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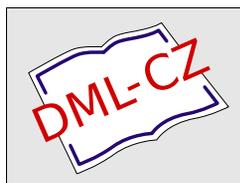
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THE NUMERICAL SOLUTION OF COMPRESSIBLE FLOWS IN TIME DEPENDENT DOMAINS*

Václav Kučera, Jan Česenek

Abstract

This work is concerned with the numerical solution of inviscid compressible fluid flow in moving domains. Specifically, we assume that the boundary part of the domain (impermeable walls) are time dependent. We consider the Euler equations, which describe the movement of inviscid compressible fluids. We present two formulations of the Euler equations in the ALE (Arbitrary Lagrangian-Eulerian) form. These two formulations are discretized in space by the discontinuous Galerkin method. We apply a semi-implicit linearization with respect to time to obtain a numerical scheme requiring the solution of only one linear system on each time level. We apply the method to the compressible flow around a moving (vibrating) profile.

1. Continuous problem

In this paper we shall be concerned with two-dimensional inviscid compressible flow in a bounded domain $\Omega_t \subset \mathbb{R}^2$ depending on time $t \in [0, T]$. We assume that the boundary of Ω_t consists of three disjoint parts $\Gamma_I, \Gamma_O, \Gamma_{W_t}$: $\partial\Omega_t = \Gamma_I \cup \Gamma_O \cup \Gamma_{W_t}$, where Γ_I and Γ_O represent the time-independent inlet and outlet, respectively, and Γ_{W_t} represents moving impermeable walls.

As the governing equations we take the Euler equations written in the conservative form

$$\frac{\partial \mathbf{w}}{\partial t} + \sum_{s=1}^2 \frac{\partial \mathbf{f}_s(\mathbf{w})}{\partial x_s} = 0 \quad \text{in } \Omega_t, \quad t \in (0, T), \quad (1)$$

where

$$\mathbf{w} = (w_1, \dots, w_4)^\top = (\rho, \rho v_1, \rho v_2, E)^\top$$

is the so-called *state vector* and

$$\mathbf{f}_s(\mathbf{w}) = (\rho v_s, \rho v_s v_1 + \delta_{s1} p, \rho v_s v_2 + \delta_{s2} p, (E + p) v_s)^\top$$

are the *Euler inviscid fluxes* of the quantity \mathbf{w} in the directions x_s , $s = 1, 2$. We use the following notation: ρ - density, p - pressure, E - total energy, $\mathbf{v} = (v_1, v_2)$ - velocity vector, $\gamma > 1$ - Poisson adiabatic constant (we take $\gamma = 1.4$ for air), $a = \sqrt{\gamma p / \rho}$ - local speed of sound.

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System (1) is completed by the thermodynamical relation arising from the equation of state

$$p = (\gamma - 1)(E - \rho |\mathbf{v}|^2 / 2),$$

furthermore by the initial condition

$$\mathbf{w}(\mathbf{x}, 0) = \mathbf{w}^0(x), \quad \mathbf{x} \in \Omega, \quad (2)$$

and boundary conditions, which are treated in Section 4

As in [6] we define the flux of the quantity \mathbf{w} in the direction $\mathbf{n} = (n_1, n_2) \in \mathbb{R}^2$, $n_1^2 + n_2^2 = 1$, by

$$\mathbf{F}(\mathbf{w}, \mathbf{n}) = \sum_{s=1}^2 \mathbf{f}_s(\mathbf{w}) n_s$$

and its Jacobi matrix

$$\mathbb{P}(\mathbf{w}, \mathbf{n}) = \frac{D\mathbf{F}(\mathbf{w}, \mathbf{n})}{D\mathbf{w}} = \sum_{s=1}^2 \mathbf{A}_s(\mathbf{w}) n_s, \quad (3)$$

where

$$\mathbf{A}_s(\mathbf{w}) = \frac{D\mathbf{f}_s(\mathbf{w})}{D\mathbf{w}}, \quad s = 1, 2,$$

are the Jacobi matrices of the mappings \mathbf{f}_s . It is possible to show that \mathbf{f}_s , $s = 1, 2$, are homogeneous mappings of order one, which implies that

$$\mathbf{f}_s(\mathbf{w}) = \mathbf{A}_s(\mathbf{w})\mathbf{w}, \quad s = 1, 2, \quad (4)$$

and

$$\mathbf{F}(\mathbf{w}, \mathbf{n}) = \mathbb{P}(\mathbf{w}, \mathbf{n})\mathbf{w}. \quad (5)$$

The matrix \mathbb{P} is diagonalizable, i.e.

$$\mathbb{P} = \mathbb{T}\Lambda\mathbb{T}^{-1}, \quad (6)$$

where $\mathbb{T} = \mathbb{T}(\mathbf{w}, \mathbf{n})$ is a nonsingular matrix and

$$\Lambda = \text{diag}(\lambda_1, \dots, \lambda_4) \quad (7)$$

is the diagonal matrix with entries

$$\lambda_1 = \mathbf{v} \cdot \mathbf{n} - a, \quad \lambda_2 = \lambda_3 = \mathbf{v} \cdot \mathbf{n}, \quad \lambda_4 = \mathbf{v} \cdot \mathbf{n} + a, \quad (8)$$

which are the eigenvalues of the matrix \mathbb{P} . (See, e.g. [6], Section 3.1.5.)

