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HYDROLOGICAL APPLICATIONS OF A MODEL-BASED APPROACH TO FUZZY SET MEMBERSHIP FUNCTIONS

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Abstract: Since the common approach to defining membership functions of fuzzy numbers is rather subjective, another, more objective method is proposed. It is applicable in situations where two models, say M_1 and M_2, share the same uncertain input parameter p. Model M_1 is used to assess the fuzziness of p, whereas the goal is to assess the fuzziness of the p-dependent output of model M_2. Simple examples are presented to illustrate the proposed approach.

Keywords: fuzzy set, membership function, uncertainty quantification
MSC: 03E72, 03E75

1. Introduction

This contribution deals with uncertain parameters represented by fuzzy sets, namely with a model-dependent definition of membership functions.

The membership function determines the membership grade of the elements of the corresponding fuzzy set [3], [4], [6], [7]. Unlike classical set theory, where the characteristic function range is limited to the bivalent set $\{0, 1\}$, the membership function range is an interval; without loss of generality, we can limit ourselves to $[0, 1]$, the commonly used range.

For fuzzy numbers, triangular or trapezoidal membership functions are widely used, for instance; see Figure 1. They are directly defined by the analyst on the basis of his or her judgment. Inevitably, strong subjective factors influence the definition. A more objective approach to the definition of a membership function is possible in situations where P, a set of uncertain input parameters, appears in two associated models, say M_1 and M_2, where the output of the model M_1 is measured and, through solving an inverse problem, enables the identification of the input parameters value. The goal is to assess the uncertainty of the output of the model M_2 via fuzzified input parameters P whose membership function is defined by means of the response of the model M_1.

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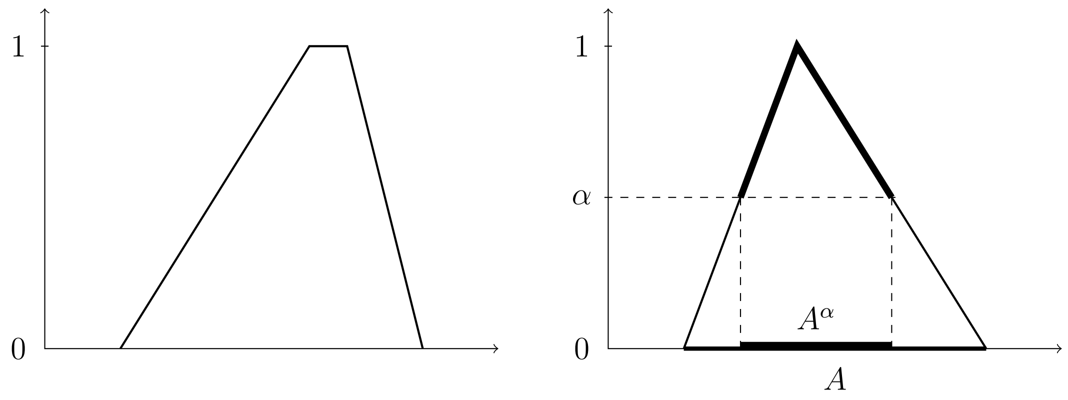

Figure 1: Left: A trapezoidal membership function. Right: A triangular membership function and an α-level set A^α.

Let us consider a space $S = \mathbb{R}^n$, where $\mathbb{R}$ stands for the field of real numbers and n is a natural number. Let μ_A be a continuous membership function defined on S and such that its support (that is, the closure of $\{a \in S |\ \mu_A(a) > 0\}$) is equal to a compact convex subset A of S. Next, we define the α-cuts of A (α-level sets) as

$$A^\alpha = \{a \in A |\ \mu_A(a) \geq \alpha\}, \quad \text{where } \alpha \in [0, 1].$$

Let us note that $A^0 \equiv A$. We assume that A^α is convex for any $\alpha \in [0, 1]$.

