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A NUMERICAL METHOD FOR THE SOLUTION OF THE NONLINEAR OBSERVER PROBLEM

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Abstract: The central part in the process of solving the observer problem for nonlinear systems is to find a solution of a partial differential equation of first order. The original method proposed to solve this equation used expansions into Taylor polynomials, however, it suffers from rather restrictive assumptions while the approach proposed here allows to generalize these requirements. Its characteristic feature is that it is based on the application of the Finite Element Method. An illustrating example is provided.

Keywords: finite element method, observer, partial differential equation

MSC: 93B51, 93C15, 35F05, 65N30

1. Introduction

Usually, knowledge of values of all variables (that means, knowledge of the entire state vector) is required to control dynamical systems. However, not all variables can be measured in practice. This concerns typically velocity in mechanical systems, other such example are some chemical or biological quantities. As their knowledge is important for the control process, one has to find a way how to recover their values from the measured ones. For instance, one needs to compute the (unmeasured) velocity in a mechanical system from the measured values of the position, forces etc. This computation is conducted by the so called observer – an algorithm that provides estimates of the unmeasured quantities from the measured ones.

2. Observer for a linear system

Consider the following linear n-dimensional system:

$$\dot{x} = Ax, \quad \text{(the state equation)} \tag{1}$$

$$y = Cx, \quad \text{(the output equation)} \tag{2}$$

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with initial conditions given as $x(0) = x_0$. Here $x\colon [0,\infty) \to \mathbb{R}^n$, $y\colon [0,\infty) \to \mathbb{R}^p$, $A \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{p\times n}$.

We assume that the state x is not measurable, only y is accessible. The question is how to gain knowledge of the entire vector $x(t)$ from the (in practice, smaller-dimensional) vector $y(t)$.

The remedy is to define another system (called the observer)

$$\dot{\hat{x}} = \varphi(\hat{x}, y),\ \ \hat{y} = \psi(\hat{x})$$

for some suitable functions $\varphi\colon \mathbb{R}^{n+p} \to \mathbb{R}^n$, $\psi\colon \mathbb{R}^n \to \mathbb{R}^p$. Then, one can introduce the following definition:

Definition 1. *The observation error e is defined as $e = x - \hat{x}$.*

The goal is to achieve $\lim_{t\to\infty} \|e(t)\| = 0$. Note that the observer may also use the output given by (2), however, it cannot utilize other quantities from the observed system. For details, see e.g. [5] or [6].

An algorithm how to estimate the non-measurable quantities can be described as follows: assume

$$\operatorname{rank}(C^T, A^TC^T, \ldots, (A^T)^{n-1}C^T) = n \tag{3}$$

holds and let also matrix $L \in \mathbb{R}^{p\times n}$ be such that eigenvalues of $A-LC$ have negative real parts. Then, consider the system

$$\dot{\hat{x}} = A\hat{x} + L(y - \hat{y}), \tag{4}$$
$$\hat{y} = C\hat{x}. \tag{5}$$

One can prove (see e.g. [1]) convergence of the observation error to zero under this assumption. The question is how to design the matrix L such that all eigenvalues of $A - LC$ have negative real parts.

The procedure described below can be easily generalized for nonlinear systems. Take $\tilde{A} \in \mathbb{R}^{n\times n}$ so that its eigenvalues have negative real parts (in practice, they must lie "further left" from the eigenvalues of A to guarantee practically acceptable behavior). Moreover, let the vector b be so that

$$\operatorname{rank}(b, \tilde{A}b, \ldots, \tilde{A}^{n-1}b) = n. \tag{6}$$

Then, an observer for system (1,2) can be found. This result is formulated in [1] in form of the following lemma:

Lemma 2. *Assume (6) as well as (3) holds and matrices A and $\tilde{A}$ have no common eigenvalues. Then there exists a matrix $\bar{\Phi}$ satisfying*

$$\bar{\Phi}A = \tilde{A}\bar{\Phi} + bC. \tag{7}$$

Moreover, matrix $\bar{\Phi}$ is nonsingular. If we define $L = \bar{\Phi}^{-1}b$, then eigenvalues of matrix $A - LC$ have negative real parts.

Equation (7) is called the Sylvester equation.

