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On uniqueness of a weak solution to the steady Navier-Stokes problem in a profile cascade with a nonlinear boundary condition on the outflow

In: Jan Chleboun and Petr Přikryl and Karel Segeth and Tomáš Vejchodský (eds.): Programs and Algorithms of Numerical Mathematics, Proceedings of Seminar. Dolní Maxov, June 1-6, 2008. Institute of Mathematics AS CR, Prague, 2008. pp. 164–169.

Persistent URL: http://dml.cz/dmlcz/702870

ON UNIQUENESS OF A WEAK SOLUTION TO THE STEADY NAVIER-STOKES PROBLEM IN A PROFILE CASCADE WITH A NONLINEAR BOUNDARY CONDITION ON THE OUTFLOW*

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1. Introduction

The paper deals with theoretical analysis of the mathematical model of viscous incompressible stationary flow through a plane cascade of profiles. The considered fluid moves around an infinite row of profiles which periodically repeat in one spatial direction. This property enables us to reduce the problem to a bounded domain which represents just one spatial period. We assume that the velocity satisfies the Dirichlet boundary condition on the inflow and on the profile, a certain "natural" nonlinear boundary condition of the "do nothing-type" on the outflow and periodic boundary conditions on the artificial boundaries which separate the chosen spatial period from other periods. We present the weak formulation of the problem and recall the theorem on the existence of a weak solution. Afterwards, we study the question of uniqueness of the weak solution. We arrive at a theorem stating that the solution is unique if the data prescribed on the boundary and the external specific body force are in certain norms "sufficiently small" with respect to the viscosity. This result is in agreement with known theorems on uniqueness in the case of the Dirichlet boundary condition on the whole boundary, see e.g. the books on the Navier-Stokes equations by R. Temam and G. P. Galdi.

2. The geometry of the problem

The considered 2D domain Ω, representing one spatial period of the flow field around the infinite and unbounded series of profiles, is sketched on Fig. 1. Its boundary consists of the curves Γ_i (the inflow), Γ_w (the surface of the profile), Γ_- and Γ_+ (the lower and the upper artificial boundaries), and Γ_o (the outflow). The reduction of the original problem for an infinite profile cascade to just one spatial period is standard and the main idea standing in the background is that the weak solution constructed in one spatial period Ω, periodically extended in the direction of the x_2-axis, becomes a solution for the whole profile cascade, see [2] for details.

*The research was supported by the research plan of the Ministry of Education of the Czech Republic No. MSM 6840770010.

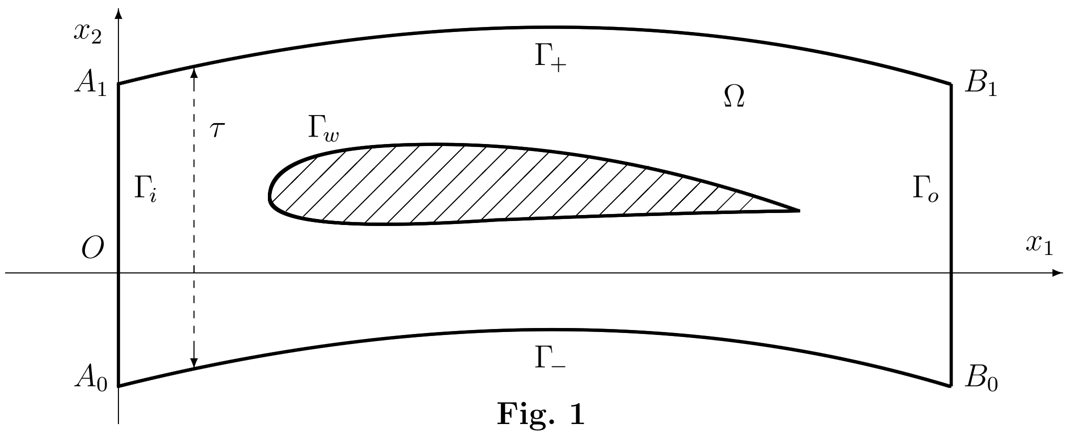

Fig. 1

3. The classical formulation of the boundary-value problem

We denote by $\boldsymbol{u} = (u_1, u_2)$ the (fluid) velocity, by p the kinematic pressure and by $\boldsymbol{n}$ the outer normal vector on the boundary. We study the flow described by 2D steady Navier-Stokes equation

$$(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = \boldsymbol{f} - \nabla p + \nu\, \Delta \boldsymbol{u}. \tag{1}$$

