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In: Karol Mikula (ed.): Proceedings of Equadiff 14, Conference on Differential Equations and Their Applications, Bratislava, July 24-28, 2017. Slovak University of Technology in Bratislava, SPEKTRUM STU Publishing, Bratislava, 2017. pp. 29–36.

Persistent URL: `http://dml.cz/dmlcz/703025`

Proceedings of EQUADIFF 2017
pp. 29–36

DYNAMICAL MODEL OF VISCOPLASTICITY

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Abstract. This paper discusses the existence theory to dynamical model of viscoplasticity and show possibility to obtain existence of solution without assuming weak safe-load condition.

Key words. viscoplasticity, coercive approximation, Yosida approximation, safe-load condition, mixed boundary condition

AMS subject classifications. 35Q74, 35A01, 74C10

1. Introduction.

Systems of equations describing an inelastic deformation of metals, under fundamental assumption of small deformations, consist of linear partial differential equations coupled with nonlinear differential inclusion.

The differential inclusion (inelastic constitutive equation) is experimental and depend on the considered material. Therefore, there are many different inelastic constitutive equations. H.-D. Alber in [1] defines a very large class of constitutive equations (of pre-monotone type) which contains all models proposed in engineering sciences known by author. However, the existence theory for such wide class of constitutive equations is not complete. Therefore, we will focus on a certain subclass of possible constitutive equations called viscoplastic models of gradient type (see Definition 1.1). For models equipped with such constitutive equation it is quite common to assume specific indirect assumption on data called *weak safe-load condition* (see Definition 1.2) as for example in: [3], [5], or [6]. However, in paper [4] authors were able to omit this indirect assumption in the case of dynamical visco-poroplasticity. We observed that similar methods can be used in case of viscoplastic models of gradient type.

1.1. Formulation of the model.

We assume that considered material (with the constant mass density $\rho > 0$) lies within the subset $\Omega \subset \mathbb{R}^3$ with smooth boundary $\partial\Omega$. The system of equations describing the inelastic deformation process can be written in the following form

$$\begin{aligned} \rho u_{tt}(x,t) - \mathrm{div}_x T(x,t) &= F(x,t), \\ T(x,t) &= \mathcal{D}\left(\varepsilon\left(u(x,t)\right) - \varepsilon^p(x,t)\right), \\ \varepsilon^p_t(x,t) &\in M(T(x,t)), \end{aligned} \tag{1.1}$$

where $\varepsilon\left(u(x,t)\right)$ denotes the symmetric part of the gradient of function $u(x,t)$ i.e.

$$\varepsilon\left(u(x,t)\right) = \frac{1}{2}\left(\nabla_x u(x,t) + \nabla_x^T u(x,t)\right).$$

The first equation $(1.1)_1$ is the balance of momentum coupled with the generalized Hooke's law (equation $(1.1)_2$). The given functions are: $F : \Omega \times [0, T_e] \to \mathbb{R}^3$ which describes a density of applied body forces and $\mathcal{D} : \mathcal{S}(3) \to \mathcal{S}(3) = \mathbb{R}^{3\times 3}_{sym}$ which is an elasticity tensor. $\mathcal{D}$ is assumed to be linear, symmetric, positive-definite and

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constant in time and space. Last equation $(1.1)_3$ is called constitutive equation, where $M : D(M) \subset \mathcal{S}(3) \to \mathcal{P}(\mathcal{S}(3))$ is a given constitutive multifunction.

For any fixed $T_e > 0$ we are interested in finding the following

- the displacement field $u : \Omega \times [0, T_e] \to \mathbb{R}^3$,
- the inelastic deformation tensor $\varepsilon^p : \Omega \times [0, T_e] \to \mathcal{S}(3) = \mathbb{R}^{3\times 3}_{sym}$,
- the Cauchy stress tensor $T : \Omega \times [0, T_e] \to \mathcal{S}(3)$,

Problem (1.1) will be considered with mixed boundary conditions

$$\begin{aligned} u(x,t) &= g_D(x,t), \qquad & x \in \Gamma_D,\ t \geqslant 0, \\ T(x,t)n(x) &= g_N(x,t), \qquad & x \in \Gamma_N,\ t \geqslant 0, \end{aligned} \tag{1.2}$$

where $n(x)$ is the outward pointing, unit normal vector at point $x \in \partial\Omega$. The sets Γ_D, Γ_N, are open subsets of $\partial\Omega$ such that $\partial\Omega = \overline{\Gamma}_D \cup \overline{\Gamma}_N$, $\Gamma_D \cap \Gamma_N = \emptyset$.

