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M.D. Rice announced in his preprint [R] that for any zero-dimensional proximally fine space X which is not precompact one can find a proximally fine space Y such that $X \times Y$ is not proximally fine. This result uses and extends the author's example from [H] where X was assumed to be uniformly discrete. Z. Frolík asked whether the assumption "zero-dimensional" in the Rice's result may be omitted. The main task of this paper is to show that it may be.

V. Kůrková, having read the preprint [R] in our seminar, indicated a probable proof of the Rice's result: X contains a countable uniformly discrete subspace N as a retract; taking Y from [H] such that $N \times Y$ is not proximally fine, we have that also $X \times Y$ is not proximally fine because $N \times Y$ is its retract.

In our general case we can find a countable uniformly discrete subspace N of a non-precompact space X and a proximally fine space Y such that $N \times Y$ is not proximally fine, but the proof that $X \times Y$ is also not proximally fine must be completely different. We know that there is a proximally continuous mapping $f: N \times Y \longrightarrow M$ which is not uniformly continuous. To prove that $X \times Y$ is not proximally fine it suffices to extend f on a proximally continuous mapping f defined on $X \times Y$. We cannot require the range of $\tilde{f}$ to be M; also, if A is a subspace of a uniform space B, $g: A \longrightarrow C$ is a proximally continuous mapping, there need not be a uniform space D containing C as a subspace and a proximally continuous $\tilde{g}: B \longrightarrow D$ extending g (e.g., if B is and A is not proximally fine, g is not uniformly continuous - for the existence of such A, B see [H],[T]). Thus in our approach, we must use special features of our case, namely that N is uniformly discrete in X.

Definition. Let I be the unit interval $[0,1]$ with the standard metric uniformity. We shall denote by Con X and call cone over a uniform space X the quotient of $X \times I$ along $X \times (0)$.

The points of Con X are pairs $\langle x,y \rangle$, $x \in X$, $y \in I - (0)$, and the point 0; sometimes we shall write $\langle x,0 \rangle$, $x \in X$, for 0.

Proposition. Con X has the following covers $\mathcal{W}(\mathcal{U},\mathcal{V})$ for a base of its uniformity:

$\mathcal{U}$ is a uniform cover of X, $\mathcal{V} = \{V_k\}_0^n$ is a uniform cover of I such that $0 \in V_0 - \bigcup_1^m V_k$,

$\mathcal{W}(\mathcal{U},\mathcal{V}) = \{ U \times V_k \mid U \in \mathcal{U}, k \neq 0 \} \cup$
$\cup \{ \langle x,y \rangle \in \text{Con } X \mid y \in V_0 \}$.

Proof. To prove that $\mathcal{W}'$s are a base for a uniformity it suffices to show that they have star-refinements. Let $\mathcal{U}'$, $\mathcal{V}'$ be star-refinements of $\mathcal{U}$, $\mathcal{V}$ and let $\mathcal{V}' = \{V_k'\}_0^m$, $0 \in V_0' - \bigcup_1^m V_k'$, $\text{st}_{\mathcal{V}'} V_0' \subset V_0$. For $\langle a,b \rangle \in \text{Con } X$, if $b \notin V_0'$, $\text{st}_{\mathcal{U}'} a \subset U \in \mathcal{U}$, $\text{st}_{\mathcal{V}'} b \subset V_k$ then $\text{st}_{\mathcal{W}(\mathcal{U}',\mathcal{V}')} \langle a,b \rangle \subset U \times V_k$, and if $b \in V_0'$ then $\text{st}_{\mathcal{W}(\mathcal{U}',\mathcal{V}')} \langle a,b \rangle \subset \{\langle x,y \rangle \in \text{Con } X \mid y \in V_0\}$.
To prove that $\mathcal{W}'$s form the uniformity of Con X we must realize that the canonical mapping $f: X \times I \longrightarrow \langle \text{Con } X, \{\mathcal{W}\} \rangle$ is uniformly continuous (clearly) and is also quotient since any $\mathcal{W}(\mathcal{U},\mathcal{V})$ refines the star of $f[\mathcal{U} \times \mathcal{V}]$.

Consequences. (1) If for $f: X \longrightarrow Y$ we define $(\text{Con } f)\langle x,y \rangle = \langle fx,y \rangle$, then Con is a functor Unif $\longrightarrow$ Unif.

(2) The mapping $i_X = \{x \longrightarrow \langle x,1 \rangle\} : X \longrightarrow \text{Con } X$ is an embedding.

(3) If p is the precompact modification, then $p \circ \text{Con} = \text{Con} \circ p$.
The third assertion follows from the facts that $p(X \times I) = pX \times I$ and that p commutes with quotients.

Now we return to our case.

Theorem. A proximally fine space X is precompact iff $X \times Y$ is proximally fine space for any proximally fine space Y.