The condition of incompressibility is

$$\operatorname{div} \boldsymbol{u} = 0. \tag{2}$$

We prescribe the inhomogeneous Dirichlet boundary condition on the inlet and the homogeneous no-slip Dirichlet boundary condition on the profile

$$\boldsymbol{u} = \boldsymbol{g} \qquad \text{on } \Gamma_i, \tag{3}$$

$$\boldsymbol{u} = \boldsymbol{0} \qquad \text{on } \Gamma_w. \tag{4}$$

We suppose that the conditions of periodicity are fulfilled on the artificial boundaries Γ_- and Γ_+

$$\boldsymbol{u}(x_1, x_2 + \tau) = \boldsymbol{u}(x_1, x_2) \qquad \text{for } (x_1, x_2) \in \Gamma_-, \tag{5}$$

$$\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}}(x_1, x_2 + \tau) = -\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}}(x_1, x_2) \qquad \text{for } (x_1, x_2) \in \Gamma_-, \tag{6}$$

$$p(x_1, x_2 + \tau) = p(x_1, x_2) \qquad \text{for } (x_1, x_2) \in \Gamma_-. \tag{7}$$

We use the nonlinear do-nothing-type boundary condition (proposed by Bruneau and Fabri in [1]) on the outflow Γ_o

$$-\nu \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}} + p\, \boldsymbol{n} - \frac{1}{2}\, (\boldsymbol{u} \cdot \boldsymbol{n})^- \, \boldsymbol{u} = \boldsymbol{h}. \tag{8}$$

4. The weak formulation of the boundary-value problem and the theorem on existence

We denote by $H^1(\Omega)$ the usual Sobolev space and by $H^1(\Omega)^2 := \big[H^1(\Omega)\big]^2$ the space of two component vector functions with both components in $H^1(\Omega)$. We define

$$\begin{aligned} V \;\; := \;\; & \big\{ \boldsymbol{v} \in H^1(\Omega)^2; \;\; \boldsymbol{v} = \boldsymbol{0} \;\text{ a.e. in } \Gamma_i \cup \Gamma_w, \;\; \boldsymbol{v}(x_1, x_2 + \tau) = \boldsymbol{v}(x_1, x_2) \\ & \text{for a.a. } \;(x_1, x_2) \in \Gamma_-, \;\text{ and }\; \operatorname{div} \boldsymbol{v} = 0 \;\text{ a.e. in } \Omega \big\}. \end{aligned}$$

V is equipped with the norm $|||\boldsymbol{v}||| := \|\nabla \boldsymbol{v}\|_{L^2(\Omega)^4}$, which is equivalent to the norm in $H^1(\Omega)^2$.

In order to realize the inhomogeneous boundary condition (3), we introduce an auxiliary function $\boldsymbol{g}^*$:

Lemma 4.1 *Let $s \in (\frac{1}{2}, 1]$, let function $\boldsymbol{g}$ belong to the Sobolev-Slobodetskiĭ space $H^s(\Gamma_i)^2$, and let $\boldsymbol{g}(A_1) = \boldsymbol{g}(A_0)$ (where A_0 and A_1 are the end points of Γ_i). Then there exists a constant $c_g > 0$ independent of $\boldsymbol{g}$ and a divergence-free extension $\boldsymbol{g}^* \in H^1(\Omega)^2$ of function $\boldsymbol{g}$ from Γ_i onto Ω such that $\boldsymbol{g}^* = \boldsymbol{0}$ on Γ_w, $\boldsymbol{g}^*$ satisfies the condition of periodicity*

$$\boldsymbol{g}^*(x_1, x_2 + \tau) \; = \; \boldsymbol{g}^*(x_1, x_2) \qquad \textit{for } (x_1, x_2) \in \Gamma_- \tag{9}$$

and the estimate

$$\|\boldsymbol{g}^*\|_{H^1(\Omega)^2} \; \leq \; c_g \, \|\boldsymbol{g}\|_{H^s(\Gamma_i)^2}. \tag{10}$$

The lemma is proved in [2].

