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In: Jan Chleboun and Petr Přikryl and Karel Segeth and Tomáš Vejchodský (eds.): Programs and Algorithms of Numerical Mathematics, Proceedings of Seminar. Dolní Maxov, June 1-6, 2008. Institute of Mathematics AS CR, Prague, 2008. pp. 59–64.

Persistent URL: http://dml.cz/dmlcz/702856

ADAPTIVE FRAME METHODS WITH CUBIC SPLINE-WAVELET BASES*

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In recent years, adaptive wavelet methods have been successfully used for solving operator equations [2, 3, 5]. It has been shown that these methods converge and that they are asymptotically optimal in the sense that storage and number of floating point operations, needed to resolve the problem with desired accuracy, remain proportional to the problem size when the resolution of the discretization is refined.

Suitable wavelet bases on bounded domains are needed for these methods. They are usually constructed in the following way: Wavelets on the real line are adapted to the interval and then by tensor product technique to the n-dimensional cube. Finally, by splitting the domain into nonoverlapping subdomains which are images of $(0, 1)^n$ under appropriate parametric mappings, one can obtain wavelet bases on a fairly general domain. However, it can be very difficult to find these parametric mappings. For this reason, more general adaptive wavelet-frame methods were proposed in [7, 10]. These methods use frames instead of wavelet bases. A frame on a bounded domain can be obtained by a union of wavelet bases on the overlapping subdomains, which are lifted tensor products of a basis on the unit interval. Thus, the construction of wavelet frames is much simpler than the construction of wavelet bases.

The effectiveness of adaptive wavelet and frame methods is strongly influenced by the choice of the wavelet basis on the interval, in particular by its conditioning. However, the conditioning of the known spline-wavelet bases [8, 9] becomes bad for primal polynomial exactness of order $N > 3$, which causes problems in practical applications. In our contribution, we focus on the cubic case, i.e. $N = 4$, and we propose a construction of cubic spline-wavelet bases on the interval adapted for complementary boundary conditions of the first order. We show that these bases are well-conditioned and that the corresponding stiffness matrices have small condition numbers. Furthermore, we show that the adaptive wavelet frame method from [7] with bases constructed in our paper realizes the optimal convergence rate.

1. Construction of boundary adapted spline-wavelet bases

In this section, we introduce a construction of stable spline-wavelet bases on the interval satisfying complementary boundary conditions of the first order. It

*The research of the first author has been supported by the MSMT project no. LC06024, while the second author has been supported by the project IGS financed by Technical University in Liberec.

means that the primal wavelet basis is adapted for homogeneous Dirichlet boundary conditions of the first order, while the dual wavelet basis preserves the full degree of polynomial exactness. This construction is based on the spline-wavelet bases from [4]. Let $\tilde{N}$ be the order of polynomial exactness of the dual MRA.

Let $\Phi_j^{old} = \{\phi_{j,k}, k = -3, \dots, 2^j - 1\}$ be the primal scaling basis on level j from [4]. The functions $\phi_{j,-3}$, $\phi_{j,2^j-1}$ are the only two functions which do not vanish at boundary points. Therefore, defining

$$\Phi_j := \left\{\phi_{j,k}, k = -2, \dots, 2^j - 2\right\} \tag{1}$$

we obtain primal scaling bases satisfying the first order Dirichlet boundary conditions.

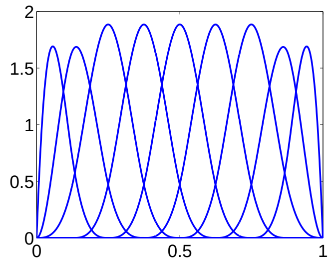

Fig. 1: *Cubic primal scaling basis for $\tilde{N} = 6$, $j = 3$ satisfying complementary boundary conditions of the first order.*

