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DGM FOR REAL OPTIONS VALUATION: OPTIONS TO CHANGE OPERATING SCALE

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Abstract: The real options approach interprets a flexibility value, embedded in a project, as an option premium. The object of interest is to valuate real options to change operating scale, typical for natural resources industry. The evolution of the project as well as option prices is decribed by partial differential equations of the Black-Scholes type, linked through a payoff function given by a type of the flexibility provided. The governing equations are discretized by the discontinuous Galerkin method over a finite element mesh and they are integrated in temporal variable by an implicit Euler scheme. The special attention is paid to the treatment of early exercise feature that is handled by additional penalty term. The capabilities of the approach presented are documented on the selected individual real options from the reference experiments using real market data.

Keywords: real option, option pricing, American option, partial differential equation, discontinuous Galerkin method, penalty method

MSC: 65M60, 35Q91, 91G60

1. Introduction

The real options approach plays an important role in the decision making process, because it provides a solution to the optimal investment decision that captures the flexibility