The weak solution $\boldsymbol{u}$ of the problem (1)–(8) can be constructed in the form $\boldsymbol{u} = \boldsymbol{g}^* + \boldsymbol{z}$ where $\boldsymbol{z} \in V$ is a new unknown function. This form of $\boldsymbol{u}$ guarantees that $\boldsymbol{u}$ satisfies the boundary condition (3) on the part Γ_i of $\partial\Omega$.

In order to arrive formally at the weak formulation of the problem (1)–(8), we multiply equation (1) by an arbitrary test function $\boldsymbol{v} = (v_1, v_2) \in V$, integrate over Ω, apply Green's theorem, and use the condition of incompressibility (2), the boundary condition (4), the conditions of periodicity (5)–(7), and the nonlinear condition (8). We obtain an equation, which can be written down in the form

$$a(\boldsymbol{u}, \boldsymbol{v}) \; = \; (\boldsymbol{f}, \boldsymbol{v}) \, + \, b(\boldsymbol{h}, \boldsymbol{v}), \tag{11}$$

with $a(\boldsymbol{u}, \boldsymbol{v}) := a_1(\boldsymbol{u}, \boldsymbol{v}) + a_2(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) + a_3(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v})$ and $b(\boldsymbol{h}, \boldsymbol{v}) := -\int_{\Gamma_o} \boldsymbol{h} \cdot \boldsymbol{v} \; \mathrm{d}S$, where

$$a_1(\boldsymbol{u}, \boldsymbol{v}) := \nu \int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{v} \; \mathrm{d}\boldsymbol{x}, \qquad a_2(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) := \int_\Omega \boldsymbol{u} \cdot \nabla \boldsymbol{v} \cdot \boldsymbol{w} \; \mathrm{d}\boldsymbol{x},$$

$$a_3(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) := \int_{\Gamma_o} \frac{1}{2} \, (\boldsymbol{u} \cdot \boldsymbol{n})^- \, \boldsymbol{v} \cdot \boldsymbol{w} \; \mathrm{d}S, \qquad (\boldsymbol{f}, \boldsymbol{v}) := \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \; \mathrm{d}\boldsymbol{x}.$$

All these forms are defined for $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w} \in V$, $\boldsymbol{f} \in L^2(\Omega)^2$ and $\boldsymbol{h} \in L^2(\Gamma_o)^2$. Thus, we arrive at the definition:

Definition 4.2 *Let $\boldsymbol{g} \in H^s(\Gamma_i)^2$ (for some $s \in (\frac{1}{2}, 1]$) satisfy the condition $\boldsymbol{g}(A_1) = \boldsymbol{g}(A_0)$. Let $\boldsymbol{f} \in L^2(\Omega)^2$ and $\boldsymbol{h} \in L^2(\Gamma_o)^2$. The* **weak solution** *of the problem (1)–(8) is the vector function $\boldsymbol{u} := \boldsymbol{g}^* + \boldsymbol{z}$, where $\boldsymbol{z} \in V$ and $\boldsymbol{u}$ satisfies the identity (11) for all test functions $\boldsymbol{v} \in V$.*

The next theorem brings the information on the existence of a weak solution $\boldsymbol{u}$.

Theorem 4.3 *There exists $\varepsilon > 0$ such that if $\|\boldsymbol{g}\|_{H^s(\Gamma_i)^2} < \varepsilon$ then there exists a solution $\boldsymbol{u} = \boldsymbol{g}^* + \boldsymbol{z}$ of the problem defined in Definition 4.2. Moreover, $\boldsymbol{z}$ satisfies the estimate*

$$|||\boldsymbol{z}||| \;\leq\; R_1, \tag{12}$$

where $R_1 = R_1\big(\nu, \|\boldsymbol{g}\|_{H^s(\Gamma_i)^2}, \|\boldsymbol{f}\|_{L^2(\Omega)^2}, \|\boldsymbol{h}\|_{L^2(\Gamma_o)^2}\big)$. Consequently, $\boldsymbol{u}$ satisfies

$$\|\nabla \boldsymbol{u}\|_{L^2(\Omega)^4} \;\leq\; R_1 + \|\nabla \boldsymbol{g}^*\|_{L^2(\Omega)^4} \;\leq\; R_1 + c_g\, \|\boldsymbol{g}\|_{H^s(\Gamma_i)^2} \;=: R_2. \tag{13}$$