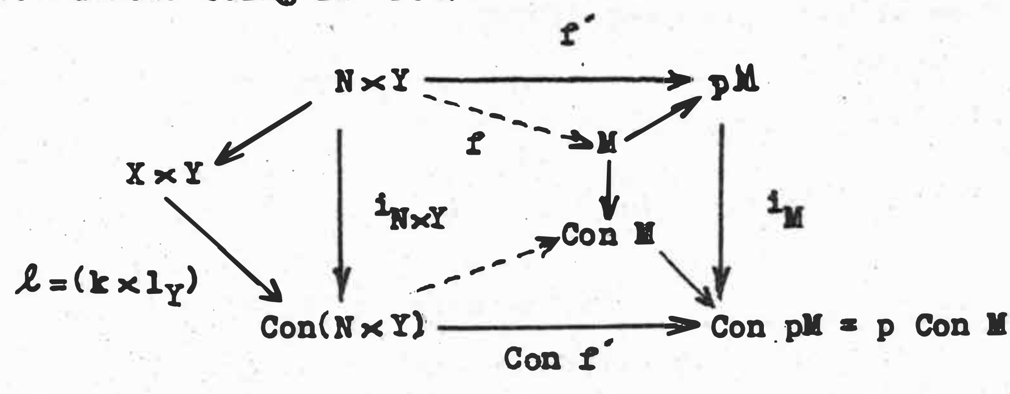
Proof. The necessity is proved in [H]. Suppose now that X is proximally fine non-precompact and N is a uniformly discrete countable subspace of X. By [H] there exists a (countable, topologically fine, with a unique accumulation point) proximally fine space Y such that $N \times Y$ is not proximally fine. Let $f\colon N \times Y \longrightarrow M$ be proximally but not uniformly continuous (hence $f'\colon N \times Y \longrightarrow pM$ is uniformly continuous). It follows from Proposition that the following two mappings are uniformly continuous:

$$\ell = \{\langle x,z,y\rangle \longrightarrow \langle x,y,z\rangle\} : (\mathrm{Con}\ N) \times Y \longrightarrow \mathrm{Con}\ (N \times Y),$$

$$k\colon X \longrightarrow \mathrm{Con}\ N,\quad kx = \begin{cases} 0 & \text{if } d\langle x,n\rangle \geqq 1 \text{ for all } n \in N \\ \langle n, 1 - d\langle x,n\rangle\rangle & \text{if } d\langle x,n\rangle < 1 \end{cases}$$

(here d is a uniformly continuous pseudometric on X such that $d\langle n,m\rangle \geqq 2$ for different $n, m \in N$ - thus k is the usual mapping into hedgehogs). We may put now

$\tilde{f} = \mathrm{Con}\ f' \circ \ell \circ (k \times 1_Y)\colon X \times Y \longrightarrow \mathrm{Con}\ (pM)$. Since $\mathrm{Con}\ (pM) = p\ \mathrm{Con}\ M$, we have received a proximally continuous mapping $\tilde{f}\colon X \times Y \longrightarrow \mathrm{Con}\ M$ which is not uniformly continuous (since its restriction f is not uniformly continuous). To see the situation, look at the diagram in Unif (except the dotted arrows being in Prox):



One can see from the proof of Theorem that we may use another concrete functor F instead of p with the property that the identity mapping $\mathrm{Con}\ FM \longrightarrow F\ \mathrm{Con}\ M$ is uniformly continuous (i.e. that $\mathrm{Con} \circ F < F \circ \mathrm{Con}$) and $F > 1_{\mathrm{Unif}}$. This condition is fulfilled if, for instance, $F > 1_{\mathrm{Unif}}$ and $FM \times I < F(M \times I)$ for any (metrizable) M (e.g., if F is an upper modification). In such a case we obtain:

If $\varkappa$ is the first cardinal such that $\varkappa \times Y$ is not F-fine for an F-fine space Y, then $X \times Y$ is not F-fine whenever the covering character of X is at least $\varkappa$.

Clearly it may happen that no such $\varkappa$ exists even if F is a reflection (e.g. a zero-dimensional modification) because the corresponding coreflective subcategory $\mathcal{K}$ of F-fine spaces consists only of uniformly discrete spaces. But if Unif $\supsetneqq \mathcal{K} \supset \{$topologically fine spaces$\}$ (if $1_{Unif} \notin F < p$), then such $\varkappa$ always exists - the proof is the same as for $F = p$ in [H]. In particular, the above generalization of Theorem can be applied to $F = e_{\alpha}$, the modification associating to X the finest uniformity containing 1-uniform covers of cardinalities smaller than α; in this case, the corresponding $\varkappa$ equals to α.

References

[H] Hušek M.: Factorization of mappings (products of proximally fine spaces), Seminar Uniform Spaces 1973/74 (Matematický ústav ČSAV, Praha 1975), 173-190.

[R] Rice M.D.: A strengthening of the proximally fine condition, preprint (1975).

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