On the dual side, we also need to omit one scaling function at each boundary, because the number of the primal scaling functions must be the same as the number of the dual scaling functions. Let $\Theta_j^{old} = \left\{\theta_{j,k}^{old}, k = -3, \dots, 2^j - 1\right\}$ be the dual scaling basis on level j before biorthogonalization from [4]. There are boundary functions of two types. The functions $\theta_{j,-3}^{old}, \dots, \theta_{j,-4+\tilde{N}}^{old}$ are left boundary functions of the first type which are defined to preserve polynomial exactness of order $\tilde{N}$. The functions $\theta_{j,-3+\tilde{N}}^{old}, \dots, \theta_{j,\tilde{N}-2}^{old}$ are left boundary functions of the second type. The right boundary scaling functions are then derived by reflection of the left boundary functions. Since we want to preserve the full degree of polynomial exactness, we omit one function of the second type at each boundary. Thus, we define

$$\begin{aligned}
\theta_{j,k} &= \theta_{j,k-1}^{old}, \quad k = -2, \dots, -3 + \tilde{N}, \\
\theta_{j,k} &= \theta_{j,k}^{old}, \quad k = -2 + \tilde{N}, \dots, 2^j - \tilde{N} - 2, \\
\theta_{j,k} &= \theta_{j,k+1}^{old}, \quad k = 2^j - \tilde{N} - 1, \dots, 2^j - 2.
\end{aligned}$$

Since the set $\Theta_j := \{\theta_{j,k} : k = -2, \dots, 2^j - 2\}$ is not biorthogonal to Φ_j, we derive a new set $\tilde{\Phi}_j$ from Θ_j by biorthogonalization. Let $\mathbf{A}_j = \left(\langle \phi_{j,k}, \theta_{j,l} \rangle\right)_{k,l=-2}^{2^j-2}$, then viewing $\tilde{\Phi}_j$ and Θ_j as column vectors we define $\tilde{\Phi}_j := \mathbf{A}_j^{-T} \Theta_j$, assuming that $\mathbf{A}_j$ is invertible, which is the case of all choices of $\tilde{N}$ in our numerical experiments.

Our next goal is to determine the corresponding sets of wavelets at the scale j, i.e. $\Psi_j := \{\psi_{j,k}, k = 1, \ldots 2^j\}$, $\tilde{\Psi}_j := \left\{\tilde{\psi}_{j,k}, k = 1, \ldots 2^j\right\}$. We follow a general principle called stable completion as in [8] with some small changes. Since this construction is quite subtle we do not go into details here.

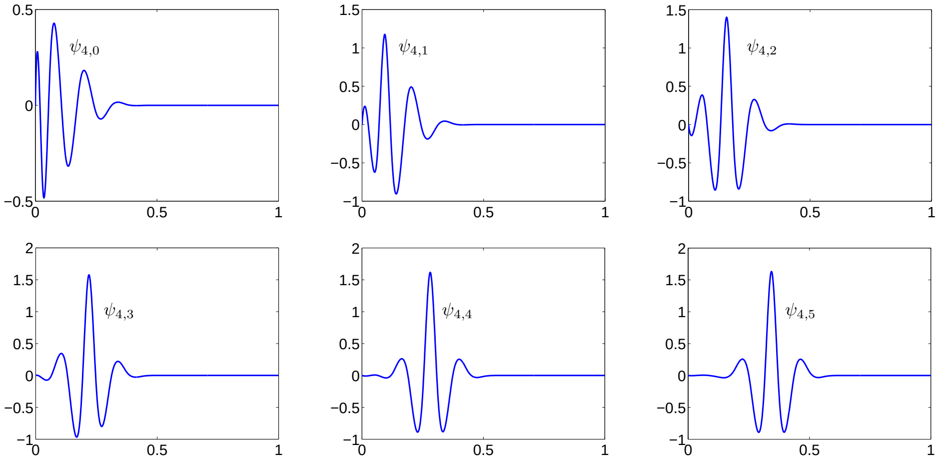

Fig. 2: *Some cubic primal wavelets for $\tilde{N} = 6$ satisfying the complementary boundary conditions of the first order.*

2. Quantitative properties of the constructed bases

In this section, quantitative properties of the constructed bases are presented. In order to further improve the condition we provide L^2-normalization of the primal functions. Then we multiply the dual functions by appropriate constants to preserve biorthogonality. The L^2-normalized bases are denoted by the superscript N. The conditioning of the resulting single-scale bases are listed in Table 1.

