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ON A FINITE ELEMENT METHOD APPLICATION IN AEROELASTICITY*

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Abstract

The subject of this paper is the numerical simulation of aeroelastic problems. The interaction of two-dimensional incompressible viscous flow and a vibrating airfoil is modelled. The solid airfoil, which can rotate around the elastic axis and oscillate in the vertical direction, is considered. The numerical simulation consists of the finite element solution of the Navier-Stokes equations coupled with the system of ordinary differential equations describing the airfoil motion. The stabilization procedure is of GLS type. The developed numerical approximation is applied on an aeroelastic problem.

1. Introduction

The mathematical model of relevant technical cases consists of (incompressible) fluid model and (elastic) structure model. In this paper mainly the numerical approximation of fluid motion is addressed. In order to approximate the Navier-Stokes equations several methods can be used. Besides finite differences, the finite volume method can be used for the approximation (for application of finite volume method to solution of incompressible flow cf. [5]). In the present paper the finite element method is used for approximation of the fluid motion. In this case one needs to treat several sources of instability: one caused by the fact that Babuška-Brezzi condition needs to be satisfied in order to guarantee the stability of the scheme, the other source of instability related to the fact that extremely large Reynolds numbers are involved in the problem (Re $\approx$ 10^5–10^6).

2. Mathematical model

The incompressible viscous air flow is described with the aid of Navier-Stokes system of equations written in so-called Arbitrary Lagrangian-Eulerian (ALE) form, cf. [6], [2]. In order to clarify the method, we start with the definition of an ALE mapping $\mathcal{A}_t$: We assume that the mapping $\mathcal{A}_t$ is a given C^1 continuous bijective mapping from the reference (original) configuration Ω_0 onto the computational domain at a time t, i.e. the current configuration Ω_t.

$$\mathcal{A}_t : \Omega_0 \mapsto \Omega_t, \qquad Y \mapsto y(t, Y) = \mathcal{A}_t(Y).$$

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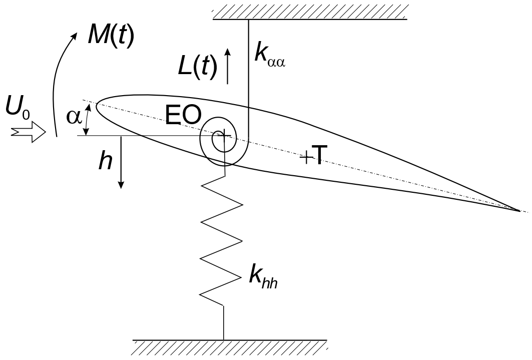

Fig. 1: *The elastic support of the airfoil hanging on translational and rotational springs.*

The time derivative with respect to the reference frame Ω_0 is called the *ALE derivative*, i.e.

$$\frac{D^{\mathcal{A}_t} f}{Dt} = \frac{\partial f}{\partial t} + (\mathbf{w}_g \cdot \nabla) f. \tag{1}$$

With the aid of the ALE derivative $D^{\mathcal{A}_t}\mathbf{u}/Dt$, the Navier-Stokes system of equations is rewritten as follows

$$\frac{D^{\mathcal{A}_t}\mathbf{u}}{Dt} - \nu \triangle \mathbf{u} + \Big((\mathbf{u} - \mathbf{w}_g) \cdot \nabla \Big) \mathbf{u} + \nabla p = 0, \qquad \nabla \cdot \mathbf{u} = 0, \qquad \text{in } \Omega_t\,, \tag{2}$$

where by Ω_t we denote the computational domain occupied by fluid at time $t \in (0, T)$, $\mathbf{u}$ denotes the velocity vector, p denotes the kinematic pressure (i.e. the dynamic pressure divided by the air density), and the domain velocity vector is denoted by $\mathbf{w}_g$. On the boundary $\partial\Omega$ we prescribe suitable boundary conditions. First, the boundary $\partial\Omega$ is decomposed into three distinct parts, i.e. $\partial\Omega = \Gamma_{W_t} \cup \Gamma_D \cup \Gamma_O$. On Γ_D and Γ_{W_t} a Dirichlet boundary conditions are prescribed, i.e.