2. ALE formulation

In order to treat the time dependence of the domain, we use the so-called *arbitrary Lagrangian-Eulerian* ALE technique. We define a reference domain Ω_0 and regular one-to-one ALE mapping of Ω_0 onto Ω_t (cf. [8], [9] and [10])

$$\mathcal{A}_t : \bar{\Omega}_0 \longrightarrow \bar{\Omega}_t, \text{ i.e. } \mathbf{X} \in \bar{\Omega}_0 \longmapsto \mathbf{x} = \mathbf{x}(\mathbf{X}, t) = \mathcal{A}_t(\mathbf{X}) \in \bar{\Omega}_t.$$

Here we use the notation \mathbf{X} for points in $\bar{\Omega}_0$ and \mathbf{x} for points in $\bar{\Omega}_t$.

Further, we define the ALE velocity:

$$\begin{aligned} \tilde{\mathbf{z}}(\mathbf{X}, t) &= \frac{\partial}{\partial t} \mathcal{A}_t(\mathbf{X}), \quad t \in [0, T], \quad \mathbf{X} \in \Omega_0, \\ \mathbf{z}(\mathbf{x}, t) &= \tilde{\mathbf{z}}(\mathcal{A}^{-1}(\mathbf{x}), t), \quad t \in [0, T], \quad \mathbf{x} \in \Omega_t, \end{aligned}$$

and the ALE derivative of a function $f = f(\mathbf{x}, t)$ defined for $\mathbf{x} \in \Omega_t$ and $t \in [0, T]$:

$$\frac{D^A}{Dt} f(\mathbf{x}, t) = \frac{\partial \tilde{f}}{\partial t}(\mathbf{X}, t), \quad (9)$$

where

$$\tilde{f}(\mathbf{X}, t) = f(\mathcal{A}_t(\mathbf{X}), t), \quad \mathbf{X} \in \Omega_0, \quad \mathbf{x} = \mathcal{A}_t(\mathbf{X}).$$

The following relations are a direct consequence of the chain rule:

$$\frac{D^A f}{Dt} = \frac{\partial f}{\partial t} + \mathbf{z} \cdot \nabla f = \frac{\partial f}{\partial t} + \operatorname{div}(\mathbf{z}f) - f \operatorname{div} \mathbf{z}.$$

This leads to two different formulations of the Euler equations in ALE form.

Formulation 1:

$$\frac{D^A \mathbf{w}}{Dt} + \sum_{s=1}^2 \frac{\partial \mathbf{f}_s(\mathbf{w})}{\partial x_s} - \mathbf{z} \cdot \nabla \mathbf{w} = 0. \quad (10)$$

Formulation 2:

$$\frac{D^A \mathbf{w}}{Dt} + \sum_{s=1}^2 \frac{\partial \mathbf{g}_s(\mathbf{w})}{\partial x_s} + \mathbf{w} \operatorname{div} \mathbf{z} = 0, \quad (11)$$

where $\mathbf{g}_s, s = 1, 2$, are modified inviscid fluxes

$$\mathbf{g}_s(\mathbf{w}) := \mathbf{f}_s(\mathbf{w}) - z_s \mathbf{w}.$$

3. Discretization of the problem in the time dependent domain

In this section, we shall describe the discretization of the initial-boundary value problem for the Euler equations written in the ALE forms (10) and (11). In the presented work we shall use the discontinuous Galerkin finite element method (DGFEM) for space semi-discretization. For an overview of various applications of the discontinuous Galerkin methods cf. [2].

3.1. Notation

In what follows we shall assume that Ω_t is a polygonal domain for all t . Let \mathcal{T}_{ht} be a partition of the closure $\overline{\Omega}_t$ into a finite number of closed triangles with mutually disjoint interiors. We shall call \mathcal{T}_{ht} a triangulation of Ω_t . We do not require the standard conforming properties of \mathcal{T}_{ht} used in the finite element method. This means that we admit the so-called hanging nodes. We shall use the following notation. By ∂K we denote the boundary of an element $K \in \mathcal{T}_{ht}$ and set $h_K = \text{diam}(K)$, $h = \max_{K \in \mathcal{T}_{ht}} h_K$. By ρ_K we denote the radius of the largest circle inscribed into K and by $|K|$ we denote the area of K .

Let $K, K' \in \mathcal{T}_{ht}$. We say that K and K' are *neighbours*, if the set $\partial K \cap \partial K'$ has positive length. We say that $\Gamma \subset K$ is a *face* of K , if it is a maximal connected open subset either of $\partial K \cap \partial K'$, where K' is a neighbour of K , or of $\partial K \cap \partial \Omega_t$. By \mathcal{F}_{ht} we denote the system of all faces of all elements $K \in \mathcal{T}_{ht}$. Further, we define the set of all inner faces by

$$\mathcal{F}_{ht}^I = \{\Gamma \in \mathcal{F}_{ht}; \Gamma \subset \Omega_t\}$$

and the set of all boundary faces by

$$\mathcal{F}_{ht}^B = \{\Gamma \in \mathcal{F}_{ht}; \Gamma \subset \partial \Omega_t\}.$$

Obviously, $\mathcal{F}_{ht} = \mathcal{F}_{ht}^I \cup \mathcal{F}_{ht}^B$.