Figure 1 depicts two (nonsymmetric) membership functions where A and A^α are closed intervals. We also observe, see Figure 1 (right), that by knowing A^α for any $\alpha \in [0, 1]$, we can reconstruct μ_A. That is,

$$\mu_A(a) = \max\{\alpha |\ a \in A^\alpha\} \tag{1}$$

for any $a \in A \subset S$.

The same idea applied to a finite sequence $\{\alpha_i\}_{i=1}^n \subset [0, 1]$ is used in numerical algorithms to approximate the membership function of a model output.

To this end, let us consider Φ, a quantity of interest whose value at a is continuously determined by an a-dependent mathematical or computational model. That is, we can view Φ as a (possibly rather complex) map from A to $\mathbb{R}$. If A is fuzzy, then $R_\Phi = \{y \in \mathbb{R} |\ \exists a \in A\ y = \Phi(a)\}$, the range of $\Phi|_A$, is also fuzzy and its membership function can be inferred by Zadeh's extension principle, see [3], [4], [7], for instance. The principle says that μ_{R_Φ}, the membership function of the fuzzy set R_Φ, can be obtained by applying the following rule

$$\mu_{R_\Phi}(y) = \max_{\{a \in A |\ y = \Phi(a)\}} \mu_A(a) \tag{2}$$

at each $y \in R_\Phi$.

Since R_Φ is an interval, it can be easier to obtain μ_Φ not directly from (2), but from (1) where A^α is replaced by R_Φ^α, the α-cut of R_Φ that coincides with the range of $\Phi|_{A^\alpha}$.

By virtue of the convexity and compactness assumptions,

$$R_\Phi^\alpha = \left[\min_{a\in A^\alpha} \Phi(a), \max_{a\in A^\alpha} \Phi(a)\right]; \tag{3}$$

see [4], for example.

Let us note that supremum appears in (1) and (2) in general if the assumptions on A and μ_A are weakened.

2. Model-driven membership function

Let us assume that a model M_1 is represented by $\psi(a, \cdot)$, a real continuous function dependent on a parameter $a \in B \subset S$. Moreover, let a be uncertain, let the output $\psi(a, \cdot)$ be measured at points $\{x_i\}_{i=1}^k$, and let the respective recorded values be denoted by $\{r_i\}_{i=1}^k$.

Next, let us identify the weighted least squares minimizer

$$a_{\min} = \arg\min_{a\in B} \omega(a), \quad \text{where} \quad \omega(a) = \sum_{i=1}^{k} w_i(r_i - \psi(a, x_i))^2 \tag{4}$$

and w_i are positive weights. It is assumed that $\omega(a_{\min}) > 0$. The quantity ω will help to define the membership function describing the fuzziness of the input of the quantity of interest Φ that is determined by a model M_2.

In [2], examples of membership functions are given, but more general options exist for the definition of the membership function. Take $0 < c_1, c_2, c_3$, c_3 odd, and

$$\mu_1(b) = 1 + c_1\left(1 - \left(\frac{\omega(b)}{\omega(b_{\min})}\right)^{c_2}\right)^{c_3}, \quad \mu_2(b) = 1 + c_1\left(\left(\frac{\omega(b_{\min})}{\omega(b)}\right)^{c_2} - 1\right)^{c_3}, \tag{5}$$

for instance. We observe that $\omega(b)/\omega(b_{\min}) \geq 1$.

For a fixed c_1, c_2, c_3 and $i \in \{1, 2\}$, the fuzzy set A is then defined by

$$A = \{a \in B |\ \mu_i(a) \in [0, 1]\}. \tag{6}$$

A natural choice might be $c_1 = 1$, $c_2 = 1/2$ or $c_2 = 1$, and $c_3 = 1$.

Once the ω-based fuzzy set A and its membership function μ_A are established, the membership function μ_{R_Φ} associated with the quantity of interest Φ is determined by Zadeh's extension principle; see Section 1.

3. Examples

Let us illustrate the above theory by simple examples.

