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STABILITY OF ALE DISCONTINUOUS GALERKIN METHOD WITH RADAU QUADRATURE

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Abstract: We assume the nonlinear parabolic problem in a time dependent domain, where the evolution of the domain is described by a regular given mapping. The problem is discretized by the discontinuous Galerkin (DG) method modified by the right Radau quadrature in time with the aid of Arbitrary Lagrangian-Eulerian (ALE) formulation. The sketch of the proof of the stability of the method is shown.

Keywords: ALE formulation, discontinuous Galerkin method, implicit Runge-Kutta method, discrete characteristic function, stability

MSC: 65N60, 65M99

1. Introduction

There are many theoretical results devoted to the stability analysis of parabolic problems with a fixed domain. Nevertheless, there is number of areas with many important applications of parabolic PDEs with time dependent domain, e.g. problems with moving boundaries, where the motion of the boundary is either prescribed or given by other means.

There are several approaches how to deal with problems in time dependent domains, e.g. fictitious domain method, see e.g. [15], or immersed boundary method, see e.g. [3]. A very popular technique is Arbitrary Lagrangian-Eulerian (ALE) method that is based on a regular one-to-one ALE mapping of the reference domain to the current one. ALE method is often applied in connection with conforming finite element method (FEM) in space and lower order time discretizations (backward Euler method, Crank-Nicolson method, BDF2), see e.g. [12] or [13]. The numerical analysis of the lower order time discretization schemes can be found in [1], [7], [8].

We present a higher order time discretization of arbitrary order based on DG method, where the integrals are evaluated by right Radau quadrature. For a survey about DG time discretization, see e.g. [16], the connection between Galerkin discretizations and implicit Runge-Kutta methods can be found in [14]. DG time

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discretization and its Radau quadrature variant are well known for their high robustness and accuracy. This is confirmed by experiments, see e.g. [4] where the results obtained by BDF and time DG are compared on the problem of vibrating airfoil. According to this comparison, DG time discretization seems to be more robust and accurate than BDF.

2. Continuous problem

Let $T > 0$. The evolution of the bounded polyhedral time dependent domain $\Omega_t \subset \mathbb{R}^d$ $(d = 1, 2, 3)$ with a Lipschitz continuous boundary $\partial\Omega_t$ in time is described by a given regular one-to-one ALE mapping

$$\mathcal{A} \colon \overline{\Omega}_0 \times [0, T] \to \overline{\Omega}_t, \tag{1}$$

where $\overline{\Omega}_0$ or $\overline{\Omega}_t$ are closures of Ω_0 or Ω_t, respectively. For the purpose of the proof of the stability we introduce following regularity assumptions on the ALE mapping $\mathcal{A}$:

$$\mathcal{A} \in W^{1,\infty}(0, T, W^{1,\infty}(\Omega_0)), \quad \mathcal{A}^{-1} \in W^{1,\infty}(0, T, W^{1,\infty}(\Omega_t)). \tag{2}$$

Moreover, we denote the Jacobi matrix of $\mathcal{A}$ by $B = \mathrm{d}\mathcal{A}/\mathrm{d}X$, the corresponding determinant by $J = \det(B)$ and the *domain velocity* by $\omega = \partial\mathcal{A}/\partial t \circ \mathcal{A}^{-1}$. From the regularity assumptions (2) it is possible to show that B, B^{-1}, J, J^{-1}, ω and $\nabla \cdot \omega$ are bounded, i.e. there exists a constant $C_{\mathcal{A}} > 0$ such that

$$\begin{aligned} &\max(\|B\|_{L^\infty(0,T,L^\infty(\Omega_0))}, \|B^{-1}\|_{L^\infty(0,T,L^\infty(\Omega_0))}, \|J\|_{W^{1,\infty}(0,T,L^\infty(\Omega_0))} \\ &\|J^{-1}\|_{L^\infty(0,T,L^\infty(\Omega_0))}, \|\omega\|_{L^\infty(0,T,L^\infty(\Omega_t))}, \|\nabla \cdot \omega\|_{L^\infty(0,T,L^\infty(\Omega_t))}) \le C_{\mathcal{A}}. \end{aligned} \tag{3}$$