Remark 3. *Denote by z the function $z = \bar{\Phi}e$. As shown in [2], the relation $\dot{z} = \tilde{A}z$ holds for this function, hence $\lim_{t\to\infty} \|z(t)\| = 0$ since all eigenvalues of $\tilde{A}$ have negative real parts. Since matrix $\bar{\Phi}$ is nonsingular, one has $\lim_{t\to\infty} \|e(t)\| = 0$.*

3. Observer for a nonlinear system

Nonlinear system with output y is defined as

$$\dot{x} = F(x), \; y = h(x) \tag{8}$$

with initial condition $x(0) = x_0$. It is assumed all functions are sufficiently smooth. Let us rewrite these equations as

$$\dot{x} = Ax + f(x), \; y = Cx + \hat{h}(x), \tag{9}$$

where functions f, $\hat{h}$ vanish at the origin together with their derivatives.

Inspired by the linear case, the goal is to find an observer defined as

$$\dot{\hat{x}} = A\hat{x} + f(\hat{x}) + L(\hat{x})(y - \hat{y}), \; \hat{y} = h(\hat{x}). \tag{10}$$

To be specific, we aim to find a continuous function $L\colon \mathbb{R}^n \to \mathbb{R}^n$ so that the observation error e converges to zero for $t \to \infty$.

The nonlinear counterpart to (7) is (see [2])

$$\frac{\partial \Phi}{\partial x}(Ax + f(x)) = \tilde{A}\Phi(x) + bh(x). \tag{11}$$

One can write the function Φ as a sum of the linear term $\bar{\Phi}x$ (this term can be computed precisely using (7)) and the $O(|x|^2)$-function ϕ which must be computed numerically. Thus, one has $\Phi(x) = \bar{\Phi}x + \phi(x)$.

The eigenvalues of matrix $\tilde{A}$ are supposed to satisfy

$$\max(\mathrm{Re}\ \lambda(\tilde{A})) < \min(\mathrm{Re}\ \lambda(A)), \tag{12}$$

where the symbol $\lambda(A)$ denotes the spectrum of matrix A. Then, the observer is given by

$$L(\hat{x}) = \left(\frac{\partial \Phi}{\partial x}(\hat{x})\right)^{-1} b. \tag{13}$$

Equation (11) has no boundary conditions as this equation is satisfied on $\mathbb{R}^n$.

Remark 4. *Again, [2] shows that $\lim_{t\to\infty} \|\Phi(x(t)) - \Phi(\hat{x}(t))\| = 0$. Since the mapping Φ is a diffeomorphism, this implies $\lim_{t\to\infty} \|e(t)\| = 0$.*

4. Solution of the nonlinear observer equation

4.1. Methods based on Taylor expansions and on successive approximations of stable manifolds

The paper [2] was the first to introduce equation (11). Taylor expansions were used for solution of this equation: all functions involved were approximated by Taylor polynomials, the solution was also sought in this form. Conditions of existence of such approximations of the solution are also given in that paper. Moreover, in [3], this method was generalized to systems with delayed measurements.

Apart from obvious drawbacks (problems with convergence, computational complexity,...) there is one very important restriction: existence of the Taylor series is guaranteed only if all eigenvalues of A have negative real parts or all have positive real parts. This is due to the fact that the proof of existence of a solution is based on the Auxiliary Lyapunov Theorem [4].

However, this is a serious issue as systems with matrix A having eigenvalues with zero real part are also often encountered in practice; the same holds for systems where matrix A has eigenvalues with both positive and negative real parts.

This very restrictive requirement was removed in [10] by proposing a different method for approximation of the solution of (11). It is based on construction of certain invariant manifolds for the system composed of (8) and (10). This method was successfully applied in practice, see e.g. [11]. However, complicated computations are required to obtain the observer by utilizing this approach.

4.2. FEM-based solution

In this section, it is described how the Finite Element Method can be used for the solution of (11). Application of FEM to the observer problem of nonlinear systems with delayed measurements was described in [7], applications to a biological system are shown in [8]. This section summarizes results of these papers.

The key theorem that allows to prove existence of a solution of (11) can be found in [9]:

Lemma 5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary, $0 \in \Omega$, let $n(x)$ be the outward normal vector at the point $x \in \partial\Omega$. Let $a > 0$, $\beta \in (C^1(\bar{\Omega}))^n$ and $g \in L^2(\Omega)$. Let $\Gamma^- = \{x \in \partial\Omega | n(x).\beta(x) < 0\}$. Further, assume there exists a constant $\omega > 0$ such that*

$$a - \frac{1}{2}\mathrm{div}\beta(x) > \omega. \tag{14}$$

Then equation

$$\beta(x).\nabla\varphi(x) + a\varphi(x) = g(x)$$

has a unique solution $\varphi \in L^2(\Omega)$ with boundary conditions $\varphi(x) = 0, x \in \Gamma^-$.