For each $\Gamma \in \mathcal{F}_{ht}$ we define a unit normal vector \mathbf{n}_Γ . We assume that for $\Gamma \in \mathcal{F}_{ht}^B$ the normal \mathbf{n}_Γ has the same orientation as the outer normal to $\partial \Omega$. For each face $\Gamma \in \mathcal{F}_{ht}^I$ the orientation of \mathbf{n}_Γ is arbitrary but fixed cf. Figure 1. Finally, by $d(\Gamma)$ we denote the length of $\Gamma \in \mathcal{F}_{ht}$.

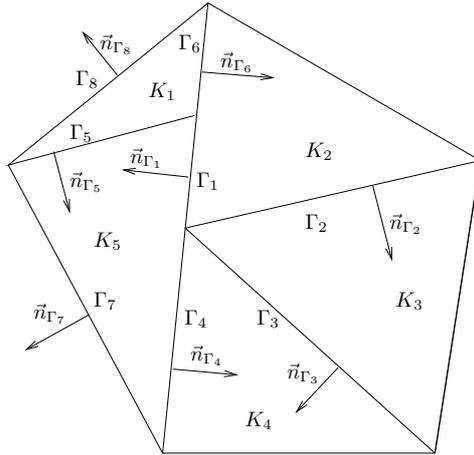


Fig. 1: Typical DG triangulation with possible hanging nodes.

3.2. Spaces of discontinuous functions

For each face $\Gamma \in \mathcal{F}_{ht}^I$ there exist two neighbours $K_\Gamma^{(L)}, K_\Gamma^{(R)} \in \mathcal{T}_{ht}$ such that $\Gamma \subset K_\Gamma^{(L)} \cap K_\Gamma^{(R)}$. We use the convention that \mathbf{n}_Γ is the outer normal to the element $K_\Gamma^{(L)}$ and the inner normal to the element $K_\Gamma^{(R)}$, see Figure 2. Let $p \geq 1$ be an integer. The approximate solution will be sought in the space of discontinuous piecewise polynomial functions

$$\mathbf{S}_{ht} = \{v; v|_K \in P^p(K), \forall K \in \mathcal{T}_{ht}\}^4,$$

where $P^p(K)$ denotes the space of all polynomials on K of degree $\leq p$. For $v \in \mathbf{S}_{ht}$ and $\Gamma \in \mathcal{F}_{ht}^I$ we introduce the following notation:

$$\begin{aligned} v|_\Gamma^{(L)} &= \text{the trace of } v|_{K_\Gamma^{(L)}} \text{ on } \Gamma, & v|_\Gamma^{(R)} &= \text{the trace of } v|_{K_\Gamma^{(R)}} \text{ on } \Gamma, \\ \langle v \rangle_\Gamma &= \frac{1}{2}(v|_\Gamma^{(L)} + v|_\Gamma^{(R)}), & [v]_\Gamma &= v|_\Gamma^{(L)} - v|_\Gamma^{(R)}. \end{aligned}$$

Now, let $\Gamma \in \mathcal{F}_{ht}^B$ and $K_\Gamma^{(L)} \in \mathcal{T}_{ht}$ be such an element that $\Gamma \subset \partial K_\Gamma^{(L)} \cap \partial\Omega_t$. For $v \in \mathbf{S}_{ht}$ we set

$$v_\Gamma = v|_\Gamma^{(L)} = v|_\Gamma^{(R)} = \text{the trace of } v|_{K_\Gamma^{(L)}} \text{ on } \Gamma,$$

i.e. we define $v|_\Gamma^{(R)}$ by extrapolation.

If $[\cdot]_\Gamma$ and $\langle \cdot \rangle_\Gamma$ appear in an integral of the form $\int_\Gamma \dots dS$, we omit the subscript Γ and write simply $[\cdot]$ and $\langle \cdot \rangle$.

3.3. Space semidiscretization

3.3.1. Formulation 1

In order to derive the discrete problem, we assume that \mathbf{w} is a sufficiently regular solution of system (10), multiply (10) by a test function $\boldsymbol{\varphi} \in \mathbf{S}_{ht}$, integrate over any element K apply Green's theorem and sum over all $K \in \mathcal{T}_{ht}$. We get the relation

$$\begin{aligned} \sum_{K \in \mathcal{T}_{ht}} \int_K \frac{D^A \mathbf{w}(t)}{Dt} \cdot \boldsymbol{\varphi} \, d\mathbf{x} &= \sum_{K \in \mathcal{T}_{ht}} \int_K \sum_{s=1}^2 \mathbf{f}_s(\mathbf{w}(t)) \cdot \frac{\partial \boldsymbol{\varphi}}{\partial x_s} \, d\mathbf{x} \\ &- \sum_{\Gamma \in \mathcal{F}_{ht}} \int_\Gamma \sum_{s=1}^2 \mathbf{f}_s(\mathbf{w}(t)) n_s \cdot [\boldsymbol{\varphi}] \, dS - \sum_{K \in \mathcal{T}_{ht}} \int_K \sum_{s=1}^2 z_s \frac{\partial \mathbf{w}}{\partial x_s} \cdot \boldsymbol{\varphi} \, d\mathbf{x}. \end{aligned}$$