$$\text{a)} \qquad \mathbf{u} = \mathbf{u}_D \text{ on } \Gamma_D, \qquad\qquad \text{b)} \qquad \mathbf{u} = \mathbf{w}_g \text{ on } \Gamma_{W_t}. \tag{3}$$

The latter part of the boundary is the only moving part of the boundary. The boundary Γ_O represents the outlet, where the following boundary condition is prescribed

$$\left[-(p - p_{ref})\mathbf{n} - \frac{1}{2}(\mathbf{u} \cdot \mathbf{n})^{-}\mathbf{u} + \nu \frac{\partial \mathbf{u}}{\partial \mathbf{n}} \right] \Bigg|_{\Gamma_O} = 0, \tag{4}$$

where p_{ref} is a reference pressure value (e.g. zero).

If Γ_O is the outflowing part of the boundary, i.e. $(\mathbf{u} \cdot \mathbf{n})^{-} = 0$, the condition (4) is equivalent to the well known *do-nothing* boundary condition. We consider the weak formulation (2–4) in the Sobolev spaces $\left(H^1(\Omega)\right)^2$ and $L^2(\Omega)$ for the velocities and pressures, respectively.

The fluid model is coupled with the nonlinear equations of motion for a flexibly supported airfoil, see [7]

$$\begin{aligned} m\,\ddot{h} + S_\alpha\,\ddot{\alpha}\,\cos\alpha - S_\alpha\,\dot{\alpha}^2\,\sin\alpha + k_{hh}\,h &= -L(t), \\ S_\alpha\,\ddot{h}\,\cos\alpha + I_\alpha \ddot{\alpha} + k_{\alpha\alpha}\,\alpha &= M(t). \end{aligned} \tag{5}$$

where h and α denotes the vertical (downwards oriented) and the rotational (clockwise oriented) displacements, respectively, whereas L and M denote the aerodynamical lift force and torsional moment. The mathematical models (5) and (2) are coupled with the evaluation of aerodynamical forces defined by

$$L = -\int_{\Gamma_{W_t}} \sum_{j=1}^{2} \sigma_{2j} n_j dS, \qquad M = -\int_{\Gamma_{W_t}} \sum_{i,j=1}^{2} \sigma_{ij} n_j r_i^{\mathrm{ort}} dS, \tag{6}$$

where $r_1^{\mathrm{ort}} = -(x_{EO2} - x_2)$, $r_2^{\mathrm{ort}} = x_{EO1} - x_1$ and σ_{ij} is the stress tensor, cf. [3].

3. Numerical approximation

First, let us start with an equidistant discretization of the time interval $[0, T]$ with the time step Δt, i.e. $t_k = k \cdot \Delta t$ for $k = 0, 1, 2, \ldots$. Let $\mathbf{u}^n, p^n$ denote approximations of the velocity vector $\mathbf{u}$ and the pressure p evaluated at the time t_n, i.e. $\mathbf{u}^n \approx \mathbf{u}(t_n)$ and $p^n \approx p(t_n)$. The ALE derivative of the velocity vector $\mathbf{u}$ is approximated by

$$\frac{D^{\mathcal{A}_t} f}{Dt} \approx \frac{3\mathbf{u}^{n+1} - 4\hat{\mathbf{u}}^n + \hat{\mathbf{u}}^{n-1}}{2\Delta t}, \tag{7}$$

where the velocity $\mathbf{u}^{n+1}$ denotes the approximate velocity at time t_{n+1} and the velocities $\hat{\mathbf{u}}^n, \hat{\mathbf{u}}^{n-1}$ are the velocities at previous time steps t_n and t_{n-1} transformed from domains $\Omega_{t_n}, \Omega_{t_{n-1}}$ onto the current computational domain $\Omega_{t_{n+1}}$, i.e., $\hat{\mathbf{u}}^n \equiv \mathbf{u}^n \left(\mathcal{A}_{t_n}\left(\mathcal{A}^{-1}_{t_{n+1}}(y)\right)\right)$,$\hat{\mathbf{u}}^{n-1} \equiv \mathbf{u}^{n-1}\left(\mathcal{A}_{t_{n-1}}\left(\mathcal{A}^{-1}_{t_{n+1}}(y)\right)\right)$. The time difference formula is then involved in the problem (2), i.e.