Furthermore, we also need to assume initial conditions in the form

$$\text{for } x \in \Omega \qquad u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \qquad \varepsilon^p(x,0) = \varepsilon^p_0(x). \tag{1.3}$$

In this paper we consider viscoplastic models of gradient type. Therefore, we assume that the inelastic constitutive multifunction M is viscoplastic of gradient type which means

Definition 1.1.
We say that constitutive multifunction $M : \mathcal{S}(3) \to \mathcal{P}(\mathcal{S}(3))$ is viscoplastic of gradient type if there exist a convex function $M_0 : \mathcal{S}(3) \to \mathbb{R}$ such that

$$M(T) = \partial M_0(T).$$

1.2. Main results.
In [3] K. Chełmiński introduce the coercive approximation process of model (1.1). Moreover, in [3] author proved that the approximate solutions converge to the solution of the original problem. However, in order to obtain needed estimates author assumed *weak safe-load condition* in the following form

Definition 1.2 (weak safe-load condition).
We say that the functions g_D, g_N satisfy the weak safe-load conditions *if there exist the initial conditions $u_0^*, u_1^* \in H^1(\Omega;\mathbb{R}^3)$ and the function $F^* \in H^1(0,T_e;L^2(\Omega;\mathbb{R}^3))$ such that, there exists a solution (u^*, T^*) of the linear system*

$$\begin{aligned} \rho u^*_{tt}(x,t) - \mathrm{div}_x T^*(x,t) &= F^*(x,t), \\ T^*(x,t) &= \mathcal{D}(\varepsilon(u^*(x,t))), \end{aligned}$$

with the initial–boundary conditions

$$\begin{array}{rcll} u^*(x,0) &=& u_0^*(x) & \text{for } x \in \Omega, \\ u_t^*(x,0) &=& u_1^*(x) & \text{for } x \in \Omega, \\ u^*(x,t) &=& g_D(x,t) & \text{for } x \in \Gamma_D, \quad t \geqslant 0, \\ T^*(x,t)n(x) &=& g_N(x,t) & \text{for } x \in \Gamma_N, \quad t \geqslant 0, \end{array}$$

and the regularity

$$u^* \in W^{2,\infty}(0,T_e;L^2(\Omega;\mathbb{R}^3)), \quad \varepsilon(u^*) \in W^{1,\infty}(0,T_e;L^2(\Omega;\mathcal{S}^3)),$$
$$T^* \in L^\infty(0,T_e;L^\infty(\Omega;\mathcal{S}^3)).$$

This indirect assumption on data is very difficult to check (especially in the case of mixed boundary conditions). Therefore, natural question arise: If such assumption is needed in case of viscoplasticity?

Now we are able to answer this question. It occurs that in order to prove the existence of solution to viscoplasticity problem the *weak safe-load condition* can be completely omitted. Namely, we are able to proof the following theorem

THEOREM 1.3 (Main result).
Consider dynamical model of viscoplasticity (1.1) *(where constitutive function is viscoplastic of gradient type) with the initial-boundary conditions* (1.2)–(1.3). *Assume that the initial conditions, boundary data and external force satisfy* (2.1)–(2.5) *then, there exists a solution* (u, ε^p, T) *in the sense of Definition 2.2.*

2. General information.

Before we start the main part of the discussion we would like to introduce regularity assumptions then it is important to define the notation of a solution. Finally in the last part of this section we introduce coercive approximation of the problem (1.1)–(1.3) along with the existence result for approximate model.

2.1. Regularity assumption on data.

First of all let us state the regularity assumptions for needed data. To obtain existence of solution to the problem (1.1)–(1.3) we assume the following (it is worth mentioning that in fact we can prove existence under slightly lower assumption on data but, for simplicity, we state them this way).

