v]\!]\,ds \end{aligned} \tag{5}$$

and linear functionals $L^+, L^- : V_h \to \mathbb{R}$ by

$$L^{\pm}(v) = \int_\Omega f v\,dx + \sum_{S\in\mathcal{E}_h^N}\int_S g_N v\,ds + \sum_{S\in\mathcal{E}_h^D}\int_S g_D\left[\frac{\sigma_S}{|S|}v \pm (A\nabla v)\cdot\boldsymbol{n}\right]ds, \tag{6}$$

where $\sigma_S \in \mathbb{R}$, $S \in \mathcal{E}_h^I \cup \mathcal{E}_h^D$, is a chosen penalty parameter. The bilinear form $a^+(\cdot,\cdot)$ introduces the *Nonsymmetric Interior Penalty Galerkin (NIPG)* variant (cf. [8]) while the bilinear form $a^-(\cdot,\cdot)$ is *symmetric* for symmetric matrix A. Therefore, we shall speak of the *Symmetric Interior Penalty Galerkin (SIPG)* variant (cf. [1]). Our discrete Discontinuous Galerkin formulation then becomes:

$$\text{find}\quad u_h \in V_h \quad\text{such that}\qquad a^{\pm}(u_h, v) = L^{\pm}(v)\qquad \forall v \in V_h. \tag{7}$$

It is well-known that there exists a unique solution of (7) if certain properties of penalty parameters are satisfied, see, e. g., [2]. Moreover, if the weak solution u of (1)–(3) satisfies $u \in H^{p+1}(\Omega)$, we are able to show that

$$|\!|\!| u-u_h |\!|\!|^2 := \sum_{T\in\mathcal{T}_h} |u-u_h|^2_{1,2,T} + \sum_{S\in\mathcal{E}_h^I\cup\mathcal{E}_h^D} \frac{1}{|S|}\|[\![u-u_h]\!]\|^2_{0,2,T} \leq C \sum_{T\in\mathcal{T}_h} h_T^{2p}|u|^2_{p+1,2,T} \tag{8}$$

with the constant $C > 0$ independent of u and h.

3. Problem with numerical integration

The core of this paper is to explain what happens if all the terms in (5) and (6) are evaluated with the aid of appropriately chosen quadrature formulae. For $\varphi \in C^0(T)$, $T \in \mathcal{T}_h$ and $\psi \in C^0(S)$, $S \in \mathcal{E}_h$, we use approximations

$$\int_T \varphi(x)\, dx \approx \sum_{\alpha=1}^{n_T} \omega_\alpha^T \varphi(x_\alpha^T), \qquad \int_S \psi\, ds \approx \sum_{\alpha=1}^{n_S} \nu_\alpha^S \psi(x_\alpha^S), \tag{9}$$

where $\omega_\alpha^T, \nu_\alpha^S > 0$ are integration weights and $x_\alpha^T \in T, x_\alpha^S \in S$ are integration points. Let us denote by $a_h^\pm(\cdot,\cdot)$ the result of application of numerical integration to the bilinear form $a^\pm(\cdot,\cdot)$ and similarly for the right-hand side. Related problem

$$\text{find} \quad \tilde{u}_h \in V_h \quad \text{such that} \qquad a_h^\pm(\tilde{u}_h, v) = L_h^\pm(v) \qquad \forall v \in V_h \tag{10}$$

makes sense assuming that all the integrands have their point values well-defined which requires higher regularity of data. The most important step in the verification of assumptions of the Lax–Milgram lemma is the proof of uniform V_h-ellipticity.

Lemma 1 *Let the quadrature formula for the integration of the first term of (5) be exact for polynomials from $P_{2p-2}(T)$ and/or let the set of quadrature points $\{x_\alpha^T\}_{\alpha=1}^{n_T}$ contain a $P_{p-1}(T)$-unisolvent subset. Let us assume that the quadrature formula for the penalty term is exact for polynomials of degree $\le 2p$ and/or let the set of quadrature points $\{x_\alpha^S\}_{\alpha=1}^{n_S}$ contain a $P_p(S)$-unisolvent subset. If penalty parameters σ_S are sufficiently large, there exists a constant $\hat{c} > 0$ independent of h such that*

$$\hat{c}|\!|\!| v |\!|\!|^2 \le a_h^\pm(v,v) \qquad \text{for all } v \in V_h.$$

Proof: According to Theorem 4.1.2 in [4] there exists a constant $c_1 > 0$ such that

$$K|v|_{1,2,T}^2 \le Kc_1 \sum\nolimits_{\alpha=1}^{n_T} \omega_\alpha^T \sum\nolimits_{i=1}^{2} |\partial_i v(x_\alpha^T)|^2 \le c_1 \sum\nolimits_{\alpha=1}^{n_T} \omega_\alpha^T \sum\nolimits_{i,j=1}^{2} (a_{ij}\partial_j v \partial_i v)(x_\alpha^T).$$

