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A note on α-universal spaces

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A NOTE ON α - UNIVERSAL SPACES.

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It is proved that the class of all pseudometric spaces with isometric embeddings forms a generalized Jónsson class. The existence of α- universal uniform spaces is derived as a consequence.

At first we repeat basic definitions.

Let $\mathcal{K}$ be a concrete category. Under a generalized Jónsson class we shall understand the class $\mathbb{K}$ of objects of $\mathcal{K}$ together with the class $\mathbb{E}$ of one-to-one morphisms of $\mathcal{K}$ ($\mathbb{K}$ -embeddings) fulfilling the following conditions:

1_E $\mathbb{E}$ (A,B) is defined for all A,B from $\mathbb{K}$.

2_E $f \in \mathbb{E}$ (A,B) whenever f is $\mathcal{K}$ -isomorphism and $A,B \in \mathbb{K}$.

3_E For $f \in \mathbb{E}$ (A,B), $g \in \mathbb{E}$ (B,C) there is $gf \in \mathbb{E}$ (A,C) .

4_E If $f \in \mathbb{E}$ (A,B) onto, then $f^{-1} \in \mathbb{E}$ (B,A) .

5_E For $A,B,C \in \mathbb{K}$, $f \in \mathbb{E}$ (A,B), $f[A] \subset C \subset B$, the inclusion $C \hookrightarrow B$ from $\mathbb{E}$ (C,B) we have $\tilde{f} \in \mathbb{E}$ (A,C), where f is the range restriction of f .

1) $\mathbb{K}$ contains objects of arbitrarilly large

cardinality

2) $A, B \in \mathbb{K}$, $f \in \mathbb{E}(A,B)$, then

i) $f[A] \in \mathbb{K}$.

ii) There is an object C in $\mathbb{K}$ and isomorphis

$g: B \rightarrow C$ such that there is an inclusion

$j: A \hookrightarrow C$ from $\mathbb{E}(A,C)$ with $gf = j$.

3) For all $A, B \in \mathbb{K}$ there is $C \in \mathbb{K}$ such that

$\mathbb{E}(A,C) \neq \emptyset$, $\mathbb{E}(B,C) \neq \emptyset$.

4) Let $A, B, C \in \mathbb{K}$, $f \in \mathbb{E}(C,A)$, $g \in \mathbb{E}(C,B)$. The

there are $D \in \mathbb{K}$, $f_1 \in \mathbb{E}(A,D)$, $g_1 \in \mathbb{E}(B,D)$

with $f_1 f = g_1 g$.

5) Let $\{A_\xi ; \xi < \zeta\}$ be a chain in $\mathbb{K}$ (that mean

ζ is an ordinal number, there is an inclusi-

on $A_\eta \hookrightarrow A_\xi$ in $\mathbb{E}$ for $\eta < \xi < \zeta$). Then there

is a unique (up to isomorphism) A in $\mathbb{K}$ on

the set $\bigcup_{\xi<\zeta} A_\xi$ such that all A_ξ $\mathbb{K}$-embed in

to A. Let $\{B_\xi ; \xi < \zeta\}$ be another chain in $\mathbb{K}$

$f_\xi \in \mathbb{E}(A_\xi, B_\xi)$ for $\xi < \zeta$, $f_\xi \subset f_\eta$ for $\xi < \eta < \zeta$

$B = \bigcup_{\xi<\zeta} B_\xi$ (in $\mathbb{K}$), then $\bigcup_{\xi<\zeta} f_\xi \in \mathbb{E}(A,B)$.

6) Let $A \in \mathbb{K}$, B be a subset of A of cardinality

less than α, where α is any infinite car-

dinal number. Then there is $C \in \mathbb{K}$ of cardina-

lity less than α, $B \subset C \subset A$, and the inclusi

on $C \hookrightarrow A$ is in $\mathbb{E}$.

Let $\mathbb{K}$ be a generalized Jónsson class. An object

A in $\mathbb{K}$ will be called

α - universal, if for all B in $\mathbb{K}$ with the c rdinality at most α we have

$\mathbb{E}(B,A) \neq \emptyset$.

α - homogeneous, if for all B in $\mathbb{K}$ with the c rdinality less than α and $f,g \in \mathbb{E}(B,)$, there is an isomorphism h of A such that $hf = g$.

Finally we define $I(\mathbb{K}_\alpha)$ as the set of all equivalence classes (under isomorphism) of objects in $\mathbb{K}$ with the cardinality less than α . The main theorem about Jónsson classes is the following:

<u>Theorem:</u> Let $\mathbb{K}$ be a generalized Jónsson class, α a c rdinal number fulfilling $\alpha = \alpha^{\underline{\alpha}}$. Let $|I(\mathbb{K}_\alpha)| \leq \alpha$. Then there is unique α- universal and α- homogeneous object in $\mathbb{K}$ of cardinality α . For details and proof of the theorem we refer to [1] .

<u>Theorem:</u> The class $\mathbb{M}$ of all nonvoid pseudometric spaces with the class of all isometric embeddings form a generalized Jónsson class.

<u>Proof:</u> The properties $1_E,\dots,5_E$, 1),2),5),6) are obvious, 3) is a consequence of 4), because the onepoint space can be isometrically embedded into any space from $\mathbb{M}$. It remains to prove the property 4) .