The proof can be found in [2]. The function $\boldsymbol{u}$ has the form $\boldsymbol{g}^* + \boldsymbol{z}$, where $\boldsymbol{z} \in V$ satisfies

$$a(\boldsymbol{g}^* + \boldsymbol{z}, \boldsymbol{v}) \;=\; (\boldsymbol{f}, \boldsymbol{v}) \;+\; b(\boldsymbol{h}, \boldsymbol{v})$$

for all $\boldsymbol{v} \in V$. The function $\boldsymbol{z}$ was constructed as a limit of an appropriate sequence of Galerkin approximations. We were able to find an explicit form of the dependence of R_1 on ν, $\|\boldsymbol{g}\|_{H^s(\Gamma_i)^2}$, $\|\boldsymbol{f}\|_{L^2(\Omega)^2}$ and $\|\boldsymbol{h}\|_{L^2(\Gamma_o)^2}$ in [2]. The restriction $\|\boldsymbol{g}\|_{H^s(\Gamma_i)^2} < \varepsilon$ follows from the requirement that the form a is coercive.

5. Uniqueness of the weak solution of the problem (1)–(8)

The next theorem presents the main result of this paper. It says that the weak solution $\boldsymbol{u}$ of the problem (1)–(8) is unique in a certain sufficiently small ball.

Theorem 5.1 (on uniqueness of the weak solution) *There exists $R > 0$ such that if $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are two weak solutions of the problem (1)–(8) defined in Definition 4.2 such that $\|\nabla \boldsymbol{u}_1\|_{L^2(\Omega)^4} \leq R$ and $\|\nabla \boldsymbol{u}_2\|_{L^2(\Omega)^4} \leq R$, then $\boldsymbol{u}_1 = \boldsymbol{u}_2$.*

Proof. Since $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are weak solutions of problem (1)–(8), they satisfy the equations

$$\begin{aligned} a(\boldsymbol{u}_1, \boldsymbol{v}) \;&=\; (\boldsymbol{f}, \boldsymbol{v}) \;+\; b(\boldsymbol{h}, \boldsymbol{v}), \\ a(\boldsymbol{u}_2, \boldsymbol{v}) \;&=\; (\boldsymbol{f}, \boldsymbol{v}) \;+\; b(\boldsymbol{h}, \boldsymbol{v}) \end{aligned}$$

for all $\boldsymbol{v} \in V$. Subtracting these equations, we get $a(\boldsymbol{u}_1, \boldsymbol{v}) - a(\boldsymbol{u}_2, \boldsymbol{v}) = 0$. Expressing the bilinear form a by means of forms a_1, a_2 and a_3 with $\boldsymbol{v} = \boldsymbol{u}_1 - \boldsymbol{u}_2$, we obtain

$$\begin{aligned} a_1(\boldsymbol{u}_1 - \boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2) + a_2(\boldsymbol{u}_1, \boldsymbol{u}_1, \boldsymbol{u}_1 - \boldsymbol{u}_2) - a_2(\boldsymbol{u}_2, \boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2) \\ + a_3(\boldsymbol{u}_1, \boldsymbol{u}_1, \boldsymbol{u}_1 - \boldsymbol{u}_2) - a_3(\boldsymbol{u}_2, \boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2) \;=\; 0. \end{aligned} \tag{14}$$