This lemma is applicable only to scalar linear equations. The equation was derived as a limit case of the diffusion equation when the diffusion coefficients decreases to 0. The proof of this lemma is based on the theory of compact operators, see [9] and references therein.

Lemma 6. *Let the following inequality holds for every $\tilde{a} \in \sigma(\tilde{A})$.*

$$\tilde{a} - \frac{1}{2}\mathrm{Trace} A > 0. \tag{15}$$

Then there exists a neighborhood of the origin $U \subset \mathbb{R}^n$ so that (14) is satisfied in U.

The following theorem is a reformulation of the above lemmas into a form suitable for the solution of (11).

Theorem 7. *Let (3) and (6) hold, all eigenvalues of $\tilde{A}$ are real and satisfy*

$$\max\{\alpha|\alpha \in \sigma(\tilde{A})\} < \min\Big(\{0\} \cup \{\mathrm{Re}\ \alpha|\alpha \in \sigma(A)\}\Big). \tag{16}$$

Assume $\Omega \subset \mathbb{R}^n$, $0 \in \Omega$, is a bounded domain such that

$$a_i - \frac{1}{2}\Big(\mathrm{Trace} A + \mathrm{div} f(x)\Big) > 0 \tag{17}$$

holds for all $x \in \Omega$ and all $i = 1, \ldots, n$. Assume also $\bar{\Phi}$ solves (7). Let also $\Gamma_i^- = \{x \in \partial\Omega | n(x).(Ax + f(x)) < 0\}$. Then

- *for every $i \in \{1, \ldots, n\}$ there exist uniquely determined functions $\phi_i \in L^2(\Omega)$, $\phi_i = 0$ on Γ_i^-, such that $\phi = (\phi_1, \ldots, \phi_n)^T$ solves*

$$(\bar{\Phi} + \frac{\partial \phi}{\partial x})(Ax + f(x)) = \tilde{A}(\bar{\Phi}x + \phi(x)) + bh(\mathcal{T}(x)). \tag{18}$$

- *the observer gain L given by*

$$L(\hat{x}) = \Big(\bar{\Phi} + \frac{\partial \phi}{\partial x}(\hat{x})\Big)^{-1} b \tag{19}$$

  is such that observer (10) guarantees $\lim_{t\to\infty} \|e(t)\| = 0$.

For the proof see [7]. This paper presents the proof even for system with delayed measurements; the formulation here is thus a special case where this delay equals zero.

Note that the restrictive condition on position of the eigenvalues of A was removed. This is the most significant contribution.

Nevertheless, there are still open problems that need to be solved before the entire theory is completed. For instance:

- The main theorem guarantees existence of a solution Φ, however not its differentiability.

- While invertibility of the Jacobi matrix of Φ is guaranteed at the origin (it coincides with the solution of the Sylvester equation which is nonsingular), invertibility for other values of $\hat{x}$ is not guaranteed.

5. Example

The plant considered in this example is a magnetic levitation system. This example was taken from [11].

$$\dot{x} = v, \ \dot{v} = g - \frac{C}{m}\frac{u + u_0}{(x + x_0 + d)^2}, \tag{20}$$

x is the vertical position of the ball to be levitated, v is its velocity. Parameters: $g = 9.81ms^{-2}$: gravitational constant, $C = 1.2281 \times 10^{-4} Nm^2A^{-2}$: electromagnetic constant, $m = 0.0661kg$: mass of the ball, $d = 0.00571m$ is the parameter of the actuator. The ball should be stabilized in the equilibrium position $x_0 = 0.007m$; $u_0 = \sqrt{\frac{mg}{C}}(x_0 + d)$ is the control signal corresponding to this equilibrium position of the ball.

Using the transformation $x_1 = x - x_0$, $x_2 = v$, one can rewrite (20) as

$$\dot{x}_1 = x_2, \ \dot{x}_2 = g(1 - \frac{(x_0 + d)^2}{(x_1 + x_0 + d)^2}) - \frac{C}{m(x + x_0 + d)^2}u. \tag{21}$$

The system is controlled by the control law $u = -268.46x_1 - 6.8332x_2$. This is the result of the LQ control design for the linearization of (21) with weighting matrices $Q = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $R = 1000$. (To find details concerning the LQ control design, see [1].)

