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# INFLUENCE OF LINEARIZATION TO THE SOLUTION OF FISHER'S EQUATION IN A PLANE

Pavol Chocholatý

## Abstract

Reaction-diffusion equations arise as mathematical models in a series of important applications. Some difference schemes to the solution of the Fisher's equation are presented.

## 1. Fisher's equation

We start our discussion of reaction-diffusion equations by considering a model arising in mathematical ecology. In order to understand the foundation of this model, we first recapture the model of population growth. This model states that the growth of a population facing limited resources is governed by an ordinary differential equation (ODE)

$$u'(t) = cu(t)(A - u(t))$$

for the population density. It is assumed that spatial variation in the density of the population is of little importance for the growth of the population. Thus, one simply assumes that the population is evenly distributed over some area  $G \subset R^d$  for all time  $t$ ,  $c > 0$  is the growth rate, and  $A > 0$  is the so-called carrying capacity of the environment. For real populations, this assumption is often quite dubious. In the next level of sophistication, it is common to take into account the tendency of a population to spread out over the area  $G$  where it is possible to live. This effect is incorporated by adding a Fickian diffusion term to the model.

Now, let  $u(x, t)$  be a function of the population density in time  $t$  and a point  $x$  of an area  $G \subset R^d$  with the boundary  $S$ . Then we get the following partial differential equation

$$u_t = \nabla(D\nabla u) + q. \tag{1}$$

Here,  $D$  is a diffusion coefficient. Equation (1) is a linear diffusion equation or the heat equation with nonhomogeneous forcing term, where  $q$  is assumed to be a continuous function in  $x$  and  $t$ . Assume now that the function  $q$  depends also on the population density  $u$ , i.e.  $q = f(x, t, u)$ , then equation (1) in the form

$$u_t - \nabla(D\nabla u) = f(x, t, u) \tag{2}$$

is a nonlinear reaction-diffusion equation.

Consider next a situation

$$f(x, t, u) = cu(A - u). \tag{3}$$

In mathematical ecology, model of population growth

$$u_t = \nabla(D\nabla u) + cu(A - u) \quad (4)$$

is called Fisher's equation. We mentioned that the first right-hand side term models the diffusion of the population. Similar terms arise in a lot of applications, where we want to capture the tendency of nature to smooth things out. For instance, if you drop a tiny amount of ink into a glass of water, you can watch how the ink spreads throughout the water by means of molecular diffusion. This situation is modeled by the diffusion equation, where Fick's law is used to state that there is a flux of ink from areas of high concentration to areas of low concentration. Similarly again, a Fickian diffusion term in a model of population density states that there is a migration from areas of high population density to areas of low population density. Human beings do not always obey this sound principle.

Fisher's equation (4) is usually studied in conjunction with Neumann-type boundary condition, i.e.

$$(\nabla u)^T \cdot \nu = 0 \quad \text{on} \quad S \times [0, T], \quad (5)$$

where  $\nu$  is meant to be the out-side normal to  $S$ .

The reason for this boundary condition is that we assume the area  $G$  to be closed, so there is no migration from the domain. Assume that  $g = g(x)$ ,  $0 \leq g(x) \leq A$  denotes the initial distribution of the population, we have the initial condition

$$u(x, 0) = g(x), \quad x \in G. \quad (6)$$

Now, (4), (5), (6) represent the initial-boundary value problem for Fisher's equation.

Since we are interested in the qualitative behaviour of this model rather than the actual quantities, we simplify the situation by putting  $D = c = A = 1$  and study the following problem

$$u_t = \Delta u + u(1 - u), \quad \frac{\partial u}{\partial \nu} \Big|_S = 0, \quad t \in [0, T], \quad u(x, 0) = g(x), \quad x \in G. \quad (7)$$

## 2. Finite difference schemes for Fisher's equation

First, we want to solve the initial-boundary value problem (7) and Cauchy problem by difference approximation. Let us start with Cauchy problem: We introduce a time-step  $l = T/m$ ,  $t_i = il$ ,  $i = 0, 1, \dots, m$  and denote by  $P(x, t, u)$  the right-hand side of differential equation in (7). The  $u_{.,i}$  are meant to be approximations to  $u(x, t_i)$ , then an explicit finite difference scheme can be written as follows:

$$u_{.,i+1} = u_{.,i} + lP(x., t_i, u_{.,i}), \quad u_{.,0} = g(x.), \quad (8)$$

which represents explicit Euler method for Cauchy problem by ODEs.