Now the exact solution $\mathbf{w}(t)$ is approximated by an element $\mathbf{w}_h(t) \in \mathbf{S}_{ht}$ and the fluxes through the faces Γ are approximated as in the finite volume method with the aid of a numerical flux $\mathbf{H}_f = \mathbf{H}_f(\mathbf{u}, \mathbf{w}, \mathbf{n})$. It means that on edge Γ

$$\sum_{s=1}^2 \mathbf{f}_s(\mathbf{w}(t)) n_s \approx \mathbf{H}_f(\mathbf{w}_h|_\Gamma^{(L)}(t), \mathbf{w}_h|_\Gamma^{(R)}(t), \mathbf{n}).$$

In this work we take \mathbf{H}_f as the Vijayasundaram numerical flux consistent with the fluxes \mathbf{f}_s , $s = 1, 2$, cf. [11]. Taking into account (3) we define the “positive” and “negative” parts of the matrix \mathbb{P} as

$$\mathbb{P}^\pm = \mathbb{T}\Lambda^\pm\mathbb{T}^{-1},$$

where

$$\Lambda^\pm = \text{diag}(\lambda_1^\pm, \dots, \lambda_4^\pm),$$

and $\lambda^+ = \max(\lambda, 0)$, $\lambda^- = \min(\lambda, 0)$. Using the above concepts, we introduce the Vijayasundaram numerical flux

$$\mathbf{H}_f(\mathbf{w}^{(L)}, \mathbf{w}^{(R)}, \mathbf{n}) = \mathbb{P}^+(\langle \mathbf{w} \rangle, \mathbf{n}) \mathbf{w}^{(L)} + \mathbb{P}^-(\langle \mathbf{w} \rangle, \mathbf{n}) \mathbf{w}^{(R)}.$$

Finally we can define the discrete forms defining the discrete form of formulation 1:

$$\begin{aligned} \left(\frac{D^A \mathbf{w}_h}{Dt}, \boldsymbol{\varphi}_h \right)_{ht} &= \int_{\Omega_{ht}} \frac{D^A \mathbf{w}_h}{Dt} \cdot \boldsymbol{\varphi}_h \, d\mathbf{x}, \\ \tilde{b}_h^{(1)}(\mathbf{w}_h, \boldsymbol{\varphi}_h) &= - \sum_{K \in \mathcal{T}_{ht}} \int_K \sum_{s=1}^2 \mathbf{f}_s(\mathbf{w}_h) \cdot \frac{\partial \boldsymbol{\varphi}}{\partial x_s} \, d\mathbf{x} \\ &\quad + \sum_{\Gamma \in \mathcal{F}_{ht}} \int_{\Gamma} \mathbf{H}_f(\mathbf{w}_h|_{\Gamma}^{(L)}, \mathbf{w}_h|_{\Gamma}^{(R)}, \mathbf{n}_{ij}) \cdot [\boldsymbol{\varphi}_h] \, dS, \\ d_h^{(1)}(\mathbf{w}_h, \boldsymbol{\varphi}_h) &= - \sum_{K \in \mathcal{T}_{ht}} \int_K \sum_{s=1}^2 z_s \frac{\partial \mathbf{w}}{\partial x_s} \cdot \boldsymbol{\varphi} \, d\mathbf{x}. \end{aligned}$$

3.3.2. Formulation 2

We proceed similarly as in the preceding section. We multiply (11) by a test function $\boldsymbol{\varphi} \in \mathbf{S}_{ht}$, integrate over any element K , apply Green’s theorem and sum over all $K \in \mathcal{T}_{ht}$. We get the relation

$$\begin{aligned} \sum_{K \in \mathcal{T}_{ht}} \int_K \frac{D^A \mathbf{w}(t)}{Dt} \cdot \boldsymbol{\varphi} \, d\mathbf{x} &= \sum_{K \in \mathcal{T}_{ht}} \int_K \sum_{s=1}^2 \mathbf{g}_s(\mathbf{w}(t)) \cdot \frac{\partial \boldsymbol{\varphi}}{\partial x_s} \, d\mathbf{x} \\ &\quad - \sum_{\Gamma \in \mathcal{F}_{ht}} \int_{\Gamma} \sum_{s=1}^2 \mathbf{g}_s(\mathbf{w}(t)) n_s \cdot [\boldsymbol{\varphi}] \, dS - \sum_{K \in \mathcal{T}_{ht}} \int_K \text{div} \mathbf{z}(\mathbf{w} \cdot \boldsymbol{\varphi}) \, d\mathbf{x}. \end{aligned}$$

Now the exact solution $\mathbf{w}(t)$ is approximated by an element $\mathbf{w}_h(t) \in \mathbf{S}_{ht}$ and the fluxes through the faces Γ are approximated with the aid of a numerical flux $\mathbf{H}_g = \mathbf{H}_g(\mathbf{u}, \mathbf{w}, \mathbf{n})$. It means that on edge Γ

$$\sum_{s=1}^2 \mathbf{g}_s(\mathbf{w}(t)) n_s \approx \mathbf{H}_g(\mathbf{w}_h|_{\Gamma}^{(L)}(t), \mathbf{w}_h|_{\Gamma}^{(R)}(t), \mathbf{n}). \quad (12)$$

