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Carmen Simerská

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REMARKS ON THE ECONOMIC CRITERION – THE INTERNAL RATE OF RETURN*

Carmen Simerská

Abstract

The internal rate of return (IRR) together with the present value (PV) is used as a popular measure for financial project. When used appropriately, it can be a valuable aid in project acceptance or selection. The purpose of this article is to survey the facts about this criterion published so far. More, we investigate the cases of multiple or nonexistent IRRs and try to choose the relevant one and explain its economic meaning.

1. Introduction

The internal rate of return (IRR) is frequently used as a valuation for investment transactions and financial securities described by a sequence of cash flows in time. We are interested in the existence, uniqueness or multiplicity of the IRR solutions. The IRR can be unambiguously used in decision making if it is unique and simple.

From time to time, the question of how we should find and interpret the IRRs, that are not unique and simple, appears among economists or even mathematicians. This was the case in the 1990's: The Canadian Institute of Actuaries came up with the problem of IRR uniqueness when trying to clarify a section of the Canadian criminal code which made it offense to lend money at an effective interest rate exceeding 60 % p.a., see [10]. In the Czech legislation, there is currently a law requiring that all providers of consumer loans include the Annual percentage rate of charge (APRC) in the loan conditions. APRC (in Czech: RPSN) is defined to be a solution of the equation

$$\sum_{j=0}^{J} \frac{C_{t_j}}{(1+i)^{t_j}} = 0,$$

for unknown i, where $C_{t_0}, \ldots, C_{t_J}$ are the cash flows, positive or negative, of the loan in time terms t_j (including all related costs of the loan to the client: fees, etc.). There are no restrictions on the APRC value but unfortunately, the law includes also loans, the advances and repayments which alternate in time (e.g. when an application fee is considered as a repayment of the loan before the loan is received). In these cases, the loans can have multiple APRC's, APRC being a double root, or not existing at all.

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2. Basic notions

Let us consider a *project* $\mathbf{B} = (B_0, \ldots, B_n)$, $B_0 \neq 0$, $B_n \neq 0$, i.e. a sequence of equally spaced (periodic) *cash flows* $B_0, \ldots, B_n$. Given the estimated market rate of interest i per period, at which the money may be borrowed or invested, a usual procedure is to accept the project if its *present value*

$$PV(i) = PV(\mathbf{B}, i) = \sum_{k=0}^{n} \frac{B_k}{(1+i)^k}$$

is greater than zero.

An *internal rate of return* (IRR) i^*, $i^* \in (-1, \infty)$, attached to the project $\mathbf{B}$ can be defined by three equivalent definitions:

- i^* is the root of the present value function $PV(\mathbf{B}, i)$, i.e. $PV(\mathbf{B}, i^*) = 0$.
- $i^* = \dfrac{1}{\nu^*} - 1$, where $\nu^* \in (0, \infty)$ is the root of the polynomial $g(\nu) = \sum_{k=0}^{n} B_k \nu^k$.
- $i^* = x^* - 1$, where $x^* \in (0, \infty)$ is the root of the polynomial $h(x) = \sum_{k=0}^{n} B_k x^{n-k}$.

Many applications require only positive IRRs; these correspond to the roots of $g(\nu)$ in the interval $(0, 1)$ and to the roots of $h(x)$ in the interval $(1, \infty)$.

Notes:

- IRR is the rate that equalizes time value of expected earnings and the investment outlays of the project. When unique, IRR defines the marginal value of interest rates (the efficiency of capital or the cost of loan) for which PV is nonnegative. Evidently, multiple or double IRRs can potentially occur.
- The value of $PV(\mathbf{B}, i)$ is an absolute criterion of the project. It is dependent on the size of the cash flows as opposed to the value and number of IRRs, which depend on the cash flows structure.
- The function $PV(i)$ is a continuous (and differentiable) function of the rate i. The IRRs, i.e. the roots of polynomials, are not continuous function of their coefficients $B_0, \ldots, B_n$, e.g., when one of the IRRs is double, a small change of a cash flows can cause the double root to disappear. In case of multiple IRRs, any numerical method to calculate them (Newton, Bairstow) can run into difficulties.
- Every finite sequence of cash flows $\mathbf{C} = (C_{t_0}, \ldots, C_{t_J})$, e.g. of an arbitrary loan, can always be considered as a periodic (e.g. daily) project $\mathbf{B} = (B_0, \ldots, B_n)$ (simply putting: $B_k = 0$ in the days with no flow). Then APRC is the effective annualized IRR of the corresponding $\mathbf{B}$ n-year period, APRC $= (1 + i^*)^{365} - 1$.
- When studying the behaviour of PV and IRR, it may be assumed, without loss of generality, that $B_0 < 0$, i.e the project requires an initial outlay. Sign reverse/identical results may be produced in the case, where the project has an initial income.