• Regularities of the external force

$$F \in H^1(0, T_e; L^2(\Omega; \mathbb{R}^3)). \tag{2.1}$$

• Regularities of the boundary conditions

$$\begin{gathered} g_D \in W^{3,\infty}(0, T_e; H^{\frac{3}{2}}(\Gamma_D; \mathbb{R}^3)), \\ g_N \in W^{2,\infty}\left(0, T_e; L^\infty\left(\Gamma_N; \mathbb{R}^3\right)\right) \cap W^{2,\infty}\left(0, T_e; H^{-\frac{1}{2}}\left(\Gamma_N; \mathbb{R}^3\right)\right), \end{gathered} \tag{2.2}$$

• Regularities of the initial conditions

$$u_0 \in H^2(\Omega; \mathbb{R}^3), \quad u_1 \in H^1(\Omega; \mathbb{R}^3), \quad \varepsilon_0^p \in L^2_{\mathrm{div}}(\Omega; \mathcal{S}(3)). \tag{2.3}$$

Moreover, we require compatibility conditions of the form

$$\begin{array}{rcll} u_0(x) & = & g_D(x, 0), & x \in \Gamma_D, \\ u_1(x) & = & g_{D,t}(x, 0), & x \in \Gamma_D, \\ T_0\,(x)\, n(x) & = & g_N(x, 0), & x \in \Gamma_N, \end{array} \tag{2.4}$$

where $T_0\,(x) := \mathcal{D}\left(\varepsilon(u_0(x)) - \varepsilon_0^p(x)\right)$ is an initial stress.

We also need to assume that the initial stress lies in a domain of the constitutive multifunction M, which means

DEFINITION 2.1.
The initial data (u_0, ε_0^p) *are said to be admissible for problem* (1.1) *if*

$$\exists M^* \in L^2\left(\Omega; \mathcal{S}\left(3\right)\right) \quad \textit{such that} \quad M^*(x) \in M\left(\mathcal{D}\left(\varepsilon(u_0(x)) - \varepsilon_0^p(x)\right)\right) \tag{2.5}$$

for almost every $x \in \Omega$.

2.2. Definition of solution.
We were able to obtain solution in the same sense as it is done in [3]. Our solution satisfy problem (1.1) almost everywhere. Namely

Definition 2.2 (Solution).
We say that (u, ε^p, T) *is a solution of the problem* (1.1)–(1.3) *if:*
1. *The following regularities are satisfied*

$$u \in W^{2,\infty}\left(0, T_e; L^2\left(\Omega; \mathbb{R}^3\right)\right), \qquad \varepsilon(u) \in W^{1,1}\left(0, T_e; L^1\left(\Omega; \mathcal{S}(3)\right)\right),$$
$$\varepsilon^p \in W^{1,1}\left(0, T_e; L^1\left(\Omega; \mathcal{S}(3)\right)\right),$$
$$T \in W^{1,\infty}\left(0, T_e; L^2\left(\Omega; \mathcal{S}(3)\right)\right), \qquad \operatorname{div} T \in L^\infty\left(0, T_e; L^2\left(\Omega; \mathbb{R}^3\right)\right).$$

2. *For almost every* $(x,t) \in \Omega \times (0, T_e)$ *the following problem is satisfied*

$$\begin{aligned} \rho u_{tt}(x,t) - \operatorname{div} T(x,t) &= F(x,t), \\ T(x,t) &= \mathcal{D}\left(\varepsilon\left(u\left(x,t\right)\right) - \varepsilon^p\left(x,t\right)\right), \\ \varepsilon^p_t(x,t) &\in M\left(T(x,t)\right). \end{aligned}$$

3. *By* γ *let us denote the trace operator. Then*

$$\begin{aligned} \gamma_{|\Gamma_D \times [0,T_e]}\left(u\right) &= g_D, \\ \gamma_{|\Gamma_N \times [0,T_e]}\left(T\, n\right) &= g_N. \end{aligned}$$

4. *For almost every* $x \in \Omega$ *initial conditions:*

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \qquad \varepsilon^p(x,0) = \varepsilon^p_0(x)$$

are satisfied.