Take (C,γ), (A,α), $(B,\beta) \in \mathbb{M}$, $f: (C,\gamma) \longrightarrow (A,\alpha)$, $g: (C,\gamma) \longrightarrow (B,\beta)$ isometric embeddings. Define on the set $D = C \vee A-f[C] \vee B-g[C]$ the function δ of two variables in the following way:

$$\rho(x,y) = \gamma(x,y) \qquad \text{for } x,y\in C$$
$$\rho(x,y) = \alpha(x,y) \qquad \text{for } x,y\in A-f[C]$$
$$\rho(x,y) = \beta(x,y) \qquad \text{for } x,y\in B-g[C]$$
$$\rho(x,y) = \alpha(fx,y) \qquad \text{for } x\in C,\ y\in A-f[C]$$
$$\rho(x,y) = \beta(gx,y) \qquad \text{for } x\in C,\ y\in B-g[C]$$
$$\rho(x,y) = \inf_{w\in C}\{\alpha(x,fw) + \beta(gw,y)\} \quad \text{for } x\in A-f[C],\ y\in B-g[C]$$

The rest will be defined symetrically.
One can easily see that ρ is a pseudometric on the set D. Only the proof of the triangular inequality is slightly unpleasant because of many possibilities which is to take into account.
Now we can define $f_1: A\longrightarrow D$ and $g_1: B\longrightarrow D$ in this way:

$$f_1 a = a \qquad \text{for } a\in A-f[C]$$
$$\quad = f^{-1}a \quad \text{for } a\in f[C]$$

g_1 will be defined analogously. Of course, f_1, g_1 are isometric embeddings with respect to correspoding pseudometrics and $f_1 f = g_1 g$.
Applying the general theorem we obtain the following:

<u>Corollary:</u> Let α be a cardinal number, $\alpha = \alpha^{\lambda} \geq 2^{\omega}$. Then there is unique α- universal and α- homogeneous pseudometric space $P^{(\alpha)}$ of the cardinality α.

Proof: There is

$$|I(\mathbb{M}_\alpha)| \leq \sum_{\beta<\alpha} (2^\omega)^{\beta \cdot \beta} \leq \sum_{\beta<\alpha} \alpha^\beta = \alpha^{\underline{\alpha}} = \alpha$$

an the t e rem 'plies.

Now we look ho this statement applies to the case of metric spaces 'n uniform spaces.

Theorem: Let α be a cardin l number, $\alpha = \alpha^{\underline{\alpha}} \geq$ $\geq 2^\omega$. Th n th e is

α - univers l and α - homogeneous metric space $M^{(\alpha)}$ of cardinality α.

Proof: e put $M^{(\alpha)}$ the associat d metric space to $P^{(\alpha)}$. Observe that an isometric embedding of a metric space N into $P^{(\alpha)}$ implies the isometric embedding o N into $M^{(\alpha)}$, hence $M^{(\alpha)}$ is α - universal of cardinality α. From the functorial nature of making a ociated metric spaces, good isomorphisms of $P^{(\alpha)}$ transla e to good isomor hisms of $M^{(\alpha)}$, hence $M^{(\alpha)}$ α - homogeneous.

For a uniform pac X, the uniform weight of X is the smallest car inality of basis of uniform covers of X. For an infinite cardinal number K we shall denote U(K) the class of all (separated) uniform spaces having the uniform weight at most K.

Theo em: For $\alpha = \alpha^{\underline{\alpha}} \geq$ $\leq K < \alpha$ cardinal numbers, there 's α - ni ersa in U(K) niform space of cardin lity α.

Proof: For $K = \omega$ take $M^{(\alpha)}$ w th ts e ri ble uni ormity, for $> \omega$ e tak th u o r -

duct $(M^{(\alpha)})^{\kappa}$. For any $X \in U(\kappa)$, $|X| \leq \alpha$ there are $\iota < \kappa$ metric with $|M_{\iota}| < |X|$ for all ι, such that X is a subspace of $\prod_{\iota<\kappa} M_{\iota}$. All M_{ι} can be embedded into $M^{(\alpha)}$, hence $\prod_{\iota<\kappa} M_{\iota}$ is a subspace of $(M^{(\alpha)})^{\kappa}$, hence $(M^{(\alpha)})^{\kappa}$ is α- universal in $U(\kappa)$. Taking into account the assumption on α, $\alpha^{\kappa} = \alpha$ hence the cardinality of $(M^{(\alpha)})^{\kappa}$ is α.

Remarks: 1) The classes $U(\kappa)$ are not generalized Jónsson classes, hence as a consequence of the general theory we can hardly obtain better results.

2) It is a classical result that every metric space of cardinality at most α can be isometrically embedded into the space $l_{\infty}(\alpha)$, hence $l_{\infty}(\alpha)$ is α - universal metric space of cardinality 2^{α}. Our theorem gives a better result, assuming that α is of special sort.

3) How strong is the condition $\alpha = \alpha^{\kappa}$? Generally it is strong. But assuming GCH (generalized continuum hypothesis) any isolated cardinal number has the property.

References:

[1] Comfort W.W., Negrepontis S.: The theory of ultrafilters
Springer - Verlag 1974