Similar technique is used to show that if the set of quadrature points $\{x_\alpha^S\}_{\alpha=1}^{n_S}$ contains a $P_p(\hat{S})$-unisolvent subset then $\|[\![v]\!]\|_{0,2,S}^2 \le c_2 \sum_{\alpha=1}^{n_S} \nu_\alpha^S [\![v(x_\alpha^S)]\!]^2$ with some $c_2 > 0$. For the *NIPG* variant the proof is finished, because other terms disappear if the same quadrature formula is used for their evaluation. The requirement $\sigma_S > 0$ is necessary. In the case of the *SIPG* formulation we take into account the inequality

$$\sum\nolimits_{\alpha=1}^{n_S} \nu_\alpha^S [\{A\nabla v\} \cdot \boldsymbol{n}[\![v]\!]](x_\alpha^S) \;\; \le \;\; c_3|S|^{-1/2}\|[\![v]\!]\|_{0,2,S} \sum\nolimits_{T:S\subset\partial T} |v|_{1,2,T},$$

where $c_3 > 0$ depends on p, shape regularity, properties of weights of quadrature formulae and properties of the matrix A. By the Young's inequality we find that

$$a_h^-(v,v) \;\; \ge \;\; \sum_{T\in\mathcal{T}_h} K\left(\frac{1}{c_1} - \frac{1}{\delta}\right)|v|_{1,2,T}^2 + \sum_{S\in\mathcal{E}_h^I\cup\mathcal{E}_h^D} \frac{1}{|S|}\|[\![v]\!]\|_{0,2,S}^2 \left(\frac{\sigma_S}{c_2} - \frac{6\delta c_3^2}{K}\right).$$

For $\delta > c_1$ and $\sigma_S > 6\delta c_2 c_3^2/K$ round brackets are positive. □

Since $a_h^{\pm}(\cdot,\cdot)$ is a continuous bilinear form and $L_h^{\pm}(\cdot)$ is a continuous linear functional on the space V_h we find by the Lax–Milgram lemma that:

Theorem 2 *There exists a unique solution of discrete problem (10).*

4. Errors of quadrature formulae

The next step is to express the error induced by the use of numerical integration. We shall denote E_T and E_S error functionals of numerical integration in a similar way as in [4, 5], i. e., $E_T(\varphi) = \int_T \varphi\, dx - \sum_{\alpha=1}^{n_T} \omega_\alpha^T \varphi(x_\alpha^T)$, etc.

Lemma 3 *Let $u, v \in P_p(T)$, $a \in W^{l+1,\infty}(T)$ and $S \subset \partial T$. Let the quadrature formula on the triangle be exact for polynomials of degree $\leq p+l-2$ and let the (edge) quadrature formula be exact for polynomials of degree $\leq p+l-1$. Then there exists a constant $C > 0$ independent of h such that*

$$|E_T(a\partial_j u \partial_i v)| \leq C h_T^l \|a\|_{l,\infty,T} \|\partial_j u\|_{p-1,2,T} \|\partial_i v\|_{0,2,T}, \quad 1 \leq i, j \leq 2, \tag{11}$$

$$\left|E_S(a\partial_j uv)\right| \leq C h_T^l \|a\|_{l,\infty,S} \|\partial_j u\|_{p-1,2,T} |S|^{-1/2} \|v\|_{0,2,S}, \quad 1 \leq i, j \leq 2, \tag{12}$$

$$|E_S(a\partial_j vu)| \leq C h_T^l \|a\|_{l+1,\infty,T} \|\partial_j v\|_{0,2,T} \|u\|_{p,2,T}, \quad 1 \leq i, j \leq 2. \tag{13}$$

Proof: Estimate (11) follows as in [4]. Other two terms are also estimated with the aid of suitable transformation to the reference edge, the Bramble-Hilbert lemma (cf. [4]) and also the estimate

$$|v|_{j,r,S} \leq c|S|^{1/r} |T|^{-1/s} |v|_{j,s,T}, \qquad 1 \leq r, s \leq +\infty \tag{14}$$

for all $v \in P_p(T)$, $S \subset \partial T$ and $j \leq p$, see proof of Lemma 1 in [7]. □

Let us now move our attention to the error arising from the integration of terms on the right-hand side. Let us focus on boundary conditions.

Lemma 4 *Let $g_N \in H^{p+1}(\Gamma_N)$ and $g_D \in H^{p+1}(\Gamma_D)$. Let the (edge) quadrature formula be exact for polynomials of degree $\leq 2p$. Then there exists a constant $C > 0$ such that*

$$\begin{aligned} |E_S(g_N v)| &\leq C|S|^{p+1/2} |g_N|_{p+1,2,S} \|v\|_{0,2,T}, \\ \frac{\sigma_S}{|S|} |E_S(g_D v)| &\leq C\sigma_S |S|^{p+1/2} |g_D|_{p+1,2,S} |S|^{-1/2} \|[\![v]\!]\|_{0,2,S}. \end{aligned}$$