$$\begin{aligned} \frac{3\mathbf{u}^{n+1} - 4\hat{\mathbf{u}}^n + \hat{\mathbf{u}}^{n-1}}{2\Delta t} - \nu \triangle \mathbf{u} + \Big((\mathbf{u} - \mathbf{w}_g) \cdot \nabla\Big)\mathbf{u} + \nabla p &= 0, \\ \nabla \cdot \mathbf{u} &= 0, \qquad \text{in } \Omega_t \end{aligned} \tag{8}$$

and the system of equations (8) is formulated weakly. The components of the approximate solution are sought in the space $X_{\boldsymbol{\Delta}}$. $X_{\boldsymbol{\Delta}}$ denotes the finite element space of Taylor-Hood elements, i.e. piecewise quadratic velocity components and linear pressures.

The *stabilized discrete problem* reads: Find $U = (\mathbf{u}, p) \in X_{\boldsymbol{\Delta}}$ such that

$$\mathbf{a}(U, U, V) + L_{\boldsymbol{\Delta}}(U, U, V) + P_{\boldsymbol{\Delta}}(U, V) = f(V) + F_{\boldsymbol{\Delta}}(V)$$

for all $V = (\mathbf{v}, q) \in X^0_{\boldsymbol{\Delta}}$ ($X^0_{\boldsymbol{\Delta}}$ denotes the space of functions from $X_{\boldsymbol{\Delta}}$ being zero on the Dirichlet part of boundary). The terms $\mathbf{a}(\cdot,\cdot,\cdot)$ and $f(\cdot)$ are the standard Galerkin terms defined as

$$\begin{aligned} \mathbf{a}(U^*, U, V) &= \frac{3}{2\Delta t}(\mathbf{u}, \mathbf{v})_\Omega + \nu (\nabla \mathbf{u}, \nabla \mathbf{v})_\Omega + \Big(\left((\mathbf{u} - \mathbf{w}_g^{n+1}) \cdot \nabla\right)\mathbf{u}, \mathbf{v}\Big)_\Omega \\ &\quad - (p, \nabla \cdot \mathbf{v})_\Omega + (\nabla \cdot \mathbf{u}, q)_\Omega, \\ f(V) &= \frac{1}{2\Delta t}\left(4\hat{\mathbf{u}}^n - \hat{\mathbf{u}}^{n-1}, \mathbf{v}\right)_\Omega - \int_{\Gamma_O} p_{\mathrm{ref}} \mathbf{v} \cdot \mathbf{n}\, dS, \end{aligned} \tag{9}$$

the terms $L_{\boldsymbol{\Delta}}(\cdot,\cdot)$ and $F_{\boldsymbol{\Delta}}(\cdot)$ are GLS (Galerkin Least Squares) additional stabilization terms defined as

$$\begin{aligned}
L_{\boldsymbol{\Delta}}(U^*, U, V) &= \sum_{K\in\tau_{\boldsymbol{\Delta}}} \delta_K \Big(\frac{3}{2\Delta t}\mathbf{u} - \nu\triangle\mathbf{u} + ((\mathbf{u}^* - \mathbf{w}_g)\cdot\nabla)\,\mathbf{u} + \nabla p, \psi(\mathbf{u}, q)\Big)_K, \\
F_{\boldsymbol{\Delta}}(V) &= \sum_{K\in\tau_{\boldsymbol{\Delta}}} \delta_K \Big(\frac{1}{2\Delta t}(4\hat{\mathbf{u}}^n - \hat{\mathbf{u}}^{n-1}), \psi(\mathbf{u}, q)\Big)_K,
\end{aligned} \tag{10}$$

where $\psi(\mathbf{u}, q) \equiv ((\mathbf{u}^* - \mathbf{w}_g)\cdot\nabla)\,\mathbf{v} + \nabla q$, and the term $P_{\boldsymbol{\Delta}}(U, V)$ is the grad-div stabilization term defined as

$$P_{\boldsymbol{\Delta}}(U, V) = \sum_{K\in\tau_{\boldsymbol{\Delta}}} \tau_K(\nabla\cdot\mathbf{u}, \nabla\cdot\mathbf{v})_K, \tag{11}$$

where $U = (\mathbf{u}, p)$, $V = (\mathbf{v}, q)$, $U^* = (\mathbf{u}^*, p)$ and δ_K and τ_K are suitably chosen parameters, cf. [4].