We consider the following nonliner initial–boundary value problem

$$\begin{aligned} \frac{\partial u}{\partial t} - \operatorname{div}(\beta(u)\nabla u) = f \quad &\text{in } \Omega_t \times (0, T), \\ u = 0 \quad &\text{in } \partial\Omega_t \times (0, T), \\ u = u^0 \quad &\text{in } \Omega_0. \end{aligned} \tag{4}$$

We assume that the right-hand side $f \in C([0, T], L^2(\Omega_t))$, the initial condition $u^0 \in L^2(\Omega_0)$ and the function $\beta \colon \mathbb{R} \to [\beta_0, \beta_1]$, where $0 < \beta_0 \le \beta_1 < \infty$, is Lipshitz continuous. We denote by $(.,.)_t$ and $\|.\|_t$ the $L^2(\Omega_t)$ scalar product and norm, respectively.

Problem (4) is usually transformed into the *ALE formulation*. To this end, we introduce ALE derivative

$$D_t u = \frac{\partial u}{\partial t} + \omega \cdot \nabla u. \tag{5}$$

Now we introduce the ALE formulation equivalent to problem (4)

$$\begin{aligned} D_t u - \operatorname{div}(\beta(u)\nabla u) - \omega \cdot \nabla u = f \quad &\text{in } \Omega_t \times (0, T), \\ u = 0 \quad &\text{in } \partial\Omega_t \times (0, T), \\ u = u^0 \quad &\text{in } \Omega_0. \end{aligned} \tag{6}$$

3. Discretization

In this section we only describe time discretization (Rothe's method) of problem (6) by discontinuous Galerkin time discretization and its right Radau quadrature variant.

3.1. Time discontinuous Galerkin

In order to discretize problem (6) in time, we consider a time partition $0 = t_0 < t_1 < \ldots < t_r = T$ with time intervals $I_m = (t_{m-1}, t_m)$, time steps $\tau_m = t_m - t_{m-1}$ and $\tau = \max_{m=1,\ldots,r} \tau_m$. We define the solution space

$$V^\tau = \{v \in L^2(0, T, H_0^1(\Omega_t)) : (v \circ \mathcal{A})|_{I_m} \in P_q(I_m, H_0^1(\Omega_0))\}.$$

For a function $v \in V^\tau$ we define the one–sided limits

$$v_\pm^m = v(t_m\pm) = \lim_{t \to t_m\pm} v(t) \tag{7}$$

and the jumps

$$\{v\}_m = v_+^m - v_-^m, \quad m \geq 1 \quad \text{and} \quad \{v\}_0 = v_+^0 - u^0. \tag{8}$$

Now, we are able to formulate the semi-discrete time discontinuous Galerkin scheme:

Definition 1. *We say that a function $U \in V^\tau$ is the discrete solution of problem (6) obtained by time discontinuous Galerkin method, if the following conditions are satisfied*

$$\begin{aligned} \int_{I_m} (D_t U, v)_t + (\beta(U)\nabla U, \nabla v)_t - (\omega \cdot \nabla U, v)_t dt + (\{U\}_{m-1}, v_+^{m-1})_{t_{m-1}} \\ = \int_{I_m} (f, v)_t dt \quad \forall m = 1, \ldots, r, \, \forall v \in V^\tau. \end{aligned} \tag{9}$$

3.2. Radau quadrature

The method (9) contains an integral of the nonlinear term that can be difficult to evaluate exactly. Therefore, it is favourable to apply a suitable quadrature formula.

