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NUMERICAL MODELLING OF RIVER FLOW (NUMERICAL SCHEMES FOR ONE TYPE OF NONCONSERVATIVE SYSTEMS)*

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Abstract

In this paper we propose a new numerical scheme to simulate the river flow in the presence of a variable bottom surface. We use the finite volume method, our approach is based on the technique described by D. L. George for shallow water equations. The main goal is to construct the scheme, which is well balanced, i.e. maintains not only some special steady states but all steady states which can occur. Furthermore this should preserve nonnegativity of some quantities, which are essentially nonnegative from their physical fundamental, for example the cross section or depth. Our scheme can be extended to the second order accuracy.

1. Introduction

We are interested in solving the problem describing the fluid flow through the channel with the general cross-section area

$$\begin{aligned} a_t + q_x &= 0, \\ q_t + \left(\frac{q^2}{a} + gI_1\right)_x &= -gaB_x + gI_2, \end{aligned} \tag{1}$$

where $a = a(x,t)$ is the unknown cross-section area, $q = q(x,t)$ is the unknown discharge, $B = B(x)$ is the function of elevation of the bottom, g is the gravitational constant and

$$I_1 = \int_0^{h(x)} [h(x) - \eta]\sigma(x,\eta)d\eta, \tag{2}$$

$$I_2 = \int_0^{h(x)} (h - \eta)\left[\frac{\partial \sigma}{\partial x}\right] d\eta, \tag{3}$$

where η is the depth integration variable, h is the water depth and $\sigma(x,\eta)$ is the width of the cross-section at the depth η.

The special case are the equations reflecting the fluid flow through the varying rectangular channel

$$\begin{aligned} a_t + q_x &= 0, \\ q_t + \left(\frac{q^2}{a} + \frac{ga^2}{2l}\right)_x &= \frac{ga^2}{2l^2} l_x - gab_x, \end{aligned} \tag{4}$$

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or the system with constant rectangular channel

$$
\begin{aligned}
h_t + (hu)_x &= 0, \\
(hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x &= -ghB_x,
\end{aligned}
\tag{5}
$$

where $h(x,t)$ is the water depth and $u(x,t)$ is the horizontal velocity.

All of the presented systems can be briefly written in the matrix form

$$
\mathbf{q}_t + [\mathbf{f}(\mathbf{q})]_x = \boldsymbol{\psi}(\mathbf{q}, x), \tag{6}
$$

where $\mathbf{q}(x,t)$ is the vector of conserved quantities, $\mathbf{f}(\mathbf{q})$ is the flux function and $\boldsymbol{\psi}(\mathbf{q}, x)$ is the source term.

There are many numerical schemes for solving (6) with different properties and possibilities of failing. For example central, upwind or central-upwind schemes. The main requirements on the numerical schemes are the consistency (in the finite volume meaning: consistency with flux function), the conservativity (if there is possibility to rewrite the problem to the conservative form it is required to have conservative numerical scheme), positive semidefiniteness, i.e. the schemes preserve nonnegativity of some quantities, which are essentially nonnegative from their physical fundamental, and the well-balancing, i.e. the schemes maintain some or all steady states which can occur. The next properties are the order of the schemes, stability and the convergence. There are, of course, related conditions to provide mentioned requirements, for example so called CFL (Courant-Friedrichs-Levy) stability condition.

2. Augmented formulations

There are several ways how to formulate the fluid flow problems. Homogeneous, autonomous, conservative formulation, which is used for standard cases, like Euler equations or fluid flow through the channel with constant cross-section and flat bottom have the form

$$
\begin{aligned}
\mathbf{q}_t + [\mathbf{f}(\mathbf{q})]_x &= \mathbf{0},\ x \in \mathbf{R},\ t \in (0,T), \\
\mathbf{q}(x,0) &= \mathbf{q}_0(x),\ x \in \mathbf{R},
\end{aligned}
\tag{7}
$$

where $\mathbf{q} = \mathbf{q}(x,t) :\ \mathbf{R} \times \langle 0,T) \to \mathbf{R}^m$, $\mathbf{q}_0 = \mathbf{q}_0(x) :\ \mathbf{R} \to \mathbf{R}^m$, $\mathbf{f} = \mathbf{f}(\mathbf{q}) :\ \mathbf{R}^m \to \mathbf{R}^m$. This formulation corresponds to (6) with zero right hand side.

The homogeneous, nonautonomous, conservative case

$$
\begin{aligned}
\mathbf{q}_t + [\mathbf{f}(\mathbf{q}, \mathbf{w}(x))]_x &= \mathbf{0},\ x \in \mathbf{R},\ t \in (0,T), \\
\mathbf{q}(x,0) &= \mathbf{q}_0(x),\ x \in \mathbf{R},
\end{aligned}
\tag{8}
$$

where $\mathbf{w} = \mathbf{w}(x) : \mathbf{R} \to \mathbf{R}^s$ is a given function.

The system (8) can be rewritten to the homogeneous, autonomous, conservative formulation (we add the equation $\mathbf{w}_t = \mathbf{0}$)

$$\begin{aligned} \tilde{\mathbf{q}}_t + [\tilde{\mathbf{f}}(\tilde{\mathbf{q}})]_x &= \mathbf{0},\ x \in \mathbf{R},\ t \in (0,T), \\ \tilde{\mathbf{q}}(x,0) &= \tilde{\mathbf{q}}_0(x),\ x \in \mathbf{R}, \end{aligned} \tag{9}$$

where $\tilde{\mathbf{q}} = [\mathbf{q}, \tilde{\mathbf{w}}]^T$, $\tilde{\mathbf{f}}(\tilde{\mathbf{q}}) = [\mathbf{f}(\mathbf{q}, \tilde{\mathbf{w}}), \mathbf{0}]^T$ and $\tilde{\mathbf{q}}_0(x) = [\mathbf{q}_0(x), \mathbf{w}(x)]^T$.