If the present value $PV(i)$ of the project is a monotonous function in $(-1, \infty)$ (most loans) and if there exists an IRR, then it is unique. Subsequently, for decision making the IRR can be simply compared to the usual opportunity cost of capital (market interest rate) to accept or reject the project. The application of this IRR criterion becomes problematic if PV is not monotonous and/or the IRR does not exist ($i^* < -1$ or complex-valued) or if there are too many of them. Evidently, the uniqueness of IRR does not imply monotonicity of PV.

3. Conditions for existence and uniqueness of IRR

In the 1970's, great effort was directed to obtain sufficient conditions for determining a unique IRR. Some "new" rules were reproved by means of old mathematical facts dealing with the roots of polynomials. We present a brief survey of the localization rules with the corresponding references.
Assuming $B_0 < 0$, it is easy to verify:

- $B_n > 0 \quad \Rightarrow \quad$ exists IRR in $(-1, \infty)$.
- $PV(0) > 0 \quad \Rightarrow \quad$ exists IRR in $(0, \infty)$.
  $(PV(0) = \displaystyle\sum_{k=0}^{n} B_k > 0$ is the minimum economic convenience of investment.)

The number and uniqueness of IRR in an interval can be guaranteed by means of the number of sign-changes in certain sequences $\mathbf{S}$.

- Descartes theorem, $\mathbf{S} = \{B_0, \ldots, B_n\}$.
  Corollary: Exactly one sign change in $\mathbf{S}$ implies a unique IRR.
- Budan-Fourier theorem, $\mathbf{S} = \{g(\nu), g'(\nu), \ldots, g^{(n)}(\nu)\} \ \rightarrow$ Jean's rule [6].
- Soper's theorem [11], $\mathbf{S} = \{B_0,\, B_0 + B_1,\, \ldots,\, \sum_{k=0}^{n} B_k\} \ \rightarrow$ Norstrom's rule [8]. Corollary: Exactly one sign change in $\mathbf{S}$ implies a unique positive IRR.
- Vincent's theorem, $\mathbf{S}$ is the diagonal of Vincent's matrix $\rightarrow$ Bernhard-de Faro condition for non-negative IRRs [2], [1].
- Sturm's theorem, $\mathbf{S}$ results from the Euclid's algorithm applied to the polynomial $g \ \rightarrow$ Kaplan's rule, exact number of IRRs ignoring multiplicity [7].

It is worth giving here in more details the Soper-Gronchi (S-G) conditions, the only ones that have a meaningfull economic interpretation. First, for the given project $\mathbf{B}$ and $i \in (-1, \infty)$, we define the project *balance stream* $\mathbf{A}(i) = (a_0(i), \ldots, a_{n-1}(i))$, where the unrecovered financial *balances* are the functions

$$a_m(i) = \sum_{k=0}^{m} B_k (1+i)^{m-k}, \qquad m = 0, \ldots, n-1.$$

This definition comes form [10]. The balances are given equivalently by the relations

$$B_0 = a_0(i)\,, \qquad B_m = a_m(i) - (1+i)a_{m-1}(i), \quad m = 1, \dots, n-1\,.$$

- Assuming any value i^* of the project IRRs was found, Soper [11] and Gronchi [3] stated the sufficient (not necessary) conditions

$$a_m(i^*) \le 0\,, \quad \forall\, m = 0, \dots, n-1\,, \tag{S-G}$$

  for the IRR value to be unique.

We recommend the following proposition that is more practical for determining the uniqueness of IRR without IRR computation.