2.3. Approximation of the model.
Observe that the free energy of (1.1) is given by

$$\rho\psi(\varepsilon, \varepsilon^p)(t) = \frac{1}{2}\mathcal{D}(\varepsilon - \varepsilon^p)(\varepsilon - \varepsilon^p).$$

The energy is only a positive semi–definite quadratic form and therefore our system is *non-coercive* (for details see [1]). The lack of coercivity significantly hinders the analysis. As a remedy we introduce a standard idea of the coercive approximation (see for example [3]) of (1.1) as follows

$$\begin{aligned} \rho\, u^k_{tt}(x,t) - \operatorname{div}_x T^k(x,t) &= F(x,t), \\ T^k(x,t) &= \mathcal{D}\left(\left(1 + \frac{1}{k}\right)\varepsilon(u^k(x,t)) - \varepsilon^{p,k}(x,t)\right), \\ \widehat{T}^k(x,t) &= T^k(x,t) - \frac{1}{k}\mathcal{D}(\varepsilon(u^k(x,t))), \\ \varepsilon^{p,k}_t(x,t) &\in \partial M_0(\widehat{T}^k(x,t)), \end{aligned} \tag{2.6}$$

where $k \geqslant 1$.

Now if we fix k, the free energy of (2.6) is given by

$$\rho\psi^k\left(\varepsilon^k, \varepsilon^{p,k}\right)(t) = \frac{1}{2}\mathcal{D}\left(\varepsilon^k - \varepsilon^{p,k}\right)\left(\varepsilon^k - \varepsilon^{p,k}\right) + \frac{1}{2k}\mathcal{D}\left(\varepsilon^k\right)\varepsilon^k.$$

One can see that now the energy is a positive-definite quadratic form. Models with that type of energy are called *coercive*. The total energy of the discussed model is in the form

$$\mathcal{E}^k(u_t^{k,\lambda},\varepsilon^{k,\lambda},\varepsilon^{p,k,\lambda})(t)=\frac{\rho}{2}\int_\Omega \left|u_t^{k,\lambda}(x,t)\right|^2 \mathrm{d}x+\int_\Omega \rho\psi^k\left(\varepsilon^{k,\lambda}(x,t),\varepsilon^{p,k,\lambda}(x,t)\right)\mathrm{d}x.$$

Now we can state the existence result for model (2.6).

THEOREM 2.3 (Existence of solution to problem (2.6)).
Assume that the initial conditions u_0, u_1, ε_0^p, given boundary data g_D, g_N and external forces F, have the regularity (2.1)–(2.3). *Moreover, suppose that initial data are admissible and along with boundary data satisfy the compatibility conditions* (2.4). *Then, for every $k\in\mathbb{N}_+$, there exists a unique solution $(u^k,\varepsilon^{p,k},T^k)$ of* (2.6) *with the initial-boundary conditions* (1.2)–(1.3) *such that*

$$\begin{aligned} u^k &\in W^{2,\infty}\left(0,T_e;L^2(\Omega;\mathbb{R}^3)\right), \quad \varepsilon(u^k)\in W^{1,\infty}\left(0,T_e;L^2(\Omega;\mathcal{S}(3))\right), \\ \varepsilon^{p,k} &\in W^{1,\infty}\left(0,T_e;L^2(\Omega;\mathcal{S}(3))\right), \quad \mathrm{div}T^k\in L^\infty\left(0,T_e;L^2\left(\Omega;\mathbb{R}^3\right)\right). \end{aligned}$$

Proof of Theorem 2.3 is very similar to the proof presented in [5, section 4 and 5] (computation is very similar however it have to be done for plasticity not for poroplasticity model). Main idea of the proof is quite simple. One have to approximate differential inclusion be sequence of differential equations given by

$$\varepsilon_t^{p,k,\lambda}(x,t)=(\partial M_0)^\lambda\left(\widehat{T}^{k,\lambda}(x,t)\right),$$

where $(\partial M_0)^\lambda$ denotes the Yosida approximation of the operator ∂M_0. This approximation is maximal-monotone and globally Lipschitz with Lipschitz constant $1/\lambda$ (for details see [2]). Then, in the case when the right hand side of a constitutive equation is globally Lipschitz vector field, one can prove existence by the same reasoning as in [5, section 4] (Galerkin approximation and fixed point method). Therefore, in order to obtain solution to (2.6) for any fixed k one have to pass to the limit with λ in its Yosida approximation which also can be done due to quite standard reasoning (see for example [5, secton 5] or [6, section 4]).

3. Passing to the limit in coercive approximation.

The main part of the classic existence proof, where *weak safe-load condition* is needed, is proving the energy estimates (see [3, Theorem 3]). In the rest of the proof [3, Theorems 4,5,6] this assumption is not essential. Therefore, here we are going to present only the quick sketch of the proof of energy estimates and the rest of reasoning will be omitted.