Now we consider the system in the form (nonhomogeneous, autonomous case)

$$\begin{aligned} \mathbf{q}_t + [\mathbf{f}(\mathbf{q}, \mathbf{w}(x))]_x &= \mathbf{B}(\mathbf{q}, \mathbf{w}(x))\mathbf{w}_x,\ x \in \mathbf{R},\ t \in (0,T), \\ \mathbf{q}(x,0) &= \mathbf{q}_0(x),\ x \in \mathbf{R}, \end{aligned} \tag{10}$$

where $\mathbf{B} = \mathbf{B}(\mathbf{q}, \mathbf{w})$ is the matrix function of the type $m \times s$.
In the case of the river flow (4) this augmented formulation has following form

$$\begin{gathered} \mathbf{q} = [a, q]^T, \quad \mathbf{w}(x) = [l(x), b(x)]^T, \\ \mathbf{f}(\mathbf{q}, \mathbf{w}) = [q, \tfrac{q^2}{a} + \tfrac{ga^2}{2l}]^T, \\ \mathbf{B}(\mathbf{q}, \mathbf{w}(x)) = \begin{bmatrix} 0 & 0 \\ \frac{ga^2}{2l^2} & -ga \end{bmatrix}. \end{gathered}$$

We can rewrite the previous system to the augmented, homogeneous, autonomous, quasilinear formulation

$$\begin{aligned} \tilde{\mathbf{q}}_t + \mathbf{C}(\tilde{\mathbf{q}})\tilde{\mathbf{q}}_x &= \mathbf{0},\ x \in \mathbf{R},\ t \in (0,T), \\ \tilde{\mathbf{q}}(x,0) &= \tilde{\mathbf{q}}_0(x),\ x \in \mathbf{R}, \end{aligned} \tag{11}$$

where

$$\mathbf{C}(\tilde{\mathbf{q}}) = \begin{bmatrix} \mathbf{f}_\mathbf{q} & \mathbf{f}_\mathbf{w} - \mathbf{B}(\mathbf{q}, \tilde{\mathbf{w}}) \\ \mathbf{0} & \mathbf{0} \end{bmatrix},$$

The following relation holds $\mathbf{f}_x = \mathbf{f}_\mathbf{q}\mathbf{q}_x + \mathbf{f}_\mathbf{w}\mathbf{w}_x$.

The next extension can be done by adding another equation in the form

$$[\mathbf{f}(\mathbf{q})]_t + \mathbf{f}_\mathbf{q}[\mathbf{f}(\mathbf{q}, \mathbf{w}(x))]_x - \mathbf{f}_\mathbf{q}\mathbf{B}(\mathbf{q}, \mathbf{w}(x))\mathbf{w}_x = \mathbf{0}.$$

The previous relation provides some theoretical insight into how the flux behaves.

The overdetermined system has now the form

$$\begin{aligned} \hat{\mathbf{q}}_t + \hat{\mathbf{D}}(\hat{\mathbf{q}})\hat{\mathbf{q}}_x &= \mathbf{0},\ x \in \mathbf{R},\ t \in (0,T), \\ \hat{\mathbf{q}}(x,0) &= \hat{\mathbf{q}}_0(x),\ x \in \mathbf{R}, \end{aligned} \tag{12}$$

where $\hat{\mathbf{q}} = [\mathbf{q}, \tilde{\mathbf{w}}, \hat{\mathbf{f}}]^T$,

$$\hat{\mathbf{D}}(\hat{\mathbf{q}}) = \begin{bmatrix} \mathbf{f}_\mathbf{q} & \mathbf{f}_\mathbf{w} - \mathbf{B}(\mathbf{q}, \tilde{\mathbf{w}}) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{f}_\mathbf{q}\mathbf{B}(\mathbf{q}, \tilde{\mathbf{w}}) & \mathbf{f}_\mathbf{q} \end{bmatrix},$$

where $\hat{\mathbf{q}}(x) = [\mathbf{q}_0(x), \mathbf{w}(x), \mathbf{f}(\mathbf{q}_0(x), \mathbf{w}(x))]^T$. The advantage of this formulation is in the conversion of the nonhomogeneous systems to the homogeneous one. For our model of the river flow the matrix has the form

$$\hat{\mathbf{D}}(\hat{\mathbf{q}}) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{-q^2}{a^2} + \frac{ga}{l} & \frac{2q}{a} & \frac{-ga^2}{l^2} & ga & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-ga^2}{2l^2} & ga & 0 & 1 \\ 0 & 0 & \frac{-gqa}{l^2} & 2gq & \frac{-q^2}{a^2} + \frac{ga}{l} & \frac{2q}{a} \end{bmatrix},$$

where $\hat{\mathbf{q}} = [a, q, l, b, q, \frac{q^2}{a} + \frac{ga^2}{2l}]$. The second and fifth equations have the same unknown quantity, so the fifth equation can be rejected.

Now we can formulate new problem in the form

$$\begin{aligned} \check{\mathbf{q}}_t + \check{\mathbf{D}}(\check{\mathbf{q}})\check{\mathbf{q}}_x &= \mathbf{0}, \; x \in \mathbf{R}, \; t \in (0, T), \\ \check{\mathbf{q}}(x, 0) &= \check{\mathbf{q}}_0(x), \; x \in \mathbf{R}, \end{aligned} \tag{13}$$

where for our model of the river flow the matrix has the form

$$\check{\mathbf{D}}(\check{\mathbf{q}}) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{-q^2}{a^2} + \frac{ga}{l} & \frac{2q}{a} & \frac{-ga^2}{l^2} & ga & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{-q^2}{a^2} + \frac{ga}{l} & \frac{-gqa}{l^2} & 2gg & \frac{2q}{a} \end{bmatrix}, \tag{14}$$

and $\check{\mathbf{q}} = [a, q, l, b, \frac{q^2}{a} + \frac{ga^2}{2l}]^T$.