Here \mathbf{H}_g is an analogy to the Vijayasundaram numerical flux consistent with the fluxes \mathbf{g}_s , $s = 1, 2$. We have

$$\frac{D\mathbf{g}_s(\mathbf{w})}{D\mathbf{w}} = \frac{D\mathbf{f}_s(\mathbf{w})}{D\mathbf{w}} - z_s\mathbb{I} = \mathbf{A}_s - z_s\mathbb{I}$$

and can write

$$\tilde{\mathbb{P}}(\mathbf{w}, \mathbf{n}) = \sum_{s=1}^2 \frac{D\mathbf{g}_s(\mathbf{w})}{D\mathbf{w}} n_s = \sum_{s=1}^2 (\mathbf{A}_s n_s - z_s n_s \mathbb{I}) = \mathbb{P}(\mathbf{w}, \mathbf{n}) - (\mathbf{z} \cdot \mathbf{n})\mathbb{I}.$$

This, (6), (7) and (8) imply that

$$\tilde{\mathbb{P}} = \mathbb{T}\tilde{\Lambda}\mathbb{T}^{-1}, \quad \tilde{\Lambda} = \text{diag}(\tilde{\lambda}_1, \dots, \tilde{\lambda}_4), \quad \tilde{\lambda}_i = \lambda_i - \mathbf{z} \cdot \mathbf{n}, \quad i = 1, \dots, 4.$$

Now we define the “positive” and “negative” parts of the matrix $\tilde{\mathbb{P}}$ as

$$\tilde{\mathbb{P}}^\pm = \mathbb{T}\tilde{\Lambda}^\pm\mathbb{T}^{-1}$$

and we introduce the modification of the Vijayasundaram numerical flux

$$\mathbf{H}_g(\mathbf{w}^{(L)}, \mathbf{w}^{(R)}, \mathbf{n}) = \tilde{\mathbb{P}}^+ (\langle \mathbf{w} \rangle, \mathbf{n}) \mathbf{w}^{(L)} + \tilde{\mathbb{P}}^- (\langle \mathbf{w} \rangle, \mathbf{n}) \mathbf{w}^{(R)}.$$

Finally we can define the discrete forms of formulation 2:

$$\begin{aligned} \tilde{b}_h^{(2)}(\mathbf{w}_h, \boldsymbol{\varphi}_h) &= - \sum_{K \in \mathcal{T}_{ht}} \int_K \sum_{s=1}^2 \mathbf{g}_s(\mathbf{w}_h) \cdot \frac{\partial \boldsymbol{\varphi}}{\partial x_s} \, d\mathbf{x} \\ &\quad + \sum_{\Gamma \in \mathcal{F}_{ht}} \int_{\Gamma} \mathbf{H}_g(\mathbf{w}_h|_{\Gamma}^{(L)}, \mathbf{w}_h|_{\Gamma}^{(R)}, \mathbf{n}_{ij}) \cdot [\boldsymbol{\varphi}_h] \, dS, \\ d_h^{(2)}(\mathbf{w}_h, \boldsymbol{\varphi}_h) &= - \sum_{K \in \mathcal{T}_{ht}} \int_K \text{div} \mathbf{z} (\mathbf{w}_h \cdot \boldsymbol{\varphi}) \, d\mathbf{x}. \end{aligned}$$

Finally we define an *approximate solution* of (10) and (11), respectively, as a function $\mathbf{w}_h = \mathbf{w}_h(t)$ satisfying the conditions

$$\begin{aligned} \text{(a)} \quad & \mathbf{w}_h(t) \in \mathcal{S}_{ht}, \quad \forall t \in [0, T], \\ \text{(b)} \quad & \left(\frac{D^{\mathcal{A}} \mathbf{w}_h(t)}{Dt}, \boldsymbol{\varphi}_h \right)_h + \tilde{b}_h(\mathbf{w}_h(t), \boldsymbol{\varphi}_h) - d_h(\mathbf{w}_h(t), \boldsymbol{\varphi}_h) = 0, \quad (13) \\ & \forall \boldsymbol{\varphi}_h \in \mathcal{S}_{ht}, \quad \forall t \in (0, T), \\ \text{(c)} \quad & \mathbf{w}_h(0) = \Pi_h \mathbf{w}^0, \end{aligned}$$

where $\Pi_h \mathbf{w}^0$ is the $L^2(\Omega_0)$ -projection of \mathbf{w}^0 from the initial condition (2) on the space \mathcal{S}_{h0} . Relation (13) represents a discrete formulation of (10) and (11), when setting $\tilde{b}_h := \tilde{b}_h^{(1)}$, $d_h := d_h^{(1)}$ and $\tilde{b}_h := \tilde{b}_h^{(2)}$, $d_h := d_h^{(2)}$, respectively.

3.4. Time discretization

In this section we shall introduce the time discretization of problem (13). Due to the similarity of the two formulations we treat here only the second formulation. The discrete problem (13) is equivalent to a large system of ordinary differential equations. In order to avoid a CFL-stability condition we apply a semi-implicit scheme, which is a generalization of the techniques proposed in [4] and [7].