THEOREM 3.1 (Energy estimates).
Assume (2.1)–(2.5) *then, for every $t\in[0,T_e]$ the following estimates hold:*

$$\operatorname*{ess\,sup}_{\tau\in(0,t)}\mathcal{E}^k\left(u_t^k,\varepsilon^k,\varepsilon^{p,k}\right)(\tau)+\int_0^t\int_\Omega \varepsilon_t^{p,k}\,\widehat{T}^k\,\mathrm{d}x\mathrm{d}\tau\leqslant C, \tag{3.1}$$

$$\operatorname*{ess\,sup}_{\tau\in(0,t)}\mathcal{E}^k\left(u_{tt}^k,\varepsilon_t^k,\varepsilon_t^{p,k}\right)(\tau)\leqslant C, \tag{3.2}$$

$$\left\|\varepsilon_t^{p,k}\right\|_{L^\infty(0,t;L^1(\Omega))} \leqslant C, \tag{3.3}$$

where $(u^k, \varepsilon^{p,k})$ *is a solution of* (2.6) *with the initial-boundary conditions* (1.2)–(1.3). *Constant* $C \geqslant 0$ *is independent of* k *and* t.

Proof.
Firstly, one have to prove the following

$$\operatorname*{ess\,sup}_{\tau\in(0,t)} \mathcal{E}^k\left(u_{tt}^k, \varepsilon_t^k, \varepsilon_t^{p,k}\right)(t) \leqslant C + C\left\|\varepsilon_t^{p,k}\right\|_{L^\infty(0,t;L^1(\Omega))} \quad \text{for a.e. } t\in(0,T_e). \tag{a}$$

To begin let us introduce a special notation for translated in time function, i.e.

$$\left(u_{t,h}^k(t), \varepsilon_h^k(t), \varepsilon_h^{p,k}(t)\right) := \left(u_t^k(t+h), \varepsilon^k(t+h), \varepsilon^{p,k}(t+h)\right),$$

where $h>0$ is a sufficiently small constant.
Then, computing $\frac{1}{h^2}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}^k\left(u_{t,h}^k - u_t^k, \varepsilon_h^k - \varepsilon^k, \varepsilon_h^{p,k} - \varepsilon^{p,k}\right)(t)$ and using similar methods as presented in [4, Theorem 7.1] in order to pass to the limit with h give

$$\begin{aligned}\mathcal{E}^k\left(u_{tt}^k, \varepsilon_t^k, \varepsilon_t^{p,k}\right)(t) \leqslant C\cdot\left(\left\|u_t^k\right\|_{L^\infty(0,t;L^1(\partial\Omega))} + \left\|T^k n\right\|_{L^\infty\left(0,t;H^{-\frac12}(\partial\Omega)\right)}\right)\\ + C(\nu) + \nu\cdot\mathcal{E}^k\left(u_{tt}^k, \varepsilon_t^k, \varepsilon_t^{p,k}\right)(t),\end{aligned} \tag{3.4}$$

where ν is an arbitrary positive constant. Using trace theorems and some elementary inequalities allows to obtain:

$$\left\|T^k(t)\, n\right\|_{H^{-\frac12}(\partial\Omega)} \leqslant C(\nu) + \nu\cdot\operatorname*{ess\,sup}_{(0,t)} \mathcal{E}^k(u_{tt}^k, \varepsilon_t^k, \varepsilon_t^{p,k})(t). \tag{3.5}$$

$$\left\|u_t^k(t)\right\|_{L^1(\partial\Omega)} \leqslant C(\nu) + C\left\|\varepsilon_t^{p,k}\right\|_{L^\infty(0,t;L^1)} + \nu\cdot\operatorname*{ess\,sup}_{(0,t)} \mathcal{E}^k(u_{tt}^k, \varepsilon_t^k, \varepsilon_t^{p,k})(t). \tag{3.6}$$

Using (3.5) and (3.6) in (3.4), taking the supremum over $(0,t)$ and fixing a sufficiently small ν finally give (a).