N	$\tilde{N}$	j	Φ_j	Φ_j^N	$\tilde{\Phi}_j$	$\tilde{\Phi}_j^N$	Ψ_j	Ψ_j^N	$\tilde{\Psi}_j$	$\tilde{\Psi}_j^N$
4	6	6	4.53	4.30	7.89	6.83	9.46	8.00	16.37	7.96
4	8	6	4.53	4.30	11.15	10.05	8.45	8.02	25.30	15.26
4	10	6	4.53	4.30	17.89	16.97	8.39	8.42	37.65	35.80

Tab. 1: *The conditioning of single-scale scaling and wavelet bases.*

The other criterion for the effectiveness of wavelet bases is the condition number of the corresponding stiffness matrix. Here, let us consider the stiffness matrix for the Poisson equation:

$$\mathbf{A}_{j_0,s} = \left(\langle \psi'_{j,k}, \psi'_{l,m} \rangle\right)_{\psi_{j,k},\psi_{l,m} \in \Psi_{j_0,s}}, \tag{2}$$

where $\Psi_{j_0,s} = \Phi_{j_0} \cup \bigcup_{j=j_0}^{j_0+s-1} \Psi_j$ denotes the multiscale basis. It is well-known that the condition number of $\mathbf{A}_{j_0,s}$ increases quadratically with the matrix size. To remedy this, we use a diagonal matrix for preconditioning

$$\mathbf{A}_{j_0,s}^{prec} = \mathbf{D}_{j_0,s}^{-1} \mathbf{A}_{j_0,s} \mathbf{D}_{j_0,s}^{-1}, \quad \mathbf{D}_{j_0,s} = \mathrm{diag}\left(\left\langle \psi'_{j,k}, \psi'_{j,k} \right\rangle^{1/2} \right)_{\psi_{j,k} \in \Psi_{j_0,s}}. \tag{3}$$

To further improve the condition number of $\mathbf{A}_{j_0,s}^{prec}$ we apply orthogonal transformation to the scaling basis on the coarsest level as in [1] and then we use diagonal matrix for preconditioning. We denote the obtained matrix by $\mathbf{A}_{j_0,s}^{ort}$. Condition numbers of the resulting matrices are listed in Table 2.

N	$\tilde{N}$	j	s	M	$\mathbf{A}_{j,s}^{prec}$	$\mathbf{A}_{j,s}^{ort}$	N	$\tilde{N}$	j	s	M	$\mathbf{A}_{j,s}^{prec}$	$\mathbf{A}_{j,s}^{ort}$
4	4	4	1	33	47.02	15.38	4	8	5	1	65	205.56	15.92
			3	129	49.56	17.40				3	257	208.37	25.04
			5	513	50.17	18.52				5	1025	209.12	27.47
			7	2049	50.28	18.91				7	4097	209.31	27.69
4	6	4	1	33	48.98	15.25	4	10	5	1	65	224.22	22.51
			3	129	49.56	15.94				3	257	226.17	81.72
			5	513	50.17	16.24				5	1025	226.42	91.26
			7	2049	50.28	16.31				7	4097	226.63	92.17

Tab. 2: *The condition numbers of stiffness matrices* $\mathbf{A}_{j,s}^{prec}$, $\mathbf{A}_{j,s}^{ort}$ *of the size* $M \times M$.

3. Adaptive frame method with the constructed bases

Adaptive frame methods are designed in particular for solving operator equations on complicated domains. However, even in some one-dimensional numerical examples the optimal convergence rate was not realized, probably due to stability problems of the used bases. Our intention is to show that the optimal convergence rates of adaptive wavelet frame methods can be achieved also for the case of cubic spline wavelets. We should emphasize that we consider the one-dimensional example as a milestone on the way to higher-dimensional problems.