If the (edge) quadrature formula is exact for polynomials of degree $\leq 2p-1$ and $A \in [W^{p+1,\infty}(S)]^{2\times 2}$ then

$$|E_S((A\nabla v)\cdot \boldsymbol{n} g_D)| \leq C\|A\|_{p+1,\infty,S} |S|^{p+1/2} \|g_D\|_{p+1,2,S} |v|_{1,2,T}.$$

Proof: It is based on results from [5]. □

5. Error estimate for the problem with numerical integration

In order to estimate the impact of the use of numerical integration, let us state the main idea of the *first Strang lemma* ([4], Theorem 4.1.1) which says that

$$\begin{aligned}\hat{c}|||\tilde{u}_h - v_h|||^2 \quad &\leq \quad a^{\pm}(u - v_h, \tilde{u}_h - v_h) + \{a^{\pm}(v_h, \tilde{u}_h - v_h) - a_h^{\pm}(v_h, \tilde{u}_h - v_h)\} \\ &\quad + \{L_h^{\pm}(\tilde{u}_h - v_h) - L^{\pm}(\tilde{u}_h - v_h)\}, \end{aligned} \tag{15}$$

where $\tilde{u}_h$ is defined by (10), v_h is arbitrary element of the space V_h and u is the weak solution of (1)–(3) and $\hat{c}$ comes from Lemma 1. Our aim is to estimate two *consistency errors* arising as the result of the numerical integration.

Theorem 5 *If the quadrature formula on triangles is exact for polynomials of degree $\leq 2p-2$, the (edge) integration formula for the second and third term in (5) is exact for polynomials of degree $\leq 2p-1$ and if the penalty term is integrated exactly, there exists a constant $C > 0$ independent of h such that*

$$|a^{\pm}(v_h, \tilde{u}_h - v_h) - a_h^{\pm}(v_h, \tilde{u}_h - v_h)| \leq C\|A\|_{p+1,\infty,\Omega} \left(\sum_{T \in \mathcal{T}_h} h_T^{2p} \|v_h\|_{p,2,T}^2 \right)^{1/2} |||\tilde{u}_h - v_h|||,$$

where $\tilde{u}_h, v_h \in V_h$.

Proof: Follows from estimates presented in Lemma 3. □

Theorem 6 *Let the quadrature formula on triangles be exact for polynomials of degree $\leq 2p-2$ and let $f \in W^{p,r}(\Omega)$ with $r \geq 2$. Let assumptions of Lemma 4 be satisfied. There exists a constant $C > 0$ independent of h such that*

$$\begin{aligned}|L_h(\tilde{u}_h - v_h) - L(\tilde{u}_h - v_h)| \quad &\leq \quad Ch^p\|f\|_{p,r,\Omega}|||\tilde{u}_h - v_h||| + Ch^{p+1/2}\Big(|g_N|_{p+1,2,\Gamma_N} \\ &\quad +|g_D|_{p+1,2,\Gamma_D} + \|A\|_{p+1,\infty,\Omega}\|g_D\|_{p+1,2,\Gamma_D}\Big)|||\tilde{u}_h - v_h|||,\end{aligned}$$

for $\tilde{u}_h, v_h \in V_h$.

Proof: Is a consequence of Lemma 4, Theorem 4.1.5 in [4] and the Broken Poincaré inequality, see [3]. We also use $|S| \leq h = \max_{T \in \mathcal{T}_h} h_T$. □

Since other terms can be estimated with the aid of the interpolation theory we are ready to write the main theorem.

Theorem 7 *Let all the assumptions of Theorem 5 and Theorem 6 be satisfied and let the approximate bilinear form $a^{\pm}(\cdot,\cdot)$ be uniformly V_h-elliptic. Then there exists a constant $C > 0$ independent of h such that*

$$\begin{aligned}|||u - \tilde{u}_h||| \quad &\leq \quad Ch^p(|u|_{p+1,2,\Omega} + \|A\|_{p+1,\infty,\Omega}\|u\|_{p+1,2,\Omega} + \|f\|_{p,r,\Omega}) \\ &\quad +Ch^{p+1/2}\Big(|g_N|_{p+1,2,\Gamma_N} + |g_D|_{p+1,2,\Gamma_D} + \|A\|_{p+1,\infty,\Omega}\|g_D\|_{p+1,2,\Gamma_D}\Big),\end{aligned}$$

where $\tilde{u}_h$ is defined in (10) and $u \in H^{p+1}(\Omega)$ is the weak solution of (1)–(3).

6. Conclusion

In this paper the effect of numerical integration in the Discontinuous Galerkin formulations for linear elliptic problem was studied. Sufficient conditions which ensure that the discrete problem is uniquely solvable were found. Moreover, if quadrature formulae of a certain precision are used then the order of accuracy (compared with the case without numerical integration) is not decreased.

If we compare these results with the conforming finite element method (see, e. g., [4]), we find that higher regularity of the matrix A is needed for the proof of error estimate. Theorem 5 has again a simple interpretation: The order of convergence is not decreased if the integration formulae yield exact integration of the bilinear form in the case that A is a constant matrix (cf. Remark 4.1.8 in [4]).

Let us also note that numerical results (not reported here) illustrate reasonable degree of agreement with presented theoretical results.

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