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In: Zdeněk Frolík (ed.): Seminar Uniform Spaces. , 1976. pp. 37–41.

Persistent URL: http://dml.cz/dmlcz/703140

Atoms and proximal fineness

J. Pelant and J. Reiterman

We are going to continue an investigation of uniform atoms on ω. The same effort is a feature of [S] in this volume, so the reader can find there all needed details; see also [PR].

In relation to problems concerning products of proximally fine spaces, M. Hušek raised the question whether here is an atom which is proximally fine. The answer is ffirmative and we tried to characterize these atoms. We hall not give a complete characterization. However, we hope that the partial results, especially relations between atoms and ultrafilters that appear in the following, are not without any interest.

Definitions. All uniform spaces are assumed to be Hausdorff. Let $(X,\mathcal{U})$ be a uniform space.

1) $(X,\mathcal{U})$ is an atom if the only uniformity which is strictly finer than $(X,\mathcal{U})$ is the uniformly discrete one.

2) $(X,\mathcal{U})$ is proximally fine if each uniformity inducing the same proximity is coarser than $(X,\mathcal{U})$.

3) $(X,\mathcal{U})$ is proximally discrete if it induces a proximal-discrete proximity, i.e. if it contains all finite covers (equivalently: all partitions into two sets).

4) Let $\mathcal{F}$ be a filter on $\omega = \{0,1,2,\dots\}$, let $X = \omega \times \{0,1\}$. Denote $S(\mathcal{F}) = (X,\mathcal{U}(\mathcal{F}))$ the uniform space such t at the base for $\mathcal{U}(\mathcal{F})$ is formed by all covers of the f m $\{\{\langle n,0\rangle,\langle n,1\rangle\};\ n\in F\}\cup\{\{x\};\ x\in X\}$ where $F\in\mathcal{F}$.

Remark. By [PR], an atom on a countable set is proximally non-discrete iff it is uniformly homeomorphic to $S(\mathcal{F})$ f r some ultrafilter $\mathcal{F}$. Further, a proximally fine atom st be proximally non-discrete (proximal discreteness and proximal fineness imply uniform discreteness). Thus, we restrict ourselves to investigation of the $S(\mathcal{F})$'s.

Consider the following properties of an ultrafilter on ω :

(PF) = $S(\mathcal{F})$ is proximally fine;
(OPF) = $S(\mathcal{F})$ is proximally fine w.r.t. 0-dimensional unif mities;
(Sel) = $\mathcal{F}$ is selective;
(R) = If $f, g: \omega \longrightarrow \omega$ are two mappings such that $f\mathcal{F} = g\mathcal{F}$ then there is $F \in \mathcal{F}$ with $f/F = g/F$;
(P) = If $f, g: \omega \longrightarrow \omega$ are two one-to-finite relations (i.e., for every $n \in \omega$, fn and gn are finite sets), suc that $fF \cap gF \neq \emptyset$ for every $F \in \mathcal{F}$, then either there is F such that $fn \cap gn \neq \emptyset$ for every $n \in F$ or there is n with $f^{-1}n \cup g^{-1}n \in \mathcal{F}$;
(Z) = If $f, g: \omega \longrightarrow \omega$ are as in (P) then either there mappings $f' \subset f$, $g' \subset g$ with $f'\mathcal{F} = g'\mathcal{F}$, or there is n wi $f^{-1}n \cup g^{-1}n \in \mathcal{F}$.

As for the above filter theoretical properties, we ha the following relations:

Proposition 1. (P)$\Longleftrightarrow$(Sel).

Proof. (P)$\Longrightarrow$(Sel). Given $\mathcal{F}$ with (P), take a parti tion $\mathcal{D} = \{D_n\}$ of ω . Suppose that $\mathcal{D} \cap \mathcal{F} = \emptyset$. Deno $\sim$ the equivalence defined by $\mathcal{D}$ and put

$$fi = \{2i\} \cup \{2j + 1;\ i > j \text{ and } i \sim j\},$$
$$gi = \{2i + 1\} \cup \{2j;\ i > j \text{ and } i \sim j\}.$$

Then f, g are one-to-finite relations such that $fi \cap gi = \emptyset$ for each i. By (P) there is $F \in \mathcal{F}$ with $fF \cap gF = \emptyset$. But t e definitions of f, g force that $|F \cap D_n| \leq 1$ for each n. Thus, $\mathcal{F}$ is selective.

(Sel) $\Rightarrow$(P). Let $f: \omega \longrightarrow \omega$ be an one-to-finite re lation. Then $\{f^{-1}n;\ n \in \omega\}$ is a point finite cover. Th if $\mathcal{F}$ is selective then, by [L], either $f^{-1}n \in \mathcal{F}$ for s n or there is $F_f \in \mathcal{F}$ such that $|f^{-1}n \cap F_f| \leq 1$ for every Now, if f, g are as above (see (P)), we shall suppose that latter case takes place (other cases are easy). Then for m $n \in F_f \cap F_g$ we have $fn \cap fm = \emptyset$ and $gn \cap gm = \emptyset$ provided $m \neq n$. may suppose that the last two equalities hold for each cou

$m, n \in \omega$, $m \neq n$. Then $\{f^{-1}gn;\ n \in \omega\}$ is a point finite cover consisting of finite sets. Using the selectivity of $\mathcal{F}$ once more, we get $F_1 \in \mathcal{F}$ such that $|F_1 \cap f^{-1}gn| \leq 1$ for every n. In other words, if $m, n \in F_1$, $m \neq n$, then $fn \cap gm = \emptyset$. Thus, denoting $F = \{n \in F_1;\ fn \cap gn \neq \emptyset\}$ and $F' = F_1 - F$, we have $fF' \cap gF' = \emptyset$ so that $F' \notin \mathcal{F}$. Then $F \in \mathcal{F}$. The proof is finished.