3. Finite volume methods

The finite volume methods are suitable for solving conservation laws, because the numerical solution is modified only by the intercell fluxes. These methods are based on the integral formulation of the problem. They use approximation of the integral averages of the unknown function instead of the approximations of the unknown functions. And the consistency of these methods is related to the flux function. See [6].

We define the following discretisation of the volume and time

$$\begin{aligned} x_j = j\Delta x, \quad j \in \mathbf{Z}, \quad \Delta x > 0, \qquad & t_n = n\Delta t, \quad n \in \mathbf{N}_0, \quad \Delta t > 0, \\ x_{j+1/2} = x_j + \Delta x/2, \qquad & t_{n+1/2} = t_n + \Delta t/2. \end{aligned}$$

We denote the conserved quantities at time t^n and point x_j: $\mathbf{q}_j^n = \mathbf{q}(x_j, t_n)$ and $\mathbf{q}_j(t) = \mathbf{q}(x_j, t)$ and its approximations: $\mathbf{Q}_j^n = \mathbf{Q}(x_j, t_n) \approx \mathbf{q}_j^n$, and $\mathbf{Q}_j(t) = \mathbf{Q}(x_j, t) \approx \mathbf{q}_j(t)$. The finite volumes mean the sets $(x_{j-1/2}, x_{j+1/2}) \times (t^n, t^{n+1})$.

We denote the integral averages of the conserved quantities over the finite volume

$$\bar{\mathbf{Q}}_j^n \approx \bar{\mathbf{q}}_j^n = \frac{1}{\Delta x} \int\limits_{x_{j-1/2}}^{x_{j+1/2}} \mathbf{q}(x, t_n)\ dx, \tag{15}$$

and the average flux along $x = x_{j+1/2}$

$$\bar{\mathbf{F}}_{j+1/2}^{n+1/2} \approx \bar{\mathbf{f}}_{j+1/2}^{n+1/2} = \frac{1}{\Delta t} \int\limits_{t_n}^{t_{n+1}} \mathbf{f}(\mathbf{q}(x_{j+1/2}, t))\ dt. \tag{16}$$

Fully discrete conservative method can be written as relation between approximations of the flux averages and approximations of the integral averages of the conserved quantities

$$\bar{\mathbf{Q}}_j^{n+1} = \bar{\mathbf{Q}}_j^n - \frac{\Delta t}{\Delta x}(\bar{\mathbf{F}}_{j+1/2}^{n+1/2} - \bar{\mathbf{F}}_{j-1/2}^{n+1/2}). \tag{17}$$

Sometimes it is useful to consider the discretisation in two steps. First step is discretisation only in the space (interval $(x_{j-1/2}, x_{j+1/2})$)

$$\bar{\mathbf{Q}}_j = \bar{\mathbf{Q}}_j(t) \approx \bar{\mathbf{q}}_j = \bar{\mathbf{q}}_j(t) = \frac{1}{\Delta x} \int\limits_{x_{j-1/2}}^{x_{j+1/2}} \mathbf{q}(x, t)\ dx. \tag{18}$$

This leads to the system of the ordinary differential equations in the time

$$\frac{d}{dt}\bar{\mathbf{Q}}_j = -\frac{1}{\Delta x}[\mathbf{F}_{j+1/2} - \mathbf{F}_{j-1/2}]. \tag{19}$$

4. Steady states

The steady states mean that the unknown quantities do not change in the time, i.e. $\mathbf{q_t} = \mathbf{0}$ and the flux function must balance the right hand side $[\mathbf{f}(\mathbf{q})]_x = \boldsymbol{\psi}(\mathbf{q}, x)$. For the augmented systems this means that, for example $\mathbf{D}(\hat{\mathbf{q}})\hat{\mathbf{q}}_x = \mathbf{0}$.

Some schemes are constructed to preserve some special steady states like so called rest at lake, i.e. there is no motion and the free surface height is constant:

$$q(x, t) = 0, \quad h(x, t) + b(x) = \text{const.} \tag{20}$$

This steady state has following form for our model (4)

$$q(x, t) = 0,\ \left(\frac{q^2}{a} + \frac{ga^2}{2l}\right)_x - \frac{ga^2}{2l^2}l_x + gab_x = 0. \tag{21}$$

Under the assumption $a = hl$ the mentioned relations can be rewritten into the form

$$ghl(h + b)_x = 0.$$

For general steady states the following equalities hold

$$q_x = 0, \quad \left(\frac{q^2}{a} + \frac{ga^2}{2l}\right)_x = \frac{ga^2}{2l^2} l_x - gab_x, \tag{22}$$

the left term in the second equality we can rewrite as

$$\left(\frac{q^2}{a} + \frac{ga^2}{2l}\right)_x = \left(-u^2 + \frac{ga}{l}\right) a_x - \frac{ga^2}{2l^2} l_x, \tag{23}$$

and together we have

$$\left(-u^2 + \frac{ga}{l}\right) a_x = \frac{ga^2}{l} l_x - gab_x. \tag{24}$$

From (24) we obtain the following relation for general steady states (the Bernoulli equation)

$$\left(\frac{1}{2}u^2 + gb + \frac{ga}{l}\right)_x = 0. \tag{25}$$

For numerical methods it is important to choose such approximation which conserved these steady states. The equation (25) means that the term $\frac{1}{2}u^2 + gb + \frac{ga}{l}$ is constant for differentiable steady states. Therefore following property has to be satisfied

$$\left(\frac{1}{2}u^2 + gb + \frac{ga}{l}\right)_j = \left(\frac{1}{2}u^2 + gb + \frac{ga}{l}\right)_{j+1}. \tag{26}$$

We rearrange (26) and we can express the discrete relation analogous to the smooth one $\phi_x = \left(-u^2 + \frac{ga}{l}\right) a_x - \frac{ga^2}{2l^2} l_x$

$$\Delta\Phi = \left(-|U_L U_R| + g\frac{\bar{A}\bar{L}}{L_L L_R}\right) \Delta A - \frac{g}{2}\frac{\tilde{A}^2}{L_L L_R} \Delta L, \tag{27}$$

where $X_L = \bar{X}_j$, $X_R = \bar{X}_{j+1}$, $\Delta X = X_R - X_L$, X represents U, L and A, $\bar{L} = (L_L + L_R)/2$, $\bar{A} = (A_L + A_R)/2$, $\tilde{A}^2 = (A_L^2 + A_R^2)/2$. The details can be found in [1].