If for a given $r \in (-1, \infty)$ *the conditions:* $a_m(r) \le 0\,, \forall\, m = 0, \dots, n-1\,,$ *are valid and* $PV(r) > 0$*, then there exists a unique IRR* i^* *of the project and* $i^* > r\,.$

Gronchi called the rate i^* for which (S-G) conditions are valid *pure lending* rate. It means that the investor does not borrow from the project at any time during its project life and only recovers its investment at the end, earning the interest i^*. The i^* of the project that has at least one period m with the balance $a_m(i^*) > 0$ should be regarded as a *lending* rate and also as a *borrowing* rate, to be paid (still by the investor) on the balance financed by the project. Then using the IRR of ambiguous meaning for this *mixed* project becomes questionable.

It was also shown that a direct comparison of the IRRs i_1 and i_2 of various projects $\mathbf{B}_1, \mathbf{B}_2$ for the purpose of ranking is not recommendable. When $i_1 > i_2$ and both are lending rates, project $\mathbf{B}_1$ should be preferable. But $\mathbf{B}_2$ is preferable if i_1 and i_2 are regarded as the borrowing ones (the lower borrowing rate is better). Hajdasinski, e.g. in [4], deals with the comparison of mutually exclusive projects by means of the IRR using the method of incremental approach. The comparison of multiple IRRs projects is an unresolved problem so far.

4. Nonuniqueness of IRR

The multiplicity or not real-values of IRR have been regarded as a fatal defect for the IRR criterion. Many objections for using IRR as a criterion of the project have been expressed till now and to use only PV criterion was often proposed. But Oehmke in [9] shows that in exactly those projects that may give either none or multiple IRRs the PV criterion exhibits anomalous behaviour as well. E.g., there are investment projects, where PV can be an increasing function in some interval, see Fig. 1. The use of market interest rate i_m smaller than the lower evaluated IRR may reduce the calculated PV. In order to investigate the cases of IRR nonuniqueness, it is useful to decompose a project by means of the proposition (e.g. [3]):

Given a project $\mathbf{B}$ *and a rate of interest* r*, there exists a set of consecutive financial operations*

$$\{A_m,\ -(1+r)A_m\}\,, \quad m = 0, \ldots, n-1\,,$$

where $A_m = a_m(r)$ *are the values of balances attached to* $\mathbf{B}$ *and* $B_n = -(1+r)A_{n-1}$, if and only if r *is an IRR of the project. Moreover, given* $\mathbf{B}$ *and* r *the set is unique.*

I.e., an IRR is an interest rate i^* uniformly applied to the single-one operations $\{B_0,\ -(1+i^*)B_0\}\,, \{A_1,\ -(1+i^*)A_1\}, \ldots,\ \{A_{n-1},\ B_n\}$, into which a project can be uniquely decomposed. Then instead of $\mathbf{B}$ we can deal with any balance stream $\mathbf{A}(i^*)$.

Hazen in [5] generalized the notion of IRR of the project $\mathbf{B}$ to any root i^* of PV (possibly complex-valued). Then $\mathbf{B}$ can always be interpreted as a result of (possibly complex-valued) balance stream $\mathbf{A}(i^*)$ for any i^*. Using the known equation

$$PV(\mathbf{B}, i) = \frac{i - i^*}{1+i}\, PV(\mathbf{A}(i^*), i)\,,$$

for a given interest rate i, he proved the following implications:

(a) *If* $PV(\mathrm{Re}(\mathrm{A}(\mathrm{i}^*)), \mathrm{i}) < 0$ *then* $PV(\mathbf{B}, i) \geq 0 \Leftrightarrow \mathrm{Re}(\mathrm{i}^*) \geq \mathrm{i}\,.$

(b) *If* $PV(\mathrm{Re}(\mathrm{A}(\mathrm{i}^*)), \mathrm{i}) > 0$ *then* $PV(\mathbf{B}, i) \geq 0 \Leftrightarrow \mathrm{Re}(\mathrm{i}^*) \leq \mathrm{i}\,.$

From his point of view, it does not matter which IRR is used to accept or reject the project. Every IRR is meaningful. All what is important is whether IRR exceeds the market rate $i = i_m$. The magnitude of i^* by itself is not significant.