We consider the same test example as in [6], i.e. the Poisson equation

$$-u'' = f \quad \text{in} \quad \Omega = (0,1), \quad u(0) = u(1) = 0, \tag{4}$$

with the functional f defined by

$$f(v) = 4v\left(\frac{1}{2}\right) - \int_0^1 \left(9\pi^2 \sin(3\pi x) + 4\right) v(x)\, dx. \tag{5}$$

Then the solution u is given by

$$u(x) = \begin{cases} -\sin(3\pi x) + 2x^2, & x \in [0, 0.5), \\ -\sin(3\pi x) + 2(1-x)^2, & x \in [0.5, 1]. \end{cases} \tag{6}$$

To test our bases, we construct a wavelet frame on Ω simply as the union of interval wavelet bases on $\Omega_1 = (0, 0.7)$ and $\Omega_2 = (0.3, 1)$. Note that the singularity is contained in the overlapping part and thus the boundary scaling functions and wavelets, which may potentially cause instabilities, are more involved in the frame than in the wavelet approach. This is the reason why we use wavelet frames instead of wavelet bases directly.

Let us define

$$\mathbf{A} = \mathbf{D}^{-1} \langle \Psi', \Psi' \rangle \mathbf{D}^{-1}, \quad \mathbf{f} = \mathbf{D}^{-1} \langle f, \Psi \rangle, \quad \mathbf{D} = \operatorname{diag} \left(\langle \psi'_{j,k}, \psi'_{j,k} \rangle^{1/2} \right)_{\psi_{j,k} \in \Psi}. \tag{7}$$

Then the variational formulation of (4) is equivalent to

$$\mathbf{AU} = \mathbf{f}, \tag{8}$$

and the solution u is given by $u = \mathbf{UD}^{-1}\Psi$. We solve the infinite dimensional problem (8) by the inexact damped Richardson iterations, for details we refer to [7].

Since the solution u has limited Sobolev regularity, $u \in H^s(\Omega) \cap H_0^1(\Omega)$ only for $s < 1.5$, the linear methods can only converge with limited order. Let $B_q^s(L^p(\Omega))$ denote a Besov space of smoothness s over $L^p(\Omega)$ with additional index q. It can be shown that $u \in B_\tau^{s+1}(L^\tau(\Omega))$ for any positive s and $\tau = (s + 0.5)^{-1}$. Therefore,

$$\|\mathbf{U} - \mathbf{U}_k\|_{l^2} \le C\, (\#\ \text{supp}\ \mathbf{U}_k)^{-n}, \tag{9}$$

where $\mathbf{U}_k$ is the k-th approximate iterand. The theoretical rate of convergence n is limited only by the polynomial exactness of the underlying wavelet bases, in our case the relation (9) holds for any $n < 3$. Figure 3 shows the logarithmic plot of the realized convergence rate for the bases designed in this contribution with $\tilde{N} = 4$ and $\tilde{N} = 6$.

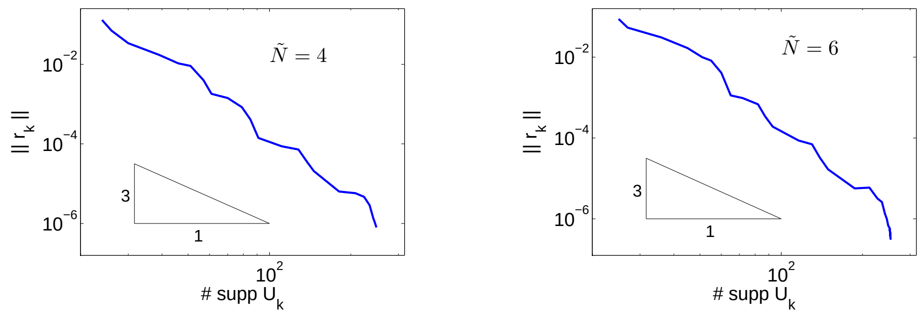

Fig. 3: *The l^2 norm of the residual $\mathbf{r}_k = \mathbf{f} - \mathbf{AU}_k$ versus the number of degrees of freedom.*

To conclude: We proposed a construction of cubic spline-wavelet bases on the interval adapted for complementary boundary conditions of the first order. As opposed to bases from [8, 9], bases constructed in our paper are well-conditioned, the corresponding stiffness matrices have small condition numbers and the adaptive wavelet frame method from [7] with our bases realizes the optimal convergence rate.

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