5. Central methods

The central methods are universal schemes for solving hyperbolic partial differential equation, see [5]. In these schemes there is not necessary to construct the characteristic decomposition of the flux f nor to compute the approximation of the Jacobian matrix. These schemes are Riemann problem free. They are robust but they are characterized by large numerical diffusion.

One example is the first-order Lax-Friedrichs scheme

$$\bar{\mathbf{Q}}_j^{n+1} = \frac{1}{2}(\bar{\mathbf{Q}}_{j-1}^n + \bar{\mathbf{Q}}_{j+1}^n) - \frac{\Delta t}{2\Delta x}[\mathbf{f}(\bar{\mathbf{Q}}_{j+1}^n) - \mathbf{f}(\bar{\mathbf{Q}}_{j-1}^n)], \tag{28}$$

where the flux function for the conservative form can be written in the form

$$\mathbf{F}^{n+1/2}_{j+1/2} = \frac{1}{2}[\mathbf{f}(\bar{\mathbf{Q}}^n_j) + \mathbf{f}(\bar{\mathbf{Q}}^n_{j+1})] - \frac{\Delta x}{2\Delta t}(\bar{\mathbf{Q}}^n_{j+1} - \bar{\mathbf{Q}}^n_j). \tag{29}$$

For our model describing fluid flow through the constant rectangular channel

$$\begin{array}{c} h_t + q_x = 0, \\ q_t + \left(\frac{q^2}{h} + \frac{1}{2}gh^2\right)_x = -ghb_x. \end{array}$$

We substitute $y = h + b$ and then we can write

$$\begin{array}{c} y_t + q_x = 0, \\ q_t + \left(\frac{q^2}{y-b} + \frac{1}{2}g(y-b)^2\right)_x = -g(y-b)b_x. \end{array} \tag{30}$$

The special steady state "rest at lake" means $y(x,t) = \text{const}$ and $q(x,t) = 0$.

The flux function and discretisation of the right hand side are in the form

$$\begin{array}{rcl} F^{n,1}_{j+1/2} &=& \frac{1}{2}(\bar{Q}^n_j + \bar{Q}^n_{j+1}) - \frac{\Delta x}{2\Delta t}(\bar{Y}^n_{j+1} - \bar{Y}^n_j), \\ F^{n,2}_{j+1/2} &=& \frac{1}{2}\left[\frac{(\bar{Q}^n_j)^2}{\bar{Y}^n_j - \bar{B}_j} + \frac{(\bar{Q}^n_{j+1})^2}{\bar{Y}^n_{j+1} - \bar{B}_{j+1}} + \frac{1}{2}g(\bar{Y}^n_j - \bar{B}_j) + \right. \\ &+& \left. \frac{1}{2}g(\bar{Y}^n_{j+1} - \bar{B}_{j+1})\right] - \frac{\Delta x}{2\Delta t}(\bar{Q}^n_{j+1} - \bar{Q}^n_j), \\ S^{1,n}_j &=& 0, \\ S^{n,2}_j &=& -\frac{g}{4\Delta x}(\bar{B}_{j+1} - \bar{B}_j) \\ && (\bar{Y}^n_{j+1} - \bar{B}_{j+1} + \bar{Y}^n_j - \bar{B}_j + \bar{Y}^n_j - \bar{B}_j + \bar{Y}^n_{j-1} - \bar{B}_{j-1}). \end{array}$$

This scheme preserves only special steady state "rest at lake". But in general these methods are not suitable for computation steady states [7]. One of their big disadvantages is the relatively large numerical dissipation.

The next type of the central method is for example Rusanovov scheme in semi-discrete form

$$\begin{array}{c} \frac{d}{dt}\bar{\mathbf{Q}}_j = -\frac{1}{2\Delta x}[\mathbf{f}(\bar{\mathbf{Q}}_{j+1}) - \mathbf{f}(\bar{\mathbf{Q}}_{j-1})] + \frac{1}{2\Delta x}[\hat{a}_{j+1/2}(\bar{\mathbf{Q}}_{j+1} - \bar{\mathbf{Q}}_j) - \\ -\hat{a}_{j-1/2}(\bar{\mathbf{Q}}_j - \bar{\mathbf{Q}}_{j-1})], \end{array} \tag{31}$$

where

$$\hat{a}_{j+1/2} = \max_p\{\max\{\lambda^p_j, \lambda^p_{j+1}\}\}.$$

This scheme can be written in the conservative form (19) where the numerical fluxes have the form

$$\mathbf{F}_{j+1/2} = \frac{1}{2}[\mathbf{f}(\bar{\mathbf{Q}}_j) + \mathbf{f}(\bar{\mathbf{Q}}_{j+1})] - \frac{1}{2}|\hat{a}_{j+1/2}|(\bar{\mathbf{Q}}_{j+1} - \bar{\mathbf{Q}}_j).$$

And as will be mentioned in the next section, this scheme can be rewritten in the fluctuation form.