In order to define further integration procedures, we have to give a rule for determining  $u_{.,i+1}$  when  $u_{.,i}, u_{.,i-1}$  and the differential equation are given. Such rule is based on Taylor's expansion and constitutes an explicit two-step formula for  $u_{.,i+1}$ :

$$u_{.,i+1} = \frac{7}{4}u_{.,i} - \frac{3}{4}u_{.,i-1} + \frac{l}{4}P(x., t_i, u_{.,i}), \quad (9)$$

$$u_{.,i+1} = \frac{4}{3}u_{.,i} - \frac{1}{3}u_{.,i-1} + \frac{2l}{3}P(x., t_i, u_{.,i}), \quad (10)$$

$$u_{.,i+1} = u_{.,i-1} + 2lP(x., t_i, u_{.,i}). \quad (11)$$

We are now going to write down the approximation for the initial-boundary value problem:

A. For  $d = 1$  the problem (7) can be written under the form

$$\begin{aligned} u_t &= u_{xx} + u(1 - u), & x \in (0, L), & \quad t \in (0, T), \\ u_x(0, t) &= u_x(L, t) = 0, & t \in [0, T], \\ u(x, 0) &= g(x), & x \in [0, L], \end{aligned} \quad (12)$$

and a mesh width  $h = L/n$ ,  $x_j = jh$ ,  $j = 0, 1, 2, \dots, n$ ,  $n$  natural number, and divide the interval  $[0, L]$  into subintervals of length  $h$ . For simplicity, put  $L = 1$  and we have on  $[0, 1]$   $(n - 1)$  inner grids  $x_1, x_2, \dots, x_{n-1}$ , and two boundary grids  $x_0 = 0$ ,  $x_n = 1$ . As usual, we assume that the solution  $u$  of (12) can be continued on the left side of boundary grid  $x_0$  and on the right side of  $x_n$ , we use the approximation by symmetric differences to describe Neumann boundary conditions.

So,  $P(x, t_i, u_j)$  can be written by using symmetric differences under the form

$$h^2 P(x_j, t_i, u_{j,i}) = u_{j+1,i} + (h^2 - 2)u_{j,i} + u_{j-1,i} - h^2 u_{j,i}^2, \quad j = 0, 1, \dots, n. \quad (13)$$

Denoting by  $u_i$  a vector function which is defined for all  $t_i = il$ ,  $i = 0, 1, 2, \dots, m$ , we approximate the Cauchy problem (8) by

$$\begin{aligned} u_{j,i+1} &= ru_{j+1,i} + (1 + rh^2 - 2r)u_{j,i} + ru_{j-1,i} - rh^2 u_{j,i}^2, \\ u_{j,0} &= g(x_j), \quad u_{-1,i} = u_{1,i}, \quad u_{n-1,i} = u_{n+1,i}, \end{aligned} \quad (14)$$

where, as usual,  $r = l/h^2$ . Tveito and Winther [2] have presented the stability condition

$$l < \frac{h^2}{2 + h^2}, \quad (15)$$

this property provided that the mesh parameters satisfy the requirement, which is slightly more restrictive than the corresponding condition for the heat equation,  $l \leq h^2/2$  or  $r \leq 1/2$ . In a paper [1] we have utilized the preceding result to construct a new stability condition

$$l < \frac{h^2}{2 + h^2/2}. \quad (16)$$

In order to verify this stability condition we have solved problem (12),  $g(x) = \cos^2 \pi x$  for some given  $h$  and computed  $l$  such that

$$(C1) \quad l < \frac{h^2}{2 + h^2},$$

$$(C2) \quad \frac{h^2}{2 + h^2} \leq l < \frac{h^2}{2 + h^2/2},$$

$$(C3) \quad l \geq \frac{h^2}{2 + h^2/2}.$$

In numerical experiments we have observed that the approximate solutions in cases (C1), (C2) always stayed within the unit interval and that they approached the state  $u = 1$  as time increased, but in case (C3) the oscillatory solution about the state  $u = 1$  as time increased was obtained.