If we denote

$$
\begin{aligned}
I_1 &:= a_1(\boldsymbol{u}_1 - \boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2) \;=\; \nu \int_\Omega |\nabla(\boldsymbol{u}_1 - \boldsymbol{u}_2)|^2 \,\mathrm{d}\boldsymbol{x} \;=\; \nu\, |||\boldsymbol{u}_1 - \boldsymbol{u}_2|||^2,\\
I_2 &:= a_2(\boldsymbol{u}_1, \boldsymbol{u}_1, \boldsymbol{u}_1 - \boldsymbol{u}_2) - a_2(\boldsymbol{u}_2, \boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2),\\
I_3 &:= a_3(\boldsymbol{u}_1, \boldsymbol{u}_1, \boldsymbol{u}_1 - \boldsymbol{u}_2) - a_3(\boldsymbol{u}_2, \boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2),
\end{aligned}
$$

then (14) takes the form

$$
I_1 \;=\; -I_2 \;-\; I_3. \tag{15}
$$

Further, we estimate the terms on the right-hand side of (15).

$$
\begin{aligned}
|I_2| \;&=\; \left| \int_\Omega \boldsymbol{u}_1 \cdot \nabla \boldsymbol{u}_1 \cdot (\boldsymbol{u}_1 - \boldsymbol{u}_2)\,\mathrm{d}\boldsymbol{x} \;-\; \int_\Omega \boldsymbol{u}_2 \cdot \nabla \boldsymbol{u}_2 \cdot (\boldsymbol{u}_1 - \boldsymbol{u}_2)\,\mathrm{d}\boldsymbol{x} \right|\\
&\le\; \left| \int_\Omega (\boldsymbol{u}_1 - \boldsymbol{u}_2) \cdot \nabla \boldsymbol{u}_1 \cdot (\boldsymbol{u}_1 - \boldsymbol{u}_2)\,\mathrm{d}\boldsymbol{x} \right| + \left| \int_\Omega \boldsymbol{u}_2 \cdot \nabla(\boldsymbol{u}_1 - \boldsymbol{u}_2) \cdot (\boldsymbol{u}_1 - \boldsymbol{u}_2)\,\mathrm{d}\boldsymbol{x} \right|\\
&\le\; \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|^2_{L^4(\Omega)^2}\, \|\nabla \boldsymbol{u}_1\|_{L^2(\Omega)^4} + \|\boldsymbol{u}_2\|_{L^4(\Omega)^2}\, \|\nabla(\boldsymbol{u}_1 - \boldsymbol{u}_2)\|_{L^2(\Omega)^4}\, \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{L^4(\Omega)^2}\\
&\le\; 2c_1^2\, R\, \|\nabla(\boldsymbol{u}_1 - \boldsymbol{u}_2)\|^2_{L^2(\Omega)^4} \;=\; 2c_1^2\, R\, |||\boldsymbol{u}_1 - \boldsymbol{u}_2|||^2,
\end{aligned} \tag{16}
$$

where the constant c_1 comes from the inequality $\|\boldsymbol{u}\|_{L^4(\Omega)^2} \;\le\; c_1\, \|\nabla \boldsymbol{u}\|_{L^2(\Omega)^4}$ (which can be found in [3]) for functions $\boldsymbol{u}$ from $H^1(\Omega)^2$. The term I_3 equals

$$
I_3 \;=\; \int_{\Gamma_o} \frac{1}{2}\,(\boldsymbol{u}_1 \cdot \boldsymbol{n})^- \,\boldsymbol{u}_1 \cdot (\boldsymbol{u}_1 - \boldsymbol{u}_2)\,\mathrm{d}S \;-\; \int_{\Gamma_o} \frac{1}{2}\,(\boldsymbol{u}_2 \cdot \boldsymbol{n})^- \,\boldsymbol{u}_2 \cdot (\boldsymbol{u}_1 - \boldsymbol{u}_2)\,\mathrm{d}S.
$$

According to the signs of $\boldsymbol{u}_1 \cdot \boldsymbol{n}$ and $\boldsymbol{u}_2 \cdot \boldsymbol{n}$ on Γ_o, we must split Γ_o into four parts $\Gamma_o = \Gamma_{o1} \cup \Gamma_{o2} \cup \Gamma_{o3} \cup \Gamma_{o4}$, where

(a) $\boldsymbol{u}_1 \cdot \boldsymbol{n} \ge 0$, $\boldsymbol{u}_2 \cdot \boldsymbol{n} \ge 0$, $(\boldsymbol{u}_1 \cdot \boldsymbol{n})^- = 0$, and $(\boldsymbol{u}_2 \cdot \boldsymbol{n})^- = 0$ on Γ_{o1};