6. Upwind methods

6.1. Scalar case

In this subsection we consider the equation

$$\begin{aligned} q_t + a q_x &= 0,\ x \in R,\ t \in (0,T),\ a \in R, \\ q(x,0) &= q_0(x),\ x \in R. \end{aligned} \tag{32}$$

This advection equation has known solution $q(x,t) = q_0(x - at)$. Usually the REA algorithm (reconstruct-evolve-average) is used for the solution. This algorithm is based on the piecewise polynomial reconstruction of the solution from the quantities $\bar{Q}_j(t)$. This reconstruction we denote $\hat{Q}_j(x,t)$ for $x \in (x_{j-1/2}, x_{j+1/2})$. This reconstruction is considered to be the initial condition for solving sets of the Riemann problems (in this case we can use the form of the solution).

Using the forward differences for time discretisation in the semidiscrete scheme (19) the numerical flux has the form

$$F_{j+1/2} = \frac{1}{2} a (Q^-_{j+1/2} + Q^+_{j+1/2}) - \frac{1}{2} |a| (Q^+_{j+1/2} - Q^-_{j+1/2}), \tag{33}$$

where

$$Q^+_{j+1/2} = \hat{Q}_{j+1}(x_{j+1/2}+, t),\ Q^-_{j+1/2} = \hat{Q}_j(x_{j+1/2}-, t).$$

This scheme can be rewritten into so called fluctuation form

$$\frac{d\bar{Q}_j}{dt} = \frac{-1}{\Delta x} (a^- \Delta Q_{j+1/2} + a \Delta Q_j + a^+ \Delta Q_{j-1/2}), \tag{34}$$

where fluctuations are defined

$$\begin{aligned} a\Delta Q_j &= a(Q^-_{j+1/2} - Q^+_{j-1/2}), \\ a^- \Delta Q_{j+1/2} &= a^-(Q^+_{j+1/2} - Q^-_{j+1/2}), \\ a^+ \Delta Q_{j-1/2} &= a^+(Q^+_{j-1/2} - Q^-_{j-1/2}), \end{aligned}$$

where $a^+ = \max\{a, 0\}$, $a^- = \min\{a, 0\}$.

For simple piecewise constant reconstruction $Q^+_{j+1/2} = \bar{Q}_{j+1},\ Q^-_{j+1/2} = \bar{Q}_j$ we obtain for $a > 0$

$$\frac{d}{dt} \bar{Q}_j = -\frac{a}{\Delta x} (\bar{Q}_j - \bar{Q}_{j-1}), \tag{35}$$

and for $a < 0$

$$\frac{d}{dt} \bar{Q}_j = -\frac{a}{\Delta x} (\bar{Q}_{j+1} - \bar{Q}_j). \tag{36}$$

6.2. Linear systems

Now we consider linear system

$$\begin{aligned} \mathbf{q}_t + \mathbf{A} \mathbf{q}_x &= \mathbf{0},\ x \in R,\ t \in (0,T), \\ \mathbf{q}(x,0) &= \mathbf{q}_0(x),\ x \in R, \end{aligned} \tag{37}$$

where $\mathbf{A}$ is real matrix $m \times m$. We suppose that the matrix $\mathbf{A}$ has distinct real eigenvalues and is diagonalisable i.e. exists regular matrix $\mathbf{R}$ such that $\mathbf{\Lambda} = \mathbf{R}^{-1}\mathbf{A}\mathbf{R}$, where $\mathbf{\Lambda}$ is diagonal matrix. So we can rewrite (37) to the form

$$\boldsymbol{\gamma}_t + \mathbf{\Lambda}\boldsymbol{\gamma}_x = \mathbf{0}, \tag{38}$$

where $\boldsymbol{\gamma}(x,t) = \mathbf{R}^{-1}\mathbf{q}(x,t)$. The system (38) represents m advection equations which can be solved analogously as in the scalar case.

After rewriting the system (37) to conservation form, where $\mathbf{f}(\mathbf{q}) = \mathbf{A}\mathbf{q}$, and solving sets of the generalized Riemann problems we get the numerical fluxes in the form

$$\mathbf{F}_{j+1/2} = \frac{1}{2}\mathbf{A}(\mathbf{Q}^-_{j+1/2} + \mathbf{Q}^+_{j+1/2}) - \frac{1}{2}|\mathbf{A}|(\mathbf{Q}^+_{j+1/2} - \mathbf{Q}^-_{j-1/2}), \tag{39}$$

and in analogy to the previous section we can write the conservative scheme in the fluctuation form

$$\frac{d\bar{\mathbf{Q}}_j}{dt} = \frac{-1}{\Delta x}(\mathbf{A}^-\Delta\mathbf{Q}_{j+1/2} + \mathbf{A}\Delta\mathbf{Q}_j + \mathbf{A}^+\Delta\mathbf{Q}_{j-1/2}), \tag{40}$$

where

$$\begin{aligned}
\mathbf{A}\Delta\mathbf{Q}_j &= \mathbf{A}(\mathbf{Q}^-_{j+1/2} - \mathbf{Q}^+_{j-1/2}),\\
\mathbf{A}^-\Delta\mathbf{Q}_{j+1/2} &= \sum_{p=1}^{m} \lambda^{-,p}\Delta\gamma^p_{j+1/2}\mathbf{r}^p,\\
\mathbf{A}^+\Delta\mathbf{Q}_{j-1/2} &= \sum_{p=1}^{m} \lambda^{+,p}\Delta\gamma^p_{j-1/2}\mathbf{r}^p,\\
\Delta\mathbf{Q}_{j+1/2} &= \sum_{p=1}^{m} \Delta\gamma^p_{j+1/2}\mathbf{r}^p,\\
\Delta\mathbf{Q}_j &= \mathbf{Q}^-_{j+1/2} - \mathbf{Q}^+_{j-1/2},
\end{aligned}$$

$\mathbf{A}^+ = \mathbf{R}\mathbf{\Lambda}^+\mathbf{R}^{-1}$, $\mathbf{A}^- = \mathbf{R}\mathbf{\Lambda}^-\mathbf{R}^{-1}$, $\mathbf{\Lambda}^+ = \mathrm{diag}(\max\{\lambda^p, 0\})$, $\mathbf{\Lambda}^- = \mathrm{diag}(\min\{\lambda^p, 0\})$, $|\mathbf{\Lambda}| = \mathrm{diag}(|\lambda^p|)$, $\Delta\boldsymbol{\gamma}_{j+1/2} = \mathbf{R}^{-1}_{j+1/2}\Delta\mathbf{Q}_{j+1/2}$.

