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FUZZY SETS AND SMALL SYSTEMS

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Abstract

Independently with [7] a corresponding fuzzy approach has been developed in [3–5] with applications in measure theory. One of the results the Egoroff theorem has been proved in an abstract form. In [1] a necessary and sufficient condition for holding the Egoroff theorem was presented in the case of a space with a monotone measure. By the help of [2] and [6] we prove a variant of the Egoroff theorem stated in [4].

1. Introduction

In [7] the notion of a fuzzy subset A of a space X has been defined as a mapping $A : X \to [0, 1]$. Especially, if $A : X \to \{0, 1\}$, then A can be identified with a classical set $B \subset X$ by the help of the equality $A = \chi_B$.

Almost at the same time the notion of a set of small measure has been characterized in [3–5] using a sequence $(\mathcal{N}_n)_{n=1}^{\infty}$ of subfamilies of a σ-algebra $\mathcal{S} \subset 2^X$ satisfying the following properties:

(i) $\emptyset \in \mathcal{N}_n, \mathcal{N}_{n+1} \subset \mathcal{N}_n$ for every $n \in \mathbb{N}$,

(ii) if $A \in \mathcal{N}_n, B \in \mathcal{S}$ and $B \subset A$, then $B \in \mathcal{N}_n$,

(iii) if $A, B, C \in \mathcal{N}_n$, then $A \cup B \cup C \in \mathcal{N}_{n-1}$,

(iv) if $A_i \supset A_{i+1}\, (i = 1, 2, \cdots)$ and $\bigcap_i A_i = \emptyset$, then to every $n \in \mathbb{N}$ there is i such that $A_i \in \mathcal{N}_n$.

The classical Egoroff theorem states that if a sequence $(f_n)_n$ of real measurable functions converges to a measurable function f almost everywhere, then it converges almost uniformly, i.e. $\forall \varepsilon > 0\ \exists A \in \mathcal{A}$ such that $\mu(A) < \varepsilon$ and $(f_n)_n$ converges uniformly to f on $X - A$.

Definition. We say that a sequence $(f_n)_n$ converges to f almost everywhere, if $\{x \in X; f_n(x)$ does not converge to $f(x)\} \in \mathcal{N}_n$ for every n. We say that $(f_n)_n$ converges to f almost uniformly, if for any $n \in \mathbb{N}$ there exists $A \in \mathcal{N}_n$ such that (f_n) converges uniformly to f on $X - A$.

2. Egoroff theorem

Theorem. Let $(\mathcal{N}_n)_n$ be a small system of subfamilies of a measurable space $(X, \mathcal{S})$. Let $(f_n)_n$ converges to f almost everywhere. Then $(f_n)_n$ converges to f almost uniformly.

Proof. First we use a result from [6]: If $(\mathcal{N}_n)_n$ satisfies (i)–(iv), then there exists a monotone continuous function $\mu : \mathcal{S} \to [0,1]$ such that

$$\mathcal{N}_n = \{A \in \mathcal{S}; \mu(A) < 3^{-n}\},$$

$n = 1, 2, 3, \ldots$ In [1] the following theorem has been proved: A monotone function $\mu : \mathcal{S} \to [0,1]$ satisfies the Egoroff theorem if and only if it satisfies the following condition (E):

For every double sequence $\left\{E_n^{(m)}\right\}$ of measurable sets which satisfies

$$E_n^{(m)} \searrow E^{(m)} \, (n \to \infty) \, , \quad \mu\left(\bigcup_{m=1}^{\infty} E^{(m)}\right) = 0$$

there exist increasing sequences $\{n_i\}_{i=1}^{\infty}$ and $\{m_i\}_{i=1}^{\infty}$ of natural numbers such that

$$\lim_{k\to\infty} \mu\left(\bigcup_{i=k}^{\infty} E_{n_i}^{(m_i)}\right) = 0.$$

We are going to prove that the monotone continuous set function μ satisfies condition (E). Let $\left\{E_n^{(m)}\right\}$ is double sequence of measurable sets for which

$$E_n^{(m)} \searrow E^{(m)} \, (n \to \infty) \, , \quad \mu\left(\bigcup_{m=1}^{\infty} E^{(m)}\right) = 0.$$

From the monotonicity it follows that

$$0 = \mu(\emptyset) \leq \mu\big(E^{(m_0)}\big) \leq \mu\left(\bigcup_{m=1}^{\infty} E^{(m)}\right) = 0.$$

We have proven that $\mu\big(E^{(m)}\big) = 0$ for arbitrary m. From this it follows that there is a natural number n_1 for which

$$\mu\big(E_{n_1}^{(1)}\big) \leq \frac{1}{3}.$$

Similarly there is a number $n_2 > n_1$ for which

$$\mu\big(E_{n_2}^{(2)}\big) \leq \frac{1}{3^2},$$

etc. Putting $m_i = i$, we get

$$\mu\left(\bigcup_{i=k}^{\infty} E_{n_i}^{(m_i)}\right) \leq \sum_{i=k}^{\infty} \frac{1}{3^i} = \frac{\frac{1}{3^k}}{1-\frac{1}{3}} = \frac{1}{2 \cdot 3^{k-1}}.$$

From this it follows that

$$\lim_{k\to\infty} \mu\left(\bigcup_{i=k}^{\infty} E_{n_i}^{(m_i)}\right) = 0.$$

□

3. Conclusion

We presented a new proof of the Egoroff theorem for small systems [4]. It follows from a representation theorem in [6] and the Egoroff theorem for monotone measures in [2].

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