(b) $\boldsymbol{u}_1 \cdot \boldsymbol{n} < 0$, $\boldsymbol{u}_2 \cdot \boldsymbol{n} \ge 0$, $(\boldsymbol{u}_1 \cdot \boldsymbol{n})^- = \boldsymbol{u}_1 \cdot \boldsymbol{n}$, and $(\boldsymbol{u}_2 \cdot \boldsymbol{n})^- = 0$ on Γ_{o2};

(c) $\boldsymbol{u}_1 \cdot \boldsymbol{n} \ge 0$, $\boldsymbol{u}_2 \cdot \boldsymbol{n} < 0$, $(\boldsymbol{u}_1 \cdot \boldsymbol{n})^- = 0$, and $(\boldsymbol{u}_2 \cdot \boldsymbol{n})^- = \boldsymbol{u}_2 \cdot \boldsymbol{n}$, on Γ_{o3};

(d) $\boldsymbol{u}_1 \cdot \boldsymbol{n} < 0$, $\boldsymbol{u}_2 \cdot \boldsymbol{n} < 0$, $(\boldsymbol{u}_1 \cdot \boldsymbol{n})^- = \boldsymbol{u}_1 \cdot \boldsymbol{n}$, and $(\boldsymbol{u}_2 \cdot \boldsymbol{n})^- = \boldsymbol{u}_2 \cdot \boldsymbol{n}$ on Γ_{o4}.

Let us denote by I_3^{o1}, I_3^{o2}, I_3^{o3}, and I_3^{o4} the same integrals as in I_3, but this time considered successively on Γ_{o1}, Γ_{o2}, Γ_{o3}, and Γ_{o4}. Obviously, $I_3^{o1} = 0$ because the integrands are equal to zero on Γ_{o1}. On Γ_{o2} we use the inequality $|\boldsymbol{u}_1 \cdot \boldsymbol{n}| \le |\boldsymbol{u}_1 \cdot \boldsymbol{n} - \boldsymbol{u}_2 \cdot \boldsymbol{n}|$, which holds because $\boldsymbol{u}_1 \cdot \boldsymbol{n} < 0$ and $\boldsymbol{u}_2 \cdot \boldsymbol{n} \ge 0$. We obtain

$$
\begin{aligned}
|I_3^{o2}| \;&=\; \left| \int_{\Gamma_{o2}} (\boldsymbol{u}_1 \cdot \boldsymbol{n})^- \boldsymbol{u}_1 \cdot (\boldsymbol{u}_1 - \boldsymbol{u}_2)\,\mathrm{d}S \right| \le \int_{\Gamma_{o2}} |\boldsymbol{u}_1 \cdot \boldsymbol{n} - \boldsymbol{u}_2 \cdot \boldsymbol{n}|\, |\boldsymbol{u}_1|\, |\boldsymbol{u}_1 - \boldsymbol{u}_2|\,\mathrm{d}S\\
&\le\; \int_{\Gamma_{o2}} |\boldsymbol{u}_1 - \boldsymbol{u}_2|^2\, |\boldsymbol{u}_1|\,\mathrm{d}S \le \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|^2_{L^4(\Gamma_{o2})^2}\, \|\boldsymbol{u}_1\|_{L^2(\Gamma_{o2})^2}\\
&\le\; \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|^2_{L^4(\Gamma_o)^2}\, \|\boldsymbol{u}_1\|_{L^2(\Gamma_o)^2} \le c_2\, \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|^2_{H^1(\Omega)^2}\, \|\boldsymbol{u}_1\|_{H^1(\Omega)^2}\\
&\le\; c_3\, |||\boldsymbol{u}_1 - \boldsymbol{u}_2|||^2\, \|\nabla \boldsymbol{u}_1\|_{L^2(\Omega)^4} \le c_3\, R\, |||\boldsymbol{u}_1 - \boldsymbol{u}_2|||^2,
\end{aligned} \tag{17}
$$