6.3. Nonlinear systems

Now we consider nonlinear system

$$\begin{aligned}
\mathbf{q}_t + [\mathbf{f}(\mathbf{q})]_x &= \mathbf{0},\ x \in R,\ t \in (0,T),\\
\mathbf{q}(x,0) &= \mathbf{q}_0(x),\ x \in R.
\end{aligned} \tag{41}$$

The fluctuation form of the conservative scheme is as follows

$$\frac{d\bar{\mathbf{Q}}_j}{dt} = \frac{-1}{\Delta x}[\mathbf{A}^-(\Delta\mathbf{Q}_{j+1/2}) + \mathbf{A}(\Delta\mathbf{Q}_j) + \mathbf{A}^+(\Delta\mathbf{Q}_{j-1/2})], \tag{42}$$

$$\begin{aligned}
\mathbf{A}(\Delta\mathbf{Q}_j) &= \mathbf{f}(\mathbf{Q}^-_{j+1/2}) - \mathbf{f}(\mathbf{Q}^+_{j-1/2}),\\
\mathbf{A}^-(\Delta\mathbf{Q}_{j+1/2}) &= \mathbf{F}^-_{j+1/2} - \mathbf{f}(\mathbf{Q}^-_{j+1/2}),\\
\mathbf{A}^+(\Delta\mathbf{Q}_{j-1/2}) &= \mathbf{f}(\mathbf{Q}^+_{j-1/2}) - \mathbf{F}^+_{j-1/2}.
\end{aligned}$$

This scheme can be written in the form

$$\frac{d}{dt}\bar{\mathbf{Q}}_j = -\frac{1}{\Delta x}[\mathbf{F}^-_{j+1/2} - \mathbf{F}^+_{j-1/2}]. \tag{43}$$

The fluctuations have following property based on the Rankine-Hugoniot condition for the discontinuities

$$\mathbf{f}(\mathbf{Q}^+_{j+1/2}) - \mathbf{f}(\mathbf{Q}^-_{j+1/2}) = \mathbf{A}^+(\Delta\mathbf{Q}_{j+1/2}) + \mathbf{A}^-(\Delta\mathbf{Q}_{j+1/2}), \tag{44}$$

this leads to $\mathbf{F}^-_{j+1/2} = \mathbf{F}^+_{j+1/2}\ \forall j \in \mathbf{Z}$.

It is difficult to solve nonlinear Riemann problems to take exact solution. It is efficient to use some approximate Riemann solvers such as HLL or Roe's solvers. Details can be found in [3] and [4].

6.3.1. Roe's solver

This approximate Riemann solver is based on the approximation of the nonlinear system $\mathbf{q}_t + [\mathbf{f}(\mathbf{q})]_x \equiv \mathbf{q}_t + \mathbf{A}(\mathbf{q})\mathbf{q}_x = 0$, where $\mathbf{A}(\mathbf{q})$ is the Jacobian matrix, by the linear system $\mathbf{q}_t + \mathbf{A}_{j+1/2}\mathbf{q}_x = 0$, where $\mathbf{A}_{j+1/2}$ is the Roe-averaged Jacobian matrix, which is defined by suitable combination of $\mathbf{A}(\mathbf{Q}_j)$ and $\mathbf{A}(\mathbf{Q}_{j+1})$.

We define intercell numerical fluxes

$$\mathbf{F}_{j+1/2} = \frac{1}{2}[\mathbf{f}(\bar{\mathbf{Q}}_j) + \mathbf{f}(\bar{\mathbf{Q}}_{j+1})] - \frac{1}{2}|\mathbf{A}_{j+1/2}|(\bar{\mathbf{Q}}_{j+1} - \bar{\mathbf{Q}}_j), \tag{45}$$

and intercell fluctuations in the scheme (42) by

$$\begin{aligned} \mathbf{A}^-(\Delta\mathbf{Q}_{j+1/2}) &= \sum_{p=1}^{m} \lambda^{-,p}_{j+1/2}\mathbf{r}^p_{j+1/2}\Delta\gamma^p_{j+1/2}, \\ \mathbf{A}^+(\Delta\mathbf{Q}_{j+1/2}) &= \sum_{p=1}^{m} \lambda^{+,p}_{j+1/2}\mathbf{r}^p_{j+1/2}\Delta\gamma^p_{j+1/2}, \end{aligned} \tag{46}$$

where $\mathbf{r}^p_{j+1/2}$ are eigenvectors of the Roe matrix $\mathbf{A}_{j+1/2}$, $\lambda^p_{j+1/2}$ are eigenvalues called Roe's speeds and $\Delta\boldsymbol{\gamma}_{j+1/2} = \mathbf{R}^{-1}_{j+1/2}\Delta\mathbf{Q}_{j+1/2}$.