Remark. We do not know which ultrafilters are characterized by the property (Z). However, the following is obvious:

Proposition 2. $(P) \Rightarrow (R)$ and $(R + Z) \Longrightarrow P \Rightarrow Z$.

Remark. (R) does not imply (P). Indeed, Alain Louveau kindly informed us that it is well known that, under CH, there exists an ultrafilter $\mathcal{F}$ on ω with the following property: There is a partition $\mathcal{D}_0$ of ω such that each other partition is $\mathcal{F}$-equivalent either to $\mathcal{D}_0$, or to $\mathcal{D}_1 = \{\omega\}$ or to $\mathcal{D}_2 = \{\{n\};\ n \in \omega\}$, while $\mathcal{D}_0$ is $\mathcal{F}$-equivalent neither to $\mathcal{D}_1$ nor to $\mathcal{D}_2$ (two partitions are $\mathcal{F}$-equivalent if there is $F \in \mathcal{F}$ such that their traces on F coincide). Note that the ultrafilter constructed in the proof of Theorem 2 in [S] has the above property. It is clear that such an ultrafilter has (R) but is not selective.

If $\mathcal{G}$ is a filter on ω, denote $\mathcal{G}^{\Delta}$ the filter on $\omega \times \omega$ whose base consists of sets $G \times G - \Delta$ (Δ being the diagonal in $\omega \times \omega$) where $G \in \mathcal{G}$.

Proposition 3. An ultrafilter $\mathcal{F}$ has not (R) iff it admits an image $\alpha\mathcal{F}$ under a mapping $\alpha : \omega \longrightarrow \omega \times \omega$ refining some $\mathcal{G}^{\Delta}$.

Proof. Easy: Take projections $p_1 : \omega \times \omega \longrightarrow \omega$, $p_2 : \omega \times \omega \longrightarrow \omega$ and consider the obvious relations between mappings $\alpha : \omega \longrightarrow \omega \times \omega$ and couples $f, g : \omega \longrightarrow \omega$ ($p_1\alpha = f$, $p_2\alpha = g$).

Theorem. $(Sel) \Longleftrightarrow (P) \Longrightarrow (PF) \Longrightarrow (OPF) \Longleftrightarrow R$.

Proof. $(OPF) \Longrightarrow (R)$. Suppose that $\mathcal{F}$ has (OPF) but has ot (R), i.e. there are $f, g : \omega \longrightarrow \omega$ such that $f\mathcal{F} = g\mathcal{F}$ nd, without loss of generality, $fn \neq gn$ for every $n \in \omega$.

Define an equivalence relation $\sim$ on $X = \omega \times \{0,1\}$ as lows:

$\langle n,0 \rangle \sim \langle m,0 \rangle$ iff $fn = fm$,

$\langle n,0 \rangle \sim \langle m,1 \rangle$ iff $fn = gm$,

$\langle n,1 \rangle \sim \langle m,1 \rangle$ iff $gn = gm$.

Then $\sim$ induces a decomposition $\mathcal{D}$ of X. Clearly, $\mathcal{D} \notin$ so that the uniformity $\mathcal{U}'$ whose subbase is $p\,\mathcal{U}(\mathcal{F}) \cup \{\mathcal{D}$ (where p denotes the praecompact modification) is not com le with $\mathcal{U}(\mathcal{F})$. On the other hand, $\mathcal{U}'$ is 0-dimensional induces the same proximity - a contradiction with (OPF).

(R) $\Rightarrow$ (OPF). Let $\mathcal{F}$ have the property (R) and let be a 0-dimensional uniformity which induces the same proxi ty as $\mathcal{U}(\mathcal{F})$. Then each partition $\mathcal{D} \in \mathcal{U}'$ satisfies following condition:

If F_0, $F_1 \in \mathcal{F}$ then there are $x_1 \in F_1$ such that $\langle x_0,0 \rangle \sim \langle x_1,1 \rangle$ where $\sim$ denotes the equivalence defi by $\mathcal{D}$.

It is sufficient to prove that $\mathcal{D} \in \mathcal{U}(\mathcal{F})$. To do denote $\mathcal{D} = \{D_n\}$ and define f, g: $\omega \longrightarrow \omega$ by

$$fm = n \text{ iff } \langle m,0 \rangle \in D_n, \quad gm = n \text{ iff } \langle m,1 \rangle \in D_n.$$

Now, by (R) there is $F \in \mathcal{F}$ such that $f/F = g/F$, i.e. su that for each $m \in F$, $\langle m,0 \rangle, \langle m,1 \rangle \in D_n$ for some n. Thus, cover $\{\{\langle m,0 \rangle, \langle m,1 \rangle\}\,;\, m \in F\} \cup \{\{x\}\,;\, x \in X\}$ refine $\mathcal{D}$ and so $\mathcal{D} \in \mathcal{U}(\mathcal{F})$. The proof is finished.

(P) $\Longrightarrow$ (PF). Remember that each uniformity on a coun able set has a base consisting of point finite covers [V], [PR]. Now, the proof is quite analogous to that one of (R) $\Longrightarrow$ (OPF): replace only the partition $\mathcal{D}$ by a point fin cover, so that f, g defined above are one-to-finite relati

Remark. We do not know whether (OPF) $\Rightarrow$ (PF) or (PF) $\Longrightarrow$ (P) holds. Of course, according to the last remark these implications cannot hold simultaneously. Observe als that if $\mathcal{F}$ has (Z) then (P), (PF), (OPF) are equivalent.

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