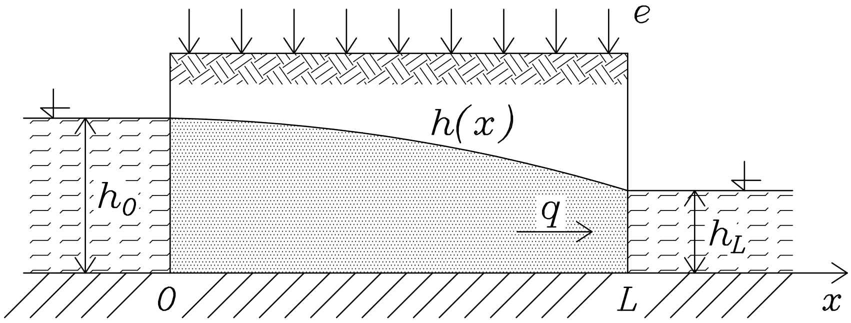

Figure 2: A permeable embankment separates two reservoirs and is subjected to infiltration or evaporation. The groundwater level function h is given by (8).

3.1. Two water levels separated by a permeable embankment

Figure 2 shows a cross section of an embankment separating two reservoirs. The embankment is L units wide and made of a permeable material. The water levels in reservoirs, namely h_0 and h_L, are different. We can assume that $h_0 > h_L$.

Due to head of water (difference of water levels), groundwater flow and also seepage through the embankment exist. The groundwater level is modeled by a smooth function h defined on the interval $[0, L]$. To add an external factor, let us introduce a constant e representing evaporation ($e > 0$) or infiltration ($e < 0$); see Figure 2, where $e < 0$.

A simple but commonly used approximation h of the true groundwater level in the embankment is based on Dupuit's postulates and solves

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(-Kh(x)\frac{\mathrm{d}h}{\mathrm{d}x}(x)\right) + e = 0, \quad h(0) = h_0, \quad h(L) = h_L, \tag{7}$$

where $0 < K \in \mathbb{R}$ is the saturated hydraulic conductivity; see [5]. Since (7) is equivalent to

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}h^2(x) = 2\frac{e}{K},$$

one can easily check that

$$h_{e,K}^2(x) = \frac{e}{K}x^2 + \left(\frac{h_L^2 - h_0^2}{L} - \frac{e}{K}L\right)x + h_0^2 \tag{8}$$

is the squared solution to (7).

We will assess seepage q (per unit length) and evaporation rate e in two steps.

3.1.1. Seepage

Seepage through the embankment at $x = L$ and consistent with (8) is (see [5]) given by

$$\Phi(K) \equiv q(L) = -\frac{eL}{2} + K\frac{h_0^2 - h_L^2}{2L}, \tag{9}$$

where Φ indicates that $q \equiv q(L)$ is the quantity of interest whose membership function $\widehat{\mu}$ will be inferred.

We can apply (8) to obtain K. To this end, let us drill two vertical boreholes into the embankment at $x_1 = L/3$ and $x_2 = 2L/3$ and assess the groundwater level h there. We obtain r_1 and r_2, respectively. Since we do not know e in (8) and since it is easier to measure infiltration rate e_{in} than evaporation rate e_{ev}, we measure e_{in} during rainfall and use $e = e_{\text{in}}$ in (8). We assume that e_{in} is measured accurately, that is, known exactly, but the values r_1 and r_2 are burdened with errors.

Let us define

$$\omega(e_{\text{in}}, K) = \sum_{i=1}^{2}(r_i - h_{e_{\text{in}},K}(x_i))^2, \quad \mu_1(K) = 2 - \sqrt{\frac{\omega(e_{\text{in}}, K)}{\omega(e_{\text{in}}, K_{\min})}}, \tag{10}$$

where $K_{\min}$ is identified by the least squares method; see (4) where $h_{e_{\text{in}},K}(x_i)$ plays the role of $\psi(a, x_i)$. As a consequence, K is fuzzified and a fuzzy interval $A = \{K \in \mathbb{R} | \ \mu_1(K) \in [0,1]\}$, see (6), is considered for the saturated hydraulic conductivity.