Now, we approximate the Cauchy problem by explicit two-step formulas (9), (10), (11) with the operator (13)

$$4u_{j,i+1} = ru_{j-1,i} + (7 - 2r + rh^2)u_{j,i} + ru_{j+1,i} - 3u_{j,i-1} - rh^2u_{j,i}^2, \quad (17)$$

$$3u_{j,i+1} = 2ru_{j-1,i} + (4 - 4r + 2rh^2)u_{j,i} + 2ru_{j+1,i} - u_{j,i-1} - 2rh^2u_{j,i}^2, \quad (18)$$

$$u_{j,i+1} = 2ru_{j-1,i} + (2rh^2 - 4r)u_{j,i} + 2ru_{j+1,i} + u_{j,i-1} - 2rh^2u_{j,i}^2, \quad (19)$$

$$l = rh^2, \quad i = 0, 1, 2, \dots, m, \quad j = 0, 1, 2, \dots, n,$$

and described conditions

$$u_{j,0} = g(x_j), \quad u_{-1,i} = u_{1,i}, \quad u_{n-1,i} = u_{n+1,i}.$$

We now consider the question of an actual error at the discrete time points  $t_i = il$ ,  $i = 0, 1, 2, \dots, m$ . The “good” dependence of actual error on  $(i + p)$ -th step to an error on  $i$ -th step is a motivation for the computational method used.

The following three tables present for  $l = 0.0049875$  and  $h = 0.1$  the propagation of the unit error at one vertex on time steps  $i + p$  for (17), (18), (19). Notice that if the error is not unite, then the presented values should be scaled by its actual size  $\varepsilon$ .

| (17) | $x_0$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ |
|------|-------|-------|-------|-------|-------|-------|-------|
| i    | 0     | 0     | 0     | 1     | 0     | 0     | 0     |
| i+1  | 0     | 0     | 0.125 | 1.5   | 0.125 | 0     | 0     |
| i+2  | 0     | 0.016 | 0.374 | 1.532 | 0.374 | 0.016 | 0     |
| i+3  | 0.002 | 0.070 | 0.662 | 1.266 | 0.662 | 0.070 | 0.002 |

| (18) | $x_0$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ |
|------|-------|-------|-------|-------|-------|-------|-------|
| i    | 0     | 0     | 0     | 1     | 0     | 0     | 0     |
| i+1  | 0     | 0     | 0.333 | 0.668 | 0.333 | 0     | 0     |
| i+2  | 0     | 0.111 | 0.445 | 0.335 | 0.445 | 0.111 | 0     |
| i+3  | 0.037 | 0.222 | 0.337 | 0.299 | 0.337 | 0.222 | 0.037 |

| (19) | $x_0$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ |
|------|-------|-------|-------|-------|-------|-------|-------|
| i    | 0     | 0     | 0     | 1     | 0     | 0     | 0     |
| i+1  | 0     | 0     | 0.998 | -1.95 | 0.998 | 0     | 0     |
| i+2  | 0     | 0.995 | -3.98 | 6.911 | -3.98 | 0.995 | 0     |
| i+3  | 0.993 | -5.95 | 16.62 | -24.1 | 16.62 | -5.95 | 0.993 |

So, it is shown that with formulas (17), (18) a good approximative solution for (12) can be computed and that the formula (19) leads to catastrophic spreading of actual error.