We introduce the partition $0 = t_0 < t_1 < \dots$ of the time interval $[0, T]$ and set $\tau_j = t_{j+1} - t_j$. The function $\mathbf{w}_h(\cdot, t_j)$ will be approximated by \mathbf{w}^j , defined in Ω_{t_j} . Let us assume that the approximate solution \mathbf{w}_h^j has already been computed for $j = 0, \dots, k$. We are interested in the computation of the approximate solution \mathbf{w}_h^{k+1} at time instant t_{k+1} . If we set

$$\hat{\mathbf{w}}_h^j(\mathbf{x}) = \mathbf{w}^j \left(\mathcal{A}_{t_j} \left(\mathcal{A}_{t_{k+1}}^{-1} \right) (\mathbf{x}) \right), \quad \mathbf{x} \in \Omega_{ht_{k+1}},$$

then, on the basis of (9), we can approximate the ALE derivative using the first order backward difference:

$$\left(\frac{D^{\mathcal{A}} \mathbf{w}_h(\mathbf{x}, t)}{Dt}, \boldsymbol{\varphi}_h \right) \Big|_{t_{k+1}} \approx \left(\frac{\mathbf{w}^{k+1}(\mathbf{x}) - \hat{\mathbf{w}}_h^k(\mathbf{x})}{\tau_k}, \boldsymbol{\varphi}_h \right), \quad \mathbf{x} \in \Omega_{ht_{k+1}}.$$

Further, on the basis of (4), (5) and the definition of the modified Vijayasundaram numerical flux we define a partial linearization $b_h^{(2)}$ of the form $\tilde{b}_h^{(2)}$:

$$\begin{aligned} b_h^{(2)}(\hat{\mathbf{w}}_h^k, \mathbf{w}_h^{k+1}, \boldsymbol{\varphi}_h) &= - \sum_{K \in \mathcal{T}_{ht_{k+1}}} \int_K \sum_{s=1}^2 (\mathbf{A}_s(\hat{\mathbf{w}}^k) - z_s) \mathbb{I} \mathbf{w}^{k+1} \cdot \frac{\partial \boldsymbol{\varphi}_h}{\partial x_s} \, d\mathbf{x} \quad (14) \\ &+ \sum_{\Gamma \in \mathcal{F}_{ht_{k+1}}} \int_{\Gamma} \left[\tilde{\mathbb{P}}^+ (\langle \hat{\mathbf{w}}_h^k \rangle, \mathbf{n}_{ij}) \mathbf{w}_h^{k+1} |_{\Gamma_{ij}} + \tilde{\mathbb{P}}^- (\langle \hat{\mathbf{w}}_h^k \rangle, \mathbf{n}_{ij}) \mathbf{w}_h^{k+1} |_{\Gamma_{ji}} \right] \cdot \boldsymbol{\varphi}_h \, dS. \end{aligned}$$

The term $d_h^{(2)}$ linear with respect to \mathbf{w}_h will be treated implicitly.

These considerations lead us to the following *semi-implicit scheme*: For $k = 0, 1, \dots$ find \mathbf{w}_h^{k+1} such that

$$\begin{aligned} \text{(a)} \quad & \mathbf{w}_h^{k+1} \in \mathcal{S}_{ht_{k+1}}, \\ \text{(b)} \quad & \left(\frac{\mathbf{w}_h^{k+1} - \hat{\mathbf{w}}_h^k}{\tau_k}, \boldsymbol{\varphi}_h \right) + b_h^{(2)}(\hat{\mathbf{w}}_h^k, \mathbf{w}_h^{k+1}, \boldsymbol{\varphi}_h) - d_h^{(2)}(\mathbf{w}_h^{k+1}, \boldsymbol{\varphi}_h) = 0, \\ & \forall \boldsymbol{\varphi}_h \in \mathcal{S}_{ht_{k+1}}, \\ \text{(c)} \quad & \mathbf{w}_h^0 = \Pi_h \mathbf{w}^0. \end{aligned}$$

This relation represents a system of linear algebraic equations on each time level which is solved either iteratively using the block-Jacobi preconditioned GMRES or a direct method (e.g. the direct unsymmetric multifrontal solver UMFPACK cf. [3]).

4. Boundary conditions

If $\Gamma \subset \partial\Omega_{ht}$, it is necessary to specify the boundary state $\mathbf{w}|_{\Gamma}^{(R)}$ appearing in the numerical flux \mathbf{H} in the definition of the inviscid form b_h .

On the inlet and outlet, which are fixed, we proceed in the same way as in [7], Section 4, where we prescribe the state $\mathbf{w}|_{\Gamma}^{(R)}$ in such a way that the locally linearized Euler equations are well posed. On the impermeable moving wall we prescribe the normal component of the velocity

$$\mathbf{v} \cdot \mathbf{n} = \mathbf{z} \cdot \mathbf{n}, \quad (15)$$

where \mathbf{n} is unit outer normal to Γ_{W_t} and \mathbf{z} is the wall velocity. This is done by prescribing the numerical flux \mathbf{H} on Γ_{W_t} . Again we shall treat only formulation 2. We define the numerical flux as the physical flux through the boundary with the assumption (15) taken into account. We write:

$$\sum_{s=1}^2 \mathbf{g}_s(\mathbf{w}) n_s = (\mathbf{v} \cdot \mathbf{n} - \mathbf{z} \cdot \mathbf{n}) \mathbf{w} + p(0, n_1, n_2, \mathbf{v} \cdot \mathbf{n})^T = p(0, n_1, n_2, \mathbf{v} \cdot \mathbf{n})^T =: \mathbf{H}_g$$

on Γ_{W_t} . We proceed similarly in formulation 1.