4. Numerical results

The presented method was applied to several practical problems and the numerical results were validated. Here, the numerical results for the coupled system (2) and (5) is presented for the case of flexibly supported airfoil NACA 0012. The solution was performed for far field velocity $U_\infty = 5\,\mathrm{m\,s^{-1}}$ and modified parameter values were taken from [1]. The critical velocity determined by NASTRAN computations was 30.4 m/s, which corresponds to the results computed by the presented method. The airfoil response can be seen in Figure 2 and 3.

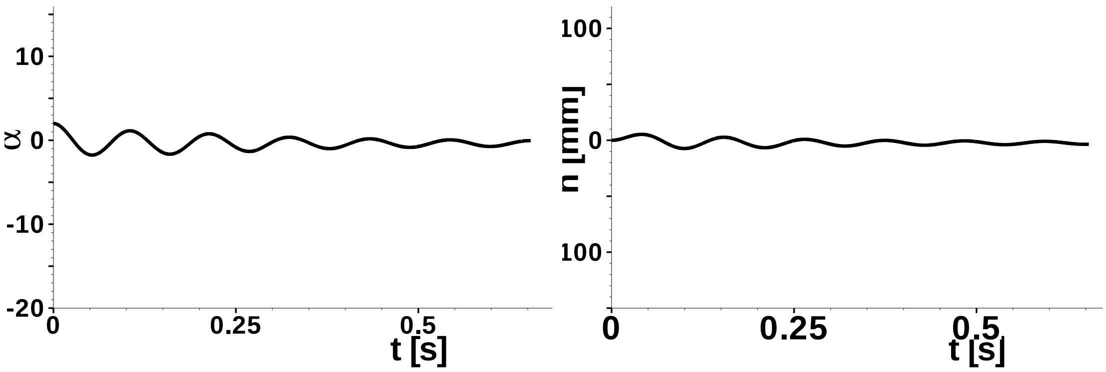

Fig. 2: *Aerodynamical forces acting on airfoil NACA 0012 for far field velocity* $U_\infty = 29\,m\,s^{-1}$ *causes damped vibrations.*

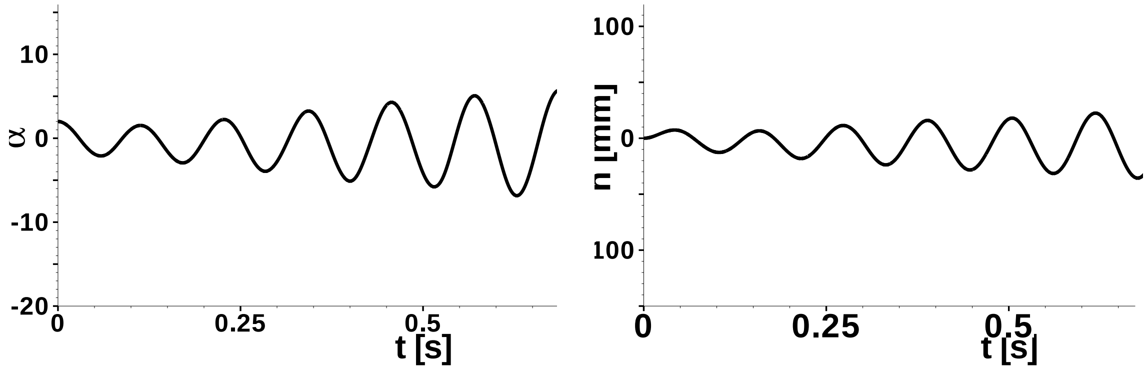

Fig. 3: *The airfoil response of the aerodynamical forces applied on the airfoil NACA 0012 for far field velocity* $U_\infty = 32\, m\, s^{-1}$.

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