B. The computational technique outlined in the previous part can be applied formally to the problem (7) also in case  $d = 2$  with domain  $G = (0, 1) \times (0, 1)$ ,  $\bar{G} = G + S$ . In this chapter, we concentrate on the solution of Cauchy problem using implicit methods

$$u_{.,i+1} = u_{.,i} + lP(x., t_{i+1}, u_{.,i+1}), \quad u_{.,0} = g(x.).$$

For a grid function to  $u(x, y, t)$ , defined on the set of grid points  $\bar{G}_h = \{(x_j, y_k); h = 1/n; 0 \leq j, k \leq n\}$  at a given time  $t_i = il \in [0, T]$ ,  $i = 0, 1, 2, \dots, m$ ,  $l = T/m$ , we frequently write  $u_{j,k,i}$ . For such functions the finite difference operator approximating the differential operator  $P(x, t, u)$  is now defined by

$$h^2 P(x_j, y_k, t_i, u_{j,k,i}) = u_{j+1,k,i} + u_{j-1,k,i} + u_{j,k-1,i} + u_{j,k+1,i} + (h^2 - 4)u_{j,k,i} + h^2 u_{j,k,i}^2 \quad (20)$$

for all interior grid points  $(x_j, y_k, t_i)$ ,  $j, k = 1, 2, \dots, n-1$ . For a Neumann condition defined on  $S$ , we frequently use forwards and backwards differences

$$u_{1,k,i} = u_{0,k,i}, \quad u_{j,1,i} = u_{j,0,i}, \quad u_{n,k,i} = u_{n-1,k,i}, \quad u_{j,n-1,i} = u_{j,n,i}, \quad j, k = 0, 1, \dots, n. \quad (21)$$

A finite difference approximation is now started by Cauchy condition

$$u_{j,k,0} = g(x_j, y_k) \quad (22)$$

and is represented on each  $t_i$ ,  $i = 1, 2, \dots, m$  by a system of nonlinear equations. But our computational approach is based on a sequential solution procedure, where the operator splitting is the key to obtain higher efficiency. This is the case for the alternating direction implicit (ADI) scheme first proposed for the implicit solution of heat flow in two geometric dimensions by Peaceman and Rachford. They have proved both stability and convergence for the method. The computational setup can be briefly described using linearization as follows: Consider the first finite-difference scheme in the linearization form

$$u_{j,k,z} - u_{j,k,i} = r(u_{j-1,k,z} - 2u_{j,k,z} + u_{j+1,k,z}) + r(u_{j,k-1,i} - 2u_{j,k,i} + u_{j,k+1,i}) + rh^2 u_{j,k,z}(1 - u_{j,k,i}) \quad (23)$$

and now formulate two approaches of linearization of the second finite-difference scheme

$$u_{j,k,i+1} - u_{j,k,z} = r(u_{j,k-1,i+1} - 2u_{j,k,i+1} + u_{j,k+1,i+1}) + r(u_{j-1,k,z} - 2u_{j,k,z} + u_{j+1,k,z}) + rh^2u_{j,k,z}(1 - u_{j,k,i+1}), \quad (24)$$

$$u_{j,k,i+1} - u_{j,k,z} = r(u_{j,k-1,i+1} - 2u_{j,k,i+1} + u_{j,k+1,i+1}) + r(u_{j-1,k,z} - 2u_{j,k,z} + u_{j+1,k,z}) + rh^2u_{j,k,i+1}(1 - u_{j,k,z}) \quad (25)$$

with both Cauchy condition (22) and boundary condition (21), where  $r = l/(2h^2)$  and  $z = i + 1/2$ ,  $i = 0, 1, \dots, m - 1$ .

A computational comparison of the above-mentioned approaches (23), (24) or (23), (25) for Fisher's equation with Cauchy condition  $g(x, y) = \cos^2(\pi(x + y))/2$  and steps  $h = 0.1$ ,  $l = 0.01$  introduces on some  $t_i = il$  at the point  $(0.7, 0.5) \in G$  the following table:

| t         | 0.01        | 0.1         | 1           |
|-----------|-------------|-------------|-------------|
| (23),(24) | 0.161929944 | 0.289809450 | 0.712927005 |
| (23),(25) | 0.162252848 | 0.290045491 | 0.714969435 |

## References

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