Secondly, one have to prove that

$$\begin{aligned}\operatorname*{ess\,sup}_{\tau\in(0,t)} \mathcal{E}^k\left(u_t^k, \varepsilon^k, \varepsilon^{p,k}\right)(\tau) + \int_0^t\int_\Omega \varepsilon_t^{p,k}\,\widehat{T}^k\,\mathrm{d}x\mathrm{d}\tau \leqslant C(\mu_1) + \mu_1\left\|\varepsilon_t^{p,k}\right\|_{L^\infty(0,t;L^1(\Omega))}\\ + C\left\|\varepsilon_t^{p,k}\right\|_{L^1(0,t;L^1(\Omega))},\end{aligned} \tag{b}$$

where μ_1 is an arbitrary positive constant.

We start by computing $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}^k\left(u_t^k, \varepsilon^k, \varepsilon^{p,k}\right)(t)$ then, after using few elementary estimates and integrating over time $(0,t)$ one can obtain

$$\begin{aligned}\mathcal{E}^k\left(u_t^k, \varepsilon^k, \varepsilon^{p,k}\right)(t) + \int_0^t\int_\Omega \varepsilon_t^{p,k}\,\widehat{T}^k\,\mathrm{d}x\mathrm{d}\tau \leqslant C(\nu) + \nu\cdot\operatorname*{ess\,sup}_{(0,t)} \mathcal{E}^k\left(u_t^k, \varepsilon^k, \varepsilon^{p,k}\right)\\ + C\left\|T^k n\right\|_{L^\infty\left(0,t;H^{-\frac12}(\partial\Omega)\right)}\\ + C\left\|u_t^k\right\|_{L^1(0,t;L^1(\partial\Omega))},\end{aligned} \tag{3.7}$$

where ν is an arbitrary positive constant. By using trace theorems and some elementary inequalities one can prove the following inequalities

$$\begin{aligned}\left\|T^k(t)n\right\|_{H^{-\frac{1}{2}}(\partial\Omega)} \leqslant & C(\nu,\mu_1)+\nu\cdot\operatorname*{ess\,sup}_{(0,t)}\mathcal{E}^k(u_t^k,\varepsilon^k,\varepsilon^{p,k})(t)\\ & +\frac{\mu_1}{2}\left\|\varepsilon_t^{p,k}\right\|_{L^\infty(0,t;L^1(\Omega))}.\end{aligned} \tag{3.8}$$

$$\begin{aligned}\left\|u_t^k\right\|_{L^1(0,t;L^1(\partial\Omega))} \leqslant & C\left(\nu,\mu_1\right)+\frac{\mu_1}{2}\left\|\varepsilon_t^{p,k}\right\|_{L^\infty(0,t;L^1(\Omega))}+C\left\|\varepsilon_t^{p,k}\right\|_{L^1(0,t;L^1(\Omega))}\\ & +\nu\cdot\operatorname*{ess\,sup}_{(0,t)}\mathcal{E}^k(u_t^k,\varepsilon^k,\varepsilon^{p,k})(t).\end{aligned} \tag{3.9}$$

Hence, by using (3.8) and (3.9) in (3.7), taking the supremum over $(0,t)$ and, fixing sufficiently small $\nu>0$ one can obtain (b).

As a third step one have to prove the following inequality:

$$\left\|\varepsilon_t^{p,k}\right\|_{L^1(0,t;L^1(\Omega))}\leqslant C\left(\mu_2\right)+\mu_2\left\|\varepsilon_t^{p,k}\right\|_{L^\infty(0,t;L^1(\Omega))}, \tag{c}$$

where μ_2 is an arbitrary positive constant.

Due to the monotonicity of ∂M_0 one can obtain that for any $\delta_0>0$ the following inequality holds

$$\left|\varepsilon_t^{p,k}\right|\leqslant\frac{1}{\delta_0}\varepsilon_t^{p,k}\widehat{T}^k+\frac{1}{\delta_0}\sup_{|\sigma|\leqslant\delta_0}\left|m\left(\partial M_0\left(\sigma\right)\right)\right|\left(\left|\widehat{T}^k\right|+\delta_0\right), \tag{3.10}$$

where $m\left(\partial M_0\left(\sigma\right)\right)$ is the element of $\partial M_0\left(\sigma\right)$ of minimal norm.
Integrating (3.10) over $\Omega\times(0,t)$ for $t\leqslant T_e$, using some elementary inequalities along with (b) and fixing a sufficiently large δ_0 (it is possible due to viscoplasticity assumption) give (c).