The observer is designed as follows: let $C = (1, 0)$ which corresponds to the practical setting (no velocity measurement is available). Choose $b = (1, 1)^T$ and $\tilde{A} = \left(\begin{smallmatrix} -50 & 0 \\ 0 & -51 \end{smallmatrix}\right)$. Then, the observer is faster than the fastest mode of the plant.

Solution of (7):

$$\overline{\Phi} = \begin{pmatrix} 0.052276 & -0.0010455 \\ 0.048229 & -0.000945667 \end{pmatrix}.$$

The function ϕ was computed using FEM on the elliptical domain

$$\Omega = \left\{ (x_1, x_2) \mid \left(\frac{x_1}{0.005}\right)^2 + \left(\frac{x_1}{0.02}\right)^2 \leq 1 \right\}.$$

The resulting observer gain reads

$$L(x_1,x_2) = \\ = \begin{pmatrix} 0.052276 - 2.42064x_1 - 0.08029x_2 & -0.0010455 - 0.08029x_1 + 0.0012199x_2 \\ 0.048229 - 3.8177x_1 - 0.069993x_2 & \end{pmatrix}^{-1} \\ \times \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

To apply the observer successfully in the control loop, the observation must be faster than any mode of the observed system. This is why the eigenvalues of matrix $\tilde{A}$ have rather large absolute values. Unfortunately, this results in a strong sensitivity of the computed solution on computational errors (incl. quality of the mesh).

Mesh number	1	2	3
N_p	172	725	3789
N_{el}	294	1352	7384
N_{dof}	1274	5602	29922

Table 1: Mesh parameters

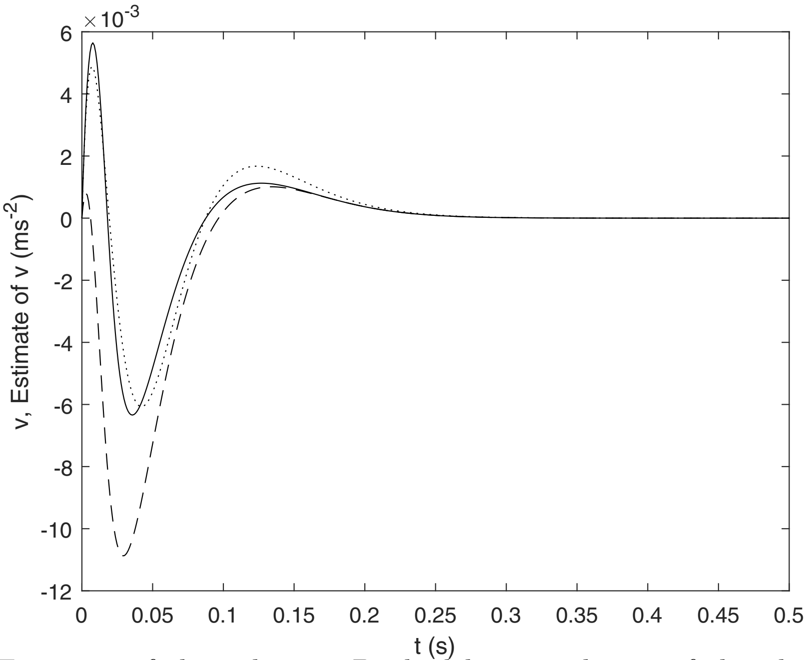

Figure 1: Estimate of the velocity. Dashed line – velocity of the observed system, solid line – estimate by the proposed observer, dotted line – estimate by the linear observer.

The mesh parameters are given in Table 1. The meaning of the columns is as follows: N_p: number of nodes, N_{el}: number of elements, N_{dof}: the total number of degrees of freedom of this discretization. The mesh depicted in Fig. 2 was used for computations above. The following mesh was created by refining this mesh uniformly as well as manually in an area around the origin. Finally, further refinement around the origin gives the mesh shown in Fig. 2. After computation of the function Φ on the domain Ω, the observer was constructed. For simulations, the initial value of the position of the ball was $x_1(0) = 4 * 10^{-4}m$, the initial velocity $x_2(0) = 0$ while the initial condition of the observer was zero. The proposed observer was tested by simulations. The estimation of the velocity is depicted in Fig. 1. Here, the dashed line is the velocity $x(t)$ of the observed system (which, in case of application to the practical system, would be inaccessible), the solid line represents the estimate of the velocity provided by the nonlinear observer. For comparison, the dotted line shows the estimate by the linear observer. One can see that the nonlinear observer yields a more accurate estimate.