When treating the problem of multiplicity, we take into account that the project (e.g. Fig. 1) can behave differently depending on the market rate i_m used. For some investor the same project can be rejected as an investment and for a borrower at different i_m rejected as a loan. Therefore, the project cannot be defined as an investment or as a loan unless i_m is specified. It is natural to determine the intervals of monotonicity $(-1, r_1\rangle\,, \ldots,\ \langle r_k, r_{k+1}\rangle\,, \ldots,\ \langle r_K, \infty)$, where $K < n$, i.e the intervals between the extremal points r_k of the function PV.

We call such an interval I the *investment (or loan) interval* of the project if

$$\frac{\partial PV}{\partial i}(i) = \sum_{k=1}^{n} \frac{-kB_k}{(1+i)^{k+1}} \leq 0 \quad \left(\text{or } \frac{\partial PV}{\partial i}(i) \geq 0\right) \quad \forall i \in I.$$

When $i_m \in I$, the type of relevant interval is given. If $i_m = r_k$, it does not matter which interval, right or left, we choose. Moreover, in case $i^* \in I$ (at most one IRR) we take it as the relevant IRR with respect to i_m. The significance of this i^* corresponds to the investment (loan) interval I. When investment, we accept (reject) the whole project if $i^* \geq i_m$ ($i^* \leq i_m$). (When loan, then vice versa.)

In the case when no IRRs are in I, we accept the project if PV is positive at the endpoints of the interval I, regardless of the magnitudes of the IRRs of the project, see Fig. 2. The presence of complex roots means the sign of derivative changes twice without crossing the i axis.)

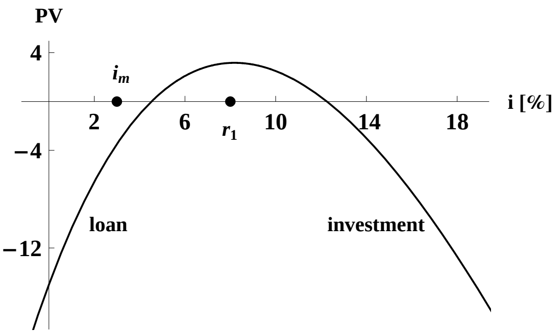

Fig. 1: *Anomalous project P1.*

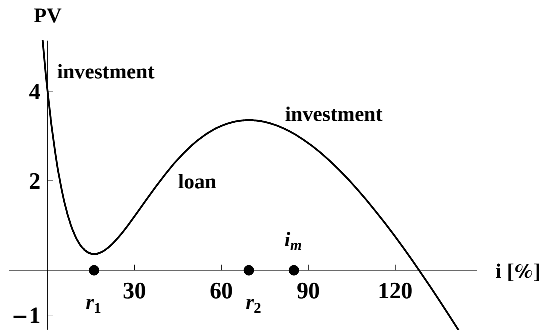

Fig. 2: *Project P2, unique IRR.*

	B_0	B_1	B_2	B_3	B_4	B_5	IRR[%]
P1	-815	900	-100	1200	-1200	0	4.5 ; 12.3
P2	-77	340	-470	252	-110	69	$-108.5 \pm 53.7\,i$; $15.1 \pm 6.9\,i$; 128.2

5. Conclusions

The problem of rating the project has been substantially simplified with the help of computers graphing the function PV and by means of special software tools (Maple, Mathematica), which can provide all existing (even complex-valued) roots. But how the financial manager should deal with a project when the symbolic programs are not available? When the cash flows have multiple sign changes, several spreadsheet calculators tend to give "ERR" (an error message) instead of IRR.

Before any calculation one should ask about the uniqueness of the IRR because if it is unique the decision making is straightforward. We found the (S-G) conditions very strong for determining the uniqueness, though they have reasonable economic meaning. There are a lot of projects with the monotonic PV and unique IRR that do not fulfill the (S-G) conditions.

When multiplicity occurs, the IRRs should not be compared even within a single project. Every real IRRs are significant, some are the measures for the investment return and others have the economic meaning as the cost of loan. If the market rate i_m is specified, the project follows the type of corresponding interval. As an investment decision tools, the IRR and PV are coherent criteria. Together give a better analysis than the former or the latter alone.

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