where the constants c_2 and c_3 come from the inequalities $\|\boldsymbol{u}\|_{L^2(\Gamma_o)^2} \leq c_2 \|\boldsymbol{u}\|_{H^1(\Omega)^2} \leq c_3 \|\nabla \boldsymbol{u}\|_{L^2(\Omega)^4}$ (which can be found in [3]) for functions $\boldsymbol{u}$ from $H^1(\Omega)^2$. The term I_3^{o3} can be estimated in the same way as I_3^{o2}. The term I_3^{o4} can be treated as follows

$$
\begin{aligned}
|I_3^{o4}| &= \left| \int_{\Gamma_{o4}} (\boldsymbol{u}_1 \cdot \boldsymbol{n})\, \boldsymbol{u}_1 \cdot (\boldsymbol{u}_1 - \boldsymbol{u}_2)\, \mathrm{d}S - \int_{\Gamma_{o4}} (\boldsymbol{u}_2 \cdot \boldsymbol{n})\, \boldsymbol{u}_2 \cdot (\boldsymbol{u}_1 - \boldsymbol{u}_2)\, \mathrm{d}S \right| \\
&= \left| \int_{\Gamma_{o4}} [(\boldsymbol{u}_1 - \boldsymbol{u}_2) \cdot \boldsymbol{n}]\ \boldsymbol{u}_1 \cdot (\boldsymbol{u}_1 - \boldsymbol{u}_2)\, \mathrm{d}S + \int_{\Gamma_{o4}} (\boldsymbol{u}_2 \cdot \boldsymbol{n})\ (\boldsymbol{u}_1 - \boldsymbol{u}_2) \cdot (\boldsymbol{u}_1 - \boldsymbol{u}_2)\, \mathrm{d}S \right| \\
&\leq \int_{\Gamma_{o4}} |\boldsymbol{u}_1 - \boldsymbol{u}_2|\, |\boldsymbol{u}_1|\, |\boldsymbol{u}_1 - \boldsymbol{u}_2|\, \mathrm{d}S + \int_{\Gamma_{o4}} |\boldsymbol{u}_2|\, |\boldsymbol{u}_1 - \boldsymbol{u}_2|\, |\boldsymbol{u}_1 - \boldsymbol{u}_2|\, \mathrm{d}S \\
&\leq \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|^2_{L^4(\Gamma_{o4})^2} \left(\|\boldsymbol{u}_1\|_{L^2(\Gamma_{o4})^2} + \|\boldsymbol{u}_2\|_{L^2(\Gamma_{o4})^2} \right) \\
&\leq \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|^2_{L^4(\Gamma_o)^2} \left(\|\boldsymbol{u}_1\|_{L^2(\Gamma_o)^2} + \|\boldsymbol{u}_2\|_{L^2(\Gamma_o)^2} \right) \\
&\leq c_4 \|\nabla \boldsymbol{u}_1 - \nabla \boldsymbol{u}_2\|_{L^2(\Omega)^4}\ c_3 \left(\|\nabla \boldsymbol{u}_1\|_{L^2(\Omega)^4} + \|\nabla \boldsymbol{u}_2\|_{L^2(\Omega)^4} \right) \leq 2c_4 \,|\!|\!| \boldsymbol{u}_1 - \boldsymbol{u}_2 |\!|\!|^2\ c_3\, R,
\end{aligned}
$$

where the constant c_4 comes from the inequality $\|\boldsymbol{u}\|_{L^4(\Gamma_o)^2} \leq c_4 \|\nabla \boldsymbol{u}\|_{L^2(\Omega)^4}$ (which can be found in [3]) for functions $\boldsymbol{u}$ from $H^1(\Omega)^2$. Substituting from (16), (17) and from the last inequality into (15), we obtain

$$\nu \,|\!|\!| \boldsymbol{u}_1 - \boldsymbol{u}_2 |\!|\!|^2 \leq (2c_1^2 + 2c_3 + 2c_3 c_4)\, R \,|\!|\!| \boldsymbol{u}_1 - \boldsymbol{u}_2 |\!|\!|^2.$$

Now it is seen that if R is so small that $\nu > (2c_1^2 + 2c_3 + 2c_3 c_4)\, R$ then $\boldsymbol{u}_1 = \boldsymbol{u}_2$. This proves the theorem. □

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