In the last step one have to prove that

$$\left\|\varepsilon_t^{p,k}\left(\tau\right)\right\|_{L^\infty(0,t;L^1(\Omega))}\leqslant C. \tag{d}$$

which finally allows to close estimates (c), (b) (a) and therefore ends the proof.

Using inequality (c) in (b) gives

$$\mathcal{E}^k\left(u_t^k,\varepsilon^k,\varepsilon^{p,k}\right)(t)+\int_0^t\int_\Omega\varepsilon_t^{p,k}\,\widehat{T}^k\,\mathrm{d}x\mathrm{d}\tau\leqslant C(\mu)+\mu\left\|\varepsilon_t^{p,k}\right\|_{L^\infty(0,t;L^1(\Omega))}, \tag{3.11}$$

where $\mu>0$ is an arbitrary constant.
After integrating (3.10) over Ω and using some elementary inequalities along with (3.11) one can obtain for almost every $t\in(0,T_e)$

$$\left\|\varepsilon_t^{p,k}(t)\right\|_{L^1(\Omega)}\leqslant\frac{1}{\delta_0}\int_\Omega\varepsilon_t^{p,k}(t)\,\widehat{T}^k(t)\,\mathrm{d}x+\mu\left\|\varepsilon_t^{p,k}\right\|_{L^\infty(0,t;L^1(\Omega))}+C(\mu,\delta_0). \tag{3.12}$$

Computing $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}^k\left(u_t^k,\varepsilon^k,\varepsilon^{p,k}\right)(t)$ and using standard inequalities lead to

$$\begin{aligned}\int_\Omega\varepsilon_t^{p,k}(t)\,\widehat{T}^k(t)\,\mathrm{d}x\leqslant & C-\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathcal{E}^k\left(u_t^k,\varepsilon^k,\varepsilon^{p,k}\right)(t)\right)+\mathcal{E}^k\left(u_t^k,\varepsilon^k,\varepsilon^{p,k}\right)(t)\\ & +C\left\|T^k(t)\,n\right\|_{H^{-\frac{1}{2}}(\partial\Omega)}+C\left\|u_t^k(t)\right\|_{L^1(\partial\Omega)}.\end{aligned} \tag{3.13}$$

Using (3.5) and (3.6) along with (a) leads to

$$\left\|T^k(t)\, n\right\|_{H^{-\frac{1}{2}}(\partial\Omega)} + \left\|u_t^k(t)\right\|_{L^1(\partial\Omega)} \leqslant C + C \left\|\varepsilon_t^{p,k}\right\|_{L^\infty(0,t;L^1(\Omega))} . \tag{3.14}$$

Using (3.11), (3.14) and (a) in (3.13) yields for almost every $t \in (0, T_e)$

$$\int_\Omega \varepsilon_t^{p,k}(t)\, \widehat{T}^k(t)\, \mathrm{d}x \leqslant C + C \left\|\varepsilon_t^{p,k}\right\|_{L^\infty(0,t;L^1(\Omega))} . \tag{3.15}$$

After inserting (3.15) into (3.12), taking the essential supremum, fixing sufficiently small $\mu > 0$ and sufficiently large $\delta_0 > 0$ (possible because constitutive function is viscoplastic) one can finally obtain (d), which ends the proof. □

REFERENCES

[1] H.-D. Alber, *Materials with memory*, volume 1682 Lecture Notes in Mathematics, Springer-Verlag, Berlin, 1998.

[2] J.-P. Aubin and A. Cellina, *Differential inclusions*, volume 264 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Springer-Verlag, Berlin, 1984.

[3] K. Chełmiński, *Coercive approximation of viscoplasticity and plasticity*, Asymptotic Anal., 26(2), pp. 105–133, 2001.

[4] K. Kisiel, *Dynamical poroplasticity model – Existence theory for gradient type nonlinearities with Lipschitz perturbations*, J. Math. Anal. Appl., 450(1), pp. 544–577, 2017.

[5] K. Kisiel and K. Kosiba, *Dynamical poroplasticity model with mixed boundary conditions – theory for $\mathcal{LM}$-type nonlinearity*, J. Math. Anal. Appl., 443(1), pp. 187–229, 2016.

[6] S. Owczarek, *Existence of solution to a non-monotone dynamic model in poroplasticity with mixed boundary conditions*, Topol. Methods Nonlinear Anal., 43(2), pp. 297–322, 2014.