5. Limiting procedure at discontinuities

For high speed flows with shock waves and contact discontinuities it is necessary to avoid the Gibbs phenomenon manifested by spurious overshoots and undershoots in computed quantities near discontinuities. In order to avoid the Gibbs phenomenon, we apply the limiting procedure from [7] based on the discontinuity indicator

$$g^k(K) = \int_{\partial K} [\hat{\rho}_h^k]^2 dS / (h_K |K|^{3/4}), \quad K \in \mathcal{T}_{ht_{k+1}},$$

introduced in [5]. The density $\hat{\rho}_h^k$ represents the first component of the state vector $\hat{\mathbf{w}}_h^k$. Then we define the discrete discontinuity indicator

$$G^k(K) = \begin{cases} 0 & \text{if } g^k(K) < 1, \\ 1 & \text{if } g^k(K) \geq 1, \end{cases}$$

and the artificial viscosity forms

$$\beta_h(\hat{\mathbf{w}}_h^k, \mathbf{w}_h^{k+1}, \boldsymbol{\varphi}_h) = \nu_1 \sum_{K \in \mathcal{T}_{ht}} h_K G^k(K) \int_K \nabla \mathbf{w}_h^{k+1} \cdot \nabla \boldsymbol{\varphi}_h d\mathbf{x},$$

and

$$J_h(\hat{\mathbf{w}}_h^k, \mathbf{w}_h^{k+1}, \boldsymbol{\varphi}_h) = \nu_2 \sum_{\Gamma \in \mathcal{F}_{ht}} \frac{1}{2} (G^k(K) + G^k(K)) \int_{\Gamma} [\mathbf{w}_h^{k+1}] \cdot [\boldsymbol{\varphi}_h] dS,$$

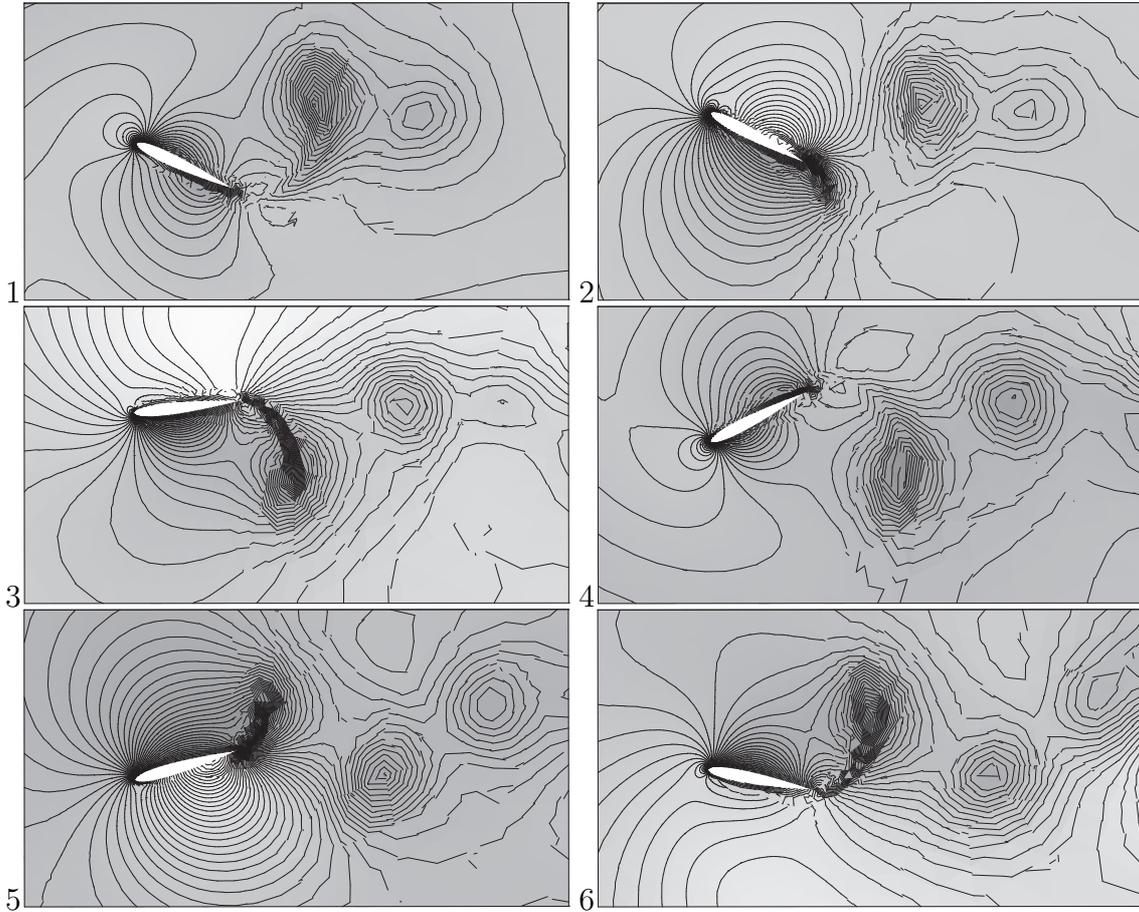


Fig. 2: Density isolines for the periodically oscillating NACA0012 profile.