We observe that $\widehat{\mu}$ is a shifted "multiple" of μ_1 in the sense that each α-cut of the fuzzy interval determined by $\widehat{\mu}$ is obtained as the $(h_0^2 - h_L^2)/(2L)$ multiple of A^α shifted by $-e_{\text{in}}L/2$; see (9). Consequently, there is no need to solve the minimization and maximization problems (3) to obtain α-cuts of the fuzzy quantity $q = \Phi(K)$ in this extremely simple example.

For $L = 10$, $h_0 = 4$, $h_L = 3$, $e_{\text{in}} = -3 \times 10^{-7}$, $r_1 = 4.41$, $r_2 = 4.09$, we obtain $\widehat{\mu}$ as depicted in Figure 3 (left).

3.1.2. Evaporation

Let us pay attention to evaporation, a new quantity of interest. To evaluate the evaporation rate e_{ev} during a dry-weather period, we again assess h at x_1 and x_2 with the respective outputs $\widetilde{r}_1$ and $\widetilde{r}_2$. Like in (10), we define

$$\widetilde{\omega}(e_{\text{ev}}, K) = \sum_{i=1}^{2}(\widetilde{r}_i - h_{e_{\text{ev}},K}(x_i))^2 \tag{11}$$

but, unlike (10), $e_{\text{ev}} \equiv e$ is not known. For each fixed K, an inverse problem can be solved, that is, the evaporation rate can be found that minimizes (11). However, since K is fuzzy, we have to consider $K \in A^\alpha$, where A^α are the α-cuts determined by μ_1 through r_i and e_{in}; see (10). The model M_1 remains unchanged, but the model M_2 becomes the K-dependent inverse problem now.

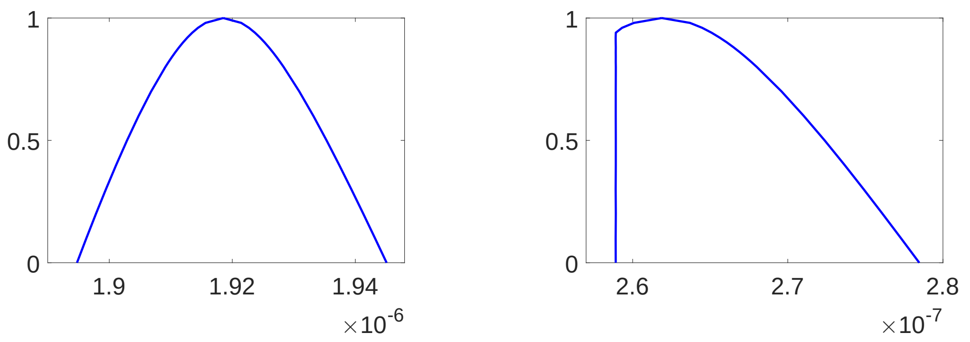

Figure 3: Left: The membership function $\widehat{\mu}$ of q. Right: The membership function $\widetilde{\mu}$ of $e_{\rm ev}$. In both graphs, the vertical axis represents α and the horizontal axis represents the quantity of interest q and $e_{\rm ev}$, respectively.

To get $R^{\alpha}_{e_{\rm ev}} = [e^{\alpha}_{\rm ev,\,min}, e^{\alpha}_{\rm ev,\,max}]$, a parallel to (3), we solve

$$K^{\alpha}_{\rm ev,\,min} = \arg\min_{K\in A^{\alpha}} \min_{e_{\rm ev}\in I_e} \widetilde{\omega}(e_{\rm ev}, K) \text{ and } K^{\alpha}_{\rm ev,\,max} = \arg\max_{K\in A^{\alpha}} \min_{e_{\rm ev}\in I_e} \widetilde{\omega}(e_{\rm ev}, K), \tag{12}$$

where I_e is a chosen sufficiently large interval bounding the search. Then

$$e^{\alpha}_{\rm ev,\,min} = \arg\min_{e_{\rm ev}\in I_e} \widetilde{\omega}(e_{\rm ev}, K^{\alpha}_{\rm ev,\,min}) \text{ and } e^{\alpha}_{\rm ev,\,max} = \arg\min_{e_{\rm ev}\in I_e} \widetilde{\omega}(e_{\rm ev}, K^{\alpha}_{\rm ev,\,max}). \tag{13}$$

Since only a finite number of levels α is used in calculations, there is no need to solve (13) in practice. The values $e^{\alpha}_{\rm ev,\,min}$ and $e^{\alpha}_{\rm ev,\,max}$ are stored in the course of solving (12).