Let $r \in P_{q+1}$ be a polynomial of degree $q + 1$ satisfying $r(0) = 1$, $r(1) = 0$ and $r \perp P_{q-1}$. We denote the zeros of this polynomial $c_0, \ldots, c_q$, where $c_q = 1$. Then we define right Radau quadrature (right Gauss-Radau quadrature) rule on $[0, 1]$ of function F by

$$\int_0^1 F(t) \, \mathrm{d}t \approx Q[F] = \sum_{i=0}^{q} b_i F(c_i), \tag{10}$$

where the weights b_i are chosen in such a way that the resulting quadrature has maximal degree, i.e. degree $2q$. Similarly, one can define right Radau quadrature on I_m by

$$\int_{I_m} F(t)\,\mathrm{d}t \approx Q_m[F] = \tau_m \sum_{i=0}^{q} b_i F(c_{m,i}), \tag{11}$$

where $c_{m,i} = t_{m-1} + \tau_m c_i$ and $c_{m,q} = t_{m-1} + \tau_m c_q = t_m$.

Now, we are able to formulate the semi-discrete quadrature variant of method (9):

Definition 2. *We say that a function $U \in V^\tau$ is the discrete solution of problem (6) obtained by quadrature variant of time discontinuous Galerkin method, if the following conditions are satisfied*

$$\begin{aligned} Q_m[(D_t U, v)_t + (\beta(U)\nabla U, \nabla v)_t - (\omega \cdot \nabla U, v)_t] + (\{U\}_{m-1}, v_+^{m-1})_{t_{m-1}} \\ = Q_m[(f, v)_t] \quad \forall m = 1, \dots, r,\ \forall v \in V^\tau. \end{aligned} \tag{12}$$

For simplicity, we assume that there exists an unique discrete solution.

The time discretization in (12) can be viewed as a generalization of some specific classical one–step methods for parabolic problems. It is possible to show that setting $q = 0$, i.e. piecewise constant approximation in time, is equivalent to backward Euler method in time. Similarly, the higher polynomial degree approximations in time lead to methods that are equivalent to Radau IIA Runge-Kutta methods. For details about the relations between Galerkin methods and Runge-Kutta methods see e.g. [9] and [14]. For the descriptions of Radau IIA Runge-Kutta methods see e.g. [6] or [10] and [11].

4. Stability analysis

The aim of this section is to show that the numerical scheme (12) is stable, i.e. the approximate solution obtained from (12) can be bounded in terms of the data f and u^0 of the problem (4). Through this section we will denote by $C_1, C_2, \ldots > 0$ constants that can depend on the data bounds β_0 and β_1, on the polynomial degree q and the regularity of ALE mapping $C_{\mathcal{A}}$.

Setting $v = U$ in (12) we get the basic identity

$$\begin{aligned} &Q_m[(D_t U, U)_t + (\beta(U)\nabla U, \nabla U)_t - (\omega \cdot \nabla U, U)_t] + (\{U\}_{m-1}, U_+^{m-1})_{t_{m-1}} \\ &= Q_m[(f, U)_t]. \end{aligned} \tag{13}$$

We can estimate individual terms in (13).

Lemma 1. *There exists constants* $C_1, C_2 > 0$ *such that*

$$\begin{aligned} & Q_m[(D_t U, U)_t] + (\{U\}_{m-1}, U_+^{m-1})_{t_{m-1}} \\ & \geq \frac{1}{2}\|U_-^m\|_{t_m}^2 - \frac{1}{2}\|U_-^{m-1}\|_{t_{m-1}}^2 - C_1 Q_m[\|U\|_t^2], \quad \forall U \in V^\tau, \\ & Q_m[(\beta(U)\nabla U, \nabla U)_t] \geq \beta_0 Q_m[\|\nabla U\|_t^2], \quad \forall U \in V^\tau, \\ & Q_m[(\beta(U)\nabla U, \nabla v)_t] \leq \beta_1 Q_m[\|\nabla U\|_t^2] + \frac{\beta_1}{4} Q_m[\|\nabla v\|_t^2], \quad \forall U, v \in V^\tau, \\ & Q_m[(\omega \cdot \nabla U, v)_t] \leq \frac{\beta_0}{2} Q_m[\|\nabla U\|_t^2] + C_2 Q_m[\|v\|_t^2], \quad \forall U, v \in V^\tau, \\ & Q_m[(f, U)_t] \leq \frac{1}{4} Q_m[\|U\|_t^2] + Q_m[\|f\|_t^2], \quad \forall v \in V^\tau. \end{aligned} \tag{14}$$