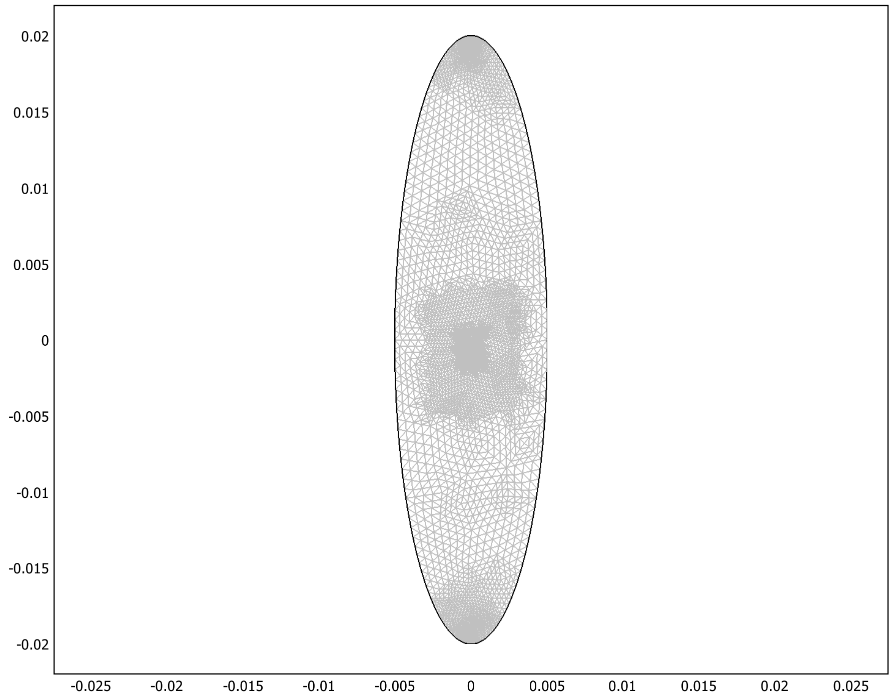

Figure 2: Mesh

Properties of the observer computed on different meshes is illustrated by Fig. 3. Initial conditions were as above. The dotted line is the velocity of the observed system. Its estimate produced by the computations on the mesh generated by the Comsol Multiphysics software (with default settings) is illustrated by the dashed line. This mesh was refined uniformly and the observer design was run again resulting in the estimate represented by the dash-dot line. Further refining of the mesh yields the mesh with parameters in Table 1. The estimate corresponds to the solid line.

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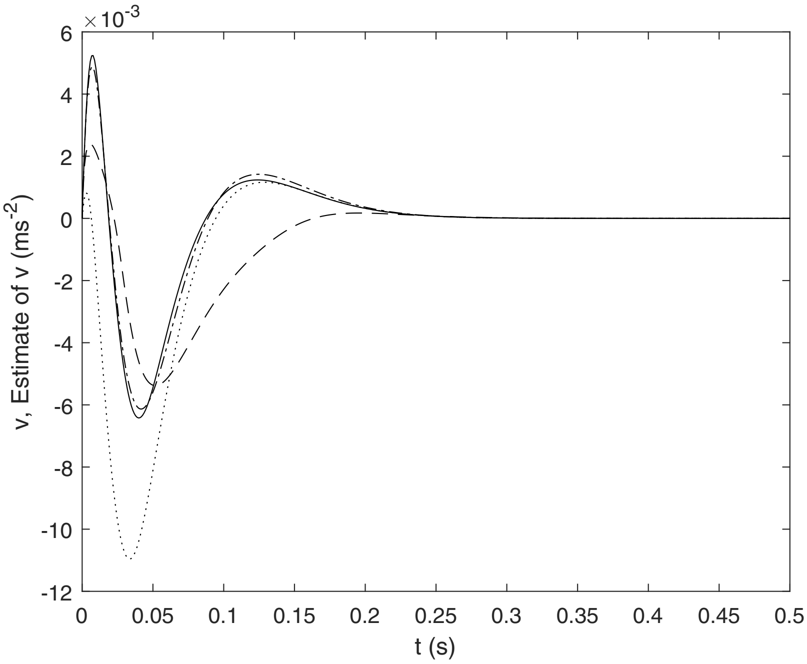

Figure 3: Estimation of the ball velocity using different meshes. Dotted line – velocity of the observed system, dashed line – estimate of the velocity computed using the initial mesh, dash-dot line – estimate of the velocity computed on the initial mesh with one uniform refinement, solid line – estimate of the velocity computed on the two times refined mesh.

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