6.3.2. HLL solver

This solver does not use the explicit linearization of the Jacobian matrix, but the solution is constructed by the consideration of two discontinuities, propagating at speeds s^1 and s^2. The middle state $\bar{\mathbf{Q}}_{j+1/2}$ is determined by conservation law

$$\mathbf{f}(\bar{\mathbf{Q}}_{j+1}) - \mathbf{f}(\bar{\mathbf{Q}}_j) = s^2_{j+1/2}(\bar{\mathbf{Q}}_{j+1} - \bar{\mathbf{Q}}_{j+1/2}) + s^1_{j+1/2}(\bar{\mathbf{Q}}_{j+1/2} - \bar{\mathbf{Q}}_j), \tag{47}$$

$$\bar{\mathbf{Q}}_{j+1/2} = \frac{\mathbf{f}(\bar{\mathbf{Q}}_{j+1}) - \mathbf{f}(\bar{\mathbf{Q}}_j) - s^2_{j+1/2}\bar{\mathbf{Q}}_{j+1} + s^1_{j+1/2}\bar{\mathbf{Q}}_j}{s^1_{j+1/2} - s^2_{j+1/2}}. \tag{48}$$

When the special choice of the characteristic speeds called Einfeld speeds is used, the solver is called HLLE. The Einfeld speeds are defined by

$$s^1_{j+1/2} = \min_p\{\min\{\lambda^p_j, \lambda^p_{j+1/2}\}\}, \qquad s^2_{j+1/2} = \max_p\{\max\{\lambda^p_{j+1}, \lambda^p_{j+1/2}\}\}, \tag{49}$$

where λ^p_j are eigenvalues of the matrix $\mathbf{A}_j = \mathbf{f}'(\bar{\mathbf{Q}}_j)$.

6.4. Augmented systems

Consider the model for river flow through the varying rectangular channel (4) as was presented in Section 1 and its augmented formulation (13) and (14) presented in Section 2. The eigencomponents for the matrix $\check{\mathbf{D}}$ are

$$\lambda_1 = 0,\ \lambda_2 = 0,\ \lambda_3 = 2u,\ \lambda_4 = u + \sqrt{\frac{ga}{l}},\ \lambda_5 = u - \sqrt{\frac{ga}{l}},$$

and

$$\begin{aligned} \mathbf{r}_1 &= [-\tfrac{ga}{\lambda_4\lambda_5}, 0, 0, -1, ga]^T, & \mathbf{r}_2 &= [-\tfrac{ga^2}{l^2\lambda_4\lambda_5}, 0, 1, 0, \tfrac{ga^2}{2l^2}]^T, \\ \mathbf{r}_3 &= [0, 0, 0, 0, 1]^T, & \mathbf{r}_4 &= [1, \lambda_4, 0, 0, \lambda_4^2]^T, \\ \mathbf{r}_5 &= [1, \lambda_5, 0, 0, \lambda_5^2]^T. \end{aligned}$$

We realize the decomposition for the augmented quasilinear formulation i.e. for the system of five equations with Einfeld speeds

$$s_1 = 0, \quad s_2 = 0, \quad s_3 = s_4 + s_5,$$

$$s_4 = \min_p\{\min\{\lambda_L^p, \lambda_{LR}^p\}\}, \qquad s_5 = \max_p\{\max\{\lambda_R^p, \lambda_{LR}^p\}\},$$

and approximation of the eigenvectors of the matrix $\check{\mathbf{D}}$

$$\begin{aligned} \mathbf{r}_1 &\approx [\tfrac{g\bar{A}}{\widetilde{s_4 s_5}}, 0, 0, -1, \tfrac{g\bar{A}\widehat{s_4 s_5}}{\widetilde{s_4 s_5}}]^T, \\ \mathbf{r}_2 &\approx [\tfrac{g\bar{A}^2}{L_L L_R \widetilde{s_4 s_5}}, 0, 1, 0, \tfrac{g\bar{A}^2 \widehat{s_4 s_5}}{L_L L_R \widetilde{s_4 s_5}} - \tfrac{g\bar{A}^2}{2L_L L_R}]^T, \\ \mathbf{r}_3 &\approx [0, 0, 0, 0, 1]^T, \\ \mathbf{r}_4 &\approx [1, s_4, 0, 0, s_4^2]^T, \\ \mathbf{r}_5 &\approx [1, s_5, 0, 0, s_5^2]^T, \end{aligned}$$

where $\widetilde{s_4 s_5} = -\bar{U}^2 + \frac{g\bar{A}\bar{L}}{L_L L_R}$, $\widehat{s_4 s_5} = -|U_L U_R| + \frac{g\bar{A}\bar{L}}{L_L L_R}$, λ_L^p and λ_R^p are eigenvalues of the Jacobian matrix for the left end right values and λ_{LR}^p are eigenvalues of the Roe's matrix.

The decomposition of the augmented system has the following form

$$\begin{bmatrix} \Delta A \\ \Delta Q \\ \Delta L \\ \Delta B \\ \Delta \Phi \end{bmatrix} = \sum_{p=1}^{5} \gamma_p \mathbf{r}^p.$$

We have five linearly independent eigenvectors. The approximation is chosen to be able to prove the consistency and provide the stability of the algorithm. In some special cases this scheme is conservative and we can prove the positive semidefiniteness, but only under the additional assumptions.

The basic version of the numerical scheme is in the form

$$\frac{d\bar{\mathbf{Q}}_j}{dt} = -\frac{1}{\Delta x}[\mathbf{A}^-(\Delta \mathbf{Q}_{j+1/2}) + \mathbf{A}^+(\Delta \mathbf{Q}_{j-1/2})], \tag{50}$$

where fluctuations are defined by

$$\begin{aligned} \mathbf{A}^-(\Delta \mathbf{Q}_{j+1/2}) &= \sum_{\substack{p=1 \\ s^{p,n}_{j+1/2}<0}}^{m} \gamma^p_{j+1/2}\mathbf{r}^p_{j+1/2}, \\ \mathbf{A}^+(\Delta \mathbf{Q}_{j+1/2}) &= \sum_{\substack{p=1 \\ s^{p,n}_{j+1/2}>0}}^{m} \gamma^p_{j+1/2}\mathbf{r}^p_{j+1/2}. \end{aligned}$$

7. Central-upwind method

Now we introduce so called central-upwind scheme. These schemes combine advantages of the upwind schemes i.e. lower numerical diffusion and usability for the steady states with advantages of the central schemes i.e. positive semidefiniteness. These schemes are Riemann solver free. This scheme can be found in [2]