Proof. The first inequality in (14) is just a modification of the similar result from [2], where the quadrature Q_m is replaced by integral over I_m. The proof is analogical. All the other inequalities in (14) are just an application of Cauchy's and Young's inequalities. □

Applying individual estimates from Lemma 1 we get

$$\begin{aligned} \frac{1}{2}\|U_-^m\|_{t_m}^2 - \frac{1}{2}\|U_-^{m-1}\|_{t_{m-1}}^2 + \frac{\beta_0}{2} Q_m[\|\nabla U\|_t^2] & \leq Q_m[\|f\|_t^2] + C_3 Q_m[\|U\|_t^2], \\ & \leq Q_m[\|f\|_t^2] + \tau_m C_3 \sup_{t \in I_m} \|U\|_t^2. \end{aligned} \tag{15}$$

To be able to get rid of the last supremum term, we need to derive a technique for estimating the values of the discrete solution inside of intervals I_m.

4.1. Discrete characteristic function

The concept of the discrete characteristic function on fixed domains comes from [5]. We will use a notation $\tilde{v} = v \circ \mathcal{A}$ for the transformation of functions from the evolving space-time cylinder to the reference space-time cylinder. From the assumptions on the ALE mapping $\mathcal{A}$ and according to the definition of space V^τ it is possible to see that this transformation is bijection between V^τ and $\tilde{V}^\tau$, where

$$\tilde{V}^\tau = \{v \in L^2(0, T, H_0^1(\Omega_0)) : v|_{I_m} \in P_q(I_m, H_0^1(\Omega_0))\}, \tag{16}$$

i.e. $\tilde{V}^\tau$ represents the space of classical piecewise polynomial functions in time.

We define the discrete characteristic function for time dependent domains in three steps. At first, the given function $U \in V^\tau$ is transformed onto the reference domain, i.e. $\tilde{U} = U \circ \mathcal{A} \in \tilde{V}^\tau$. Second step is the construction of discrete characteristic function in fixed domains, i.e. $\tilde{U}_s \in \tilde{V}^\tau$ such that

$$\begin{aligned} \tilde{U}_{s+}^{m-1} &= \tilde{U}_+^{m-1}, \\ \int_{I_m} \left(\tilde{U}_s, \frac{\partial v}{\partial t}\right)_0 \mathrm{d}t &= \int_{t_{m-1}}^{s} \left(\tilde{U}, \frac{\partial v}{\partial t}\right)_0 \mathrm{d}t \quad \forall v \in \tilde{V}_h^\tau. \end{aligned} \tag{17}$$

The last step is the transformation back to the current domain, i.e. $U_s = \tilde{U}_s \circ \mathcal{A}^{-1} \in V^\tau$.

Now, we want to show the properties of the discrete characteristic function.