with $\nu_1, \nu_2 = O(1)$. Then the resulting scheme obtained by limiting of (14), (b) has the form

$$\begin{aligned}
 & \text{(a)} \quad \mathbf{w}_h^{k+1} \in \mathcal{S}_{ht_{k+1}}, \\
 & \text{(b)} \quad \left(\frac{\mathbf{w}_h^{k+1} - \hat{\mathbf{w}}_h^k}{\tau_k}, \boldsymbol{\varphi}_h \right)_h + b_h(\hat{\mathbf{w}}_h^k, \mathbf{w}_h^{k+1}, \boldsymbol{\varphi}_h) + \beta_h(\hat{\mathbf{w}}_h^k, \mathbf{w}_h^{k+1}, \boldsymbol{\varphi}_h) \\
 & \quad + J_h(\hat{\mathbf{w}}_h^k, \mathbf{w}_h^{k+1}, \boldsymbol{\varphi}_h) = 0, \quad \forall \boldsymbol{\varphi}_h \in \mathcal{S}_{ht_{k+1}}, \quad k = 0, 1, \dots, \\
 & \text{(c)} \quad \mathbf{w}_h^0 = \Pi_h \mathbf{w}^0.
 \end{aligned}$$

Remark. In practical computations, the integrals appearing in the definition of the approximated solution are evaluated with the aid of quadrature formulae. In order to obtain an accurate, physically admissible solution, it is necessary to use isoparametric elements near curved boundaries (see [1] or [6], Section 4.6.8). In our computations we proceed in such a way that a reference triangle is transformed

by a bilinear mapping onto the approximation of a curved triangle adjacent to the boundary $\partial\Omega$.

6. Numerical experiments

We consider inviscid compressible flow around the NACA0012 profile, which is periodically moving in the vertical direction with a periodically varying angle of attack. Figure 2 shows density contours plotted at six time frames distributed over one period of the flow. The presented plots represent a part of the computational domain, which is chosen large in order to eliminate the role of artificial boundaries.

The ALE mapping is defined using two concentric circles with a center at the center of gravity of the profile. Outside the outer circle the ALE mapping is chosen as the identity mapping, i.e. no motion of the computational domain takes place. Inside the inner circle the ALE mapping is defined so that the motion of the computational domain coincides with the prescribed movement of the profile. In the space between the two circles the movement of the ALE mapping is linearly interpolated to yield a globally regular mapping.

In the solution we can observe the formation of vortices in the wake of the airfoil due to the vibrating motion, which introduces vorticity into the fluid flow. These vortices are then convected out of the computational domain. The Mach number at infinity of the flow is $M_\infty = 0.1$.

References

- [1] F. Bassi, S. Rebay: *High-order accurate discontinuous finite element solution of the 2D Euler equations*, J. Comput. Phys. **138** (1997), 251–285.
- [2] B. Cockburn, G.E. Karniadakis, C.-W. Shu (Eds.): *Discontinuous Galerkin methods*, Lecture Notes in Computational Science and Engineering 11, Springer, Berlin (2000).
- [3] T.A. Davis, I.S. Duff: *A combined unifrontal/multifrontal method for unsymmetric sparse matrices*, ACM Trans. Math. Software, **25** (1999), 1–19.
- [4] V. Dolejší, M. Feistauer: *A Semi-implicit discontinuous Galerkin finite element method for the numerical solution of inviscid compressible flow*, J. Comput. Phys. **198** (2004), 727–746.
- [5] V. Dolejší, M. Feistauer, C. Schwab: *On some aspects of the discontinuous Galerkin finite element method for conservation laws*. Math. Comput. Simul. **61** (2003), 333–346.
- [6] M. Feistauer, J. Felcman, I. Straškaraba: *Mathematical and computational methods for compressible flow*, Clarendon Press, Oxford (2003).

- [7] M. Feistauer, V. Kučera: *On a robust discontinuous Galerkin technique for the solution of compressible flow*, J. Comput. Phys. **224** (2007), 208–221.
- [8] T. Nomura, T.J.R. Hughes: *An arbitrary Lagrangian-Eulerian finite element method for interaction of fluid and a rigid body*, Comput. Methods Appl. Mech. Engrg. **95** (1992), 115–138.
- [9] P. Sváček, M. Feistauer, J. Horáček: *Numerical simulation of flow induced airfoil vibrations with large amplitudes*, J. Fluids Struct. **23** (2007), 391–411.
- [10] J.J.W. van der Vegt, H. van der Ven: *Space-time discontinuous Galerkin finite element method with dynamic grid motion for inviscid compressible flow*, J. Comput. Phys. **182** (2002), 546–585.
- [11] G. Vijayasundaram: *Transonic flow simulation using upstream centered scheme of Godunov type in finite elements*, J. Comput. Phys. **63** (1986), 416–433.