One simple method in the conservative form (19) has the numerical flux in the form

$$\mathbf{F}_{j+1/2} = \frac{a^+_{j+1/2}\mathbf{f}(\mathbf{Q}_j) - a^-_{j+1/2}\mathbf{f}(\mathbf{Q}_{j+1})}{a^+_{j+1/2} - a^-_{j+1/2}} + \frac{a^+_{j+1/2}a^-_{j+1/2}}{a^+_{j+1/2} - a^-_{j+1/2}}\left[\mathbf{Q}_{j+1} - \mathbf{Q}_j\right], \tag{51}$$

where

$$a^+_{j+1/2} = \max\left\{\lambda_N\left(\mathbf{f}'(\mathbf{Q}_j)\right), \lambda_N\left(\mathbf{f}'(\mathbf{Q}_{j+1})\right), 0\right\},$$

$$a^-_{j+1/2} = \min\left\{\lambda_1\left(\mathbf{f}'(\mathbf{Q}_j)\right), \lambda_1\left(\mathbf{f}'(\mathbf{Q}_{j+1})\right), 0\right\},$$

represent maximal speeds of the propagation of the waves in the points $x_{j+1/2}$ and we suppose $\lambda_1 < \lambda_2, \ \ldots, \ \lambda_N$.

8. Decomposition of the flux function

Described schemes can be represented and understood by the same way. The amount of information about the structure of the solution of the Riemann problem included into schemes causes the differences between schemes. This information is employed in decomposition of the difference of the flux function.

Central schemes for example Lax-Friedrichs scheme are based on the following decomposition

$$\mathbf{f}(\mathbf{Q}^{+}_{j+1/2})-\mathbf{f}(\mathbf{Q}^{-}_{j+1/2}) = s^1(\mathbf{Q}^{+}_{j+1/2}-\mathbf{Q}^{*}_{j+1/2})+s^2(\mathbf{Q}^{*}_{j+1/2}-\mathbf{Q}^{-}_{j+1/2}) = \sum_{p=1}^{2}\mathbf{Z}^{p}_{j+1/2}, \quad (52)$$

where $s^1 = \frac{\Delta x}{\Delta t}$ and $s^2 = -\frac{\Delta x}{\Delta t}$. Next we define fluctuations

$$\begin{aligned} \mathbf{A}^{-}(\Delta\mathbf{Q}_{j+1/2}) &= \sum_{\substack{p=1\\ s^p<0}}^{2} \mathbf{Z}^{p}_{j+1/2}, \\ \mathbf{A}^{+}(\Delta\mathbf{Q}_{j+1/2}) &= \sum_{\substack{p=1\\ s^p>0}}^{2} \mathbf{Z}^{p}_{j+1/2}. \end{aligned} \quad (53)$$

We can use the relation (42) and we can derive the scheme in the conservative form. These schemes are not suitable for the semidiscrete formulation because of the infinite speed ($\Delta t \to 0$) of the propagating discontinuities which is typical for the parabolic type of the equations.

The semidiscrete central methods suitable for the semidiscrete formulation use estimate of upper bound of maximal speed of the propagating discontinuities. They are based on the following decomposition

$$\mathbf{f}(\mathbf{Q}^{+}_{j+1/2})-\mathbf{f}(\mathbf{Q}^{-}_{j+1/2}) = s_{j+1/2}(\mathbf{Q}^{+}_{j+1/2}-\mathbf{Q}^{*}_{j+1/2})-s_{j+1/2}(\mathbf{Q}^{*}_{j+1/2}-\mathbf{Q}^{-}_{j+1/2}) = \sum_{p=1}^{2}\mathbf{Z}^{p}_{j+1/2}, \quad (54)$$

where

$$s_{j+1/2} = \max_{p}\{\max\{|\lambda^p(\mathbf{Q}^{-}_{j+1/2})|, |\lambda^p(\mathbf{Q}^{+}_{j+1/2})|\}\}.$$

We define

$$\begin{aligned} \mathbf{A}^{-}(\Delta\mathbf{Q}_{j+1/2}) &= \sum_{\substack{p=1\\ s^p_{j+1/2}<0}}^{2} \mathbf{Z}^{p}_{j+1/2}, \\ \mathbf{A}^{+}(\Delta\mathbf{Q}_{j+1/2}) &= \sum_{\substack{p=1\\ s^p_{j+1/2}>0}}^{2} \mathbf{Z}^{p}_{j+1/2}. \end{aligned} \quad (55)$$

The central-upwind methods can be identified with HLL solver. The decomposition has the form

$$\mathbf{f}(\bar{\mathbf{Q}}_{j+1}) - \mathbf{f}(\bar{\mathbf{Q}}_j) = s^2_{j+1/2}(\bar{\mathbf{Q}}_{j+1} - \bar{\mathbf{Q}}_{j+1/2}) + s^1_{j+1/2}(\bar{\mathbf{Q}}_{j+1/2} - \bar{\mathbf{Q}}_j) = \sum_{p=1}^{2} \mathbf{Z}^p_{j+1/2}, \quad (56)$$

where $s^1_{j+1/2} = a^+_{j+1/2}$ and $s^1_{j+1/2} = a^-_{j+1/2}$. And we define

$$\begin{aligned} \mathbf{A}^-(\Delta\mathbf{Q}_{j+1/2}) &= \sum_{\substack{p=1 \\ s^p_{j+1/2}<0}}^{2} \mathbf{Z}^p_{j+1/2}, \\ \mathbf{A}^+(\Delta\mathbf{Q}_{j+1/2}) &= \sum_{\substack{p=1 \\ s^p_{j+1/2}>0}}^{2} \mathbf{Z}^p_{j+1/2}. \end{aligned} \quad (57)$$

All described schemes can be understood in the same way.

9. Conclusion

We presented various numerical schemes for solving fluid flow problems with various properties. We show that all described schemes can be understood in the same way.

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