Lemma 2. *Let $U \in V^\tau$ and $U_s \in V^\tau$ be its discrete characteristic function associated with $s \in I_m$. Then there exists constants $C_4, C_5 > 0$ such that*

$$Q_m[(D_t U, U_s)_t] + (\{U\}_{m-1}, U_{s+}^{m-1})_{t_{m-1}} \geq \frac{1}{2}\|U(s)\|_s^2 - \frac{1}{2}\|U_-^{m-1}\|_{t_{m-1}}^2 - C_4 \tau_m \sup_{t \in I_m} \|U\|_t^2, \tag{18}$$

$$Q_m[\|U_s\|_t^2] \leq C_5 Q_m[\|U\|_t^2],$$
$$Q_m[\|\nabla U_s\|_t^2] \leq C_5 Q_m[\|\nabla U\|_t^2]\,\mathrm{d}t. \tag{19}$$

Proof. An analogical result is proved in [2]. Since the proof is long and technical, it is skipped in this paper. □

4.2. Main result

Now, we are ready to formulate the main result.

Theorem 3. *Let $U \in V^\tau$ be an approximate solution obtained by scheme (12). Then there exist constants $C > 0$ and $C^* > 0$ such that $\tau < C^*$ implies*

$$\sup_{I_m} \|U\|_t^2 \leq C(\|u^0\|_0^2 + T\|f\|_{L^\infty(0,T,L^2(\Omega_t))}^2). \tag{20}$$

Proof. Setting $v = U_s$ in the left-hand side of (12), where $s \in [t_{m-1}, t_m]$ such that $\|U(s)\|_s = \sup_{t \in I_m} \|U\|_t$, and Lemma 1 and Lemma 2 we get

$$\begin{aligned} Q_m[(D_t U, U_s)_t + (\beta(U)\nabla U, \nabla U_s)_t - (\omega \cdot \nabla U, U_s)_t] + (\{U\}_{m-1}, U_+^{m-1})_{t_{m-1}} \\ \geq \frac{1}{2}\sup_{I_m}\|U\|_t^2 - \frac{1}{2}\sup_{I_{m-1}}\|U\|_t^2 - C_4\tau_m \sup_{t\in I_m}\|U\|_t^2 - \beta_1 Q_m[\|\nabla U\|_t^2] \\ - \frac{\beta_1 C_5}{4} Q_m[\|\nabla U\|_t^2] - \frac{\beta_0}{2} Q_m[\|\nabla U\|_t^2] - C_2 C_5 \tau_m \sup_{t\in I_m}\|U\|_t^2, \end{aligned} \tag{21}$$

where we use the notation $\sup_{I_0} \|U\|_t^2 = \|u^0\|_0^2$. Similarly, setting $v = U_s$ in the right-hand side of (9) we get

$$Q_m[(f, U_s)_t] \leq Q_m[\|f\|_t^2] + \frac{C_5}{4}\tau_m \sup_{t \in I_m} \|U\|_t^2. \tag{22}$$

Using these relations we get

$$\frac{1}{2}\sup_{I_m}\|U\|_t^2 - \frac{1}{2}\sup_{I_{m-1}}\|U\|_t^2 \leq Q_m[\|f\|_t^2] + C_6\tau_m \sup_{t\in I_m}\|U\|_t^2 + C_7 Q_m[\|\nabla U\|_t^2],$$

where $C_6 = C_4 + (4C_2 + 1)C_5/4$ and $C_7 = (4\beta_1 + \beta_1 C_5 + 2\beta_0)/4$.

Multiplying (15) by $C_8 = 2C_7/\beta_0$ and summing with (23) we get

$$\frac{1}{2}\left(C_8\|U_-^m\|_{t_m}^2 + \sup_{I_m}\|U\|_t^2\right) - \frac{1}{2}\left(C_8\|U_-^{m-1}\|_{t_m}^2 + \sup_{I_{m-1}}\|U\|_t^2\right) \tag{23}$$
$$\le (C_8+1)Q_m[\|f\|_t^2] + (C_3C_8 + C_6)\tau_m \sup_{t\in I_m}\|U\|_t^2.$$

Setting $C^* = (8C_1C_3 + 2C_2)^{-1}$ we get $(C_3C_8 + C_6)\tau_m < 1/2$ and the statement of the theorem follows from the application of the discrete Gronwall lemma. □

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