For $\widetilde{r}_1 = 2.90$ and $\widetilde{r}_2 = 2.60$ entering the calculations, the membership function $\widetilde{\mu}$ of $e_{\rm ev}$ is depicted in Figure 3 (right).

The graph, which might seem strange at first glance, shows that $e_{\rm ev}$ is represented by a crisp value at the level $\alpha = 1$ because also the 1-cut of A is a singleton set comprising a unique K. If we start to increase the amount of uncertainty in K by decreasing α, we also decrease $e^{\alpha}_{\rm ev,\,min}$ as the solution of the min-min problem (12)-(13). For $\alpha < 0.94$, the condition $K \in A^{\alpha}$ is no longer an active constraint in the minimization of $\widetilde{\omega}$ with respect to $e_{\rm ev}$ and the minimizer $e^{\alpha}_{\rm ev,\,min}$ is no longer dependent on α.

Problem (12) is, in fact, a sort of best- and worst-case scenario problems. Indeed, in the min-min problem, $e_{\rm ev}$ and K "cooperate" to minimize (11), whereas K is an "antagonist" of $e_{\rm ev}$ in the max-min problem (12) in which the minimizer of $\widetilde{\omega}$ is sought under the worst conditions that K can produce.

4. Conclusions

The ideas presented in Section 1 are applicable to parameters belonging to other spaces than $\mathbb{R}$ or $\mathbb{R}^n$. We can, for instance, take $S \subset C([d_1, d_2])$, where $C([d_1, d_2])$ stands for the space of continuous functions on an interval $[d_1, d_2]$, and consider a problem M_1 represented by, say, an ordinary differential equation (ODE) $D_a u = f$ supplemented by initial or boundary conditions, where D_a is an a-dependent differential operator, $a \in S$. Let us assume that inaccurate measurements $\{r_i\}_{i=1}^n$ are associated with $u_a(x_i)$, the ODE solution at $\{x_i\}_{i=1}^n$. Under some assumptions, a function $b_{\min} \in S$ can be identified by the least squares method as in (4). Consequently, the fuzzification of the identified parameter-function can be done as in Section 2.

If a scalar quantity of interest represents the output of an a-dependent Model 2, Zadeh's principle can again be applied to obtain the membership function associated with the the quantity of interest. Besides a, Model 2 can depend on other parameters either crisp or fuzzy. In calculations, S is approximated by a set of functions controlled by a finite number of parameters. As a consequence, the approximate problem is formulated in terms of finite dimensional fuzzy sets and their α-cuts. Dealing with the latter can still be a rather hard task because A^α will enter the minimization (maximization) problem (3) as a constraint determined by (5) and the Model 1 output. Such constraint can be (and usually will be) non-linear.

The common concept of membership functions is sometimes awkward. Traditionally, the range of membership functions is limited to (subsets of) $[0, 1]$. This limits flexibility in the grading of fuzzy uncertainty. To make things easier, we can adopt the approach presented in [1] within the framework of info-gap decision theory and use membership functions in an "upside down" form where the amount of uncertainty is minimal at $\alpha = 0$ and increases with increasing α. In this approach, the upper bound of α is not limited to 1, but can be arbitrary large and can even increase in the course of computing. An example can be inferred from μ_1 in (5) as follows

$$\widehat{\mu}_A(b) = c_1 \left(\left(\frac{\omega(b)}{\omega(b_{\min})} \right)^{c_2} - 1 \right)^{c_3},$$

where c_1, c_2, and c_3 are positive constants.

The α-cuts associated with such "upside down" membership functions are defined by $A^\alpha = \{a \in A |\ \mu_A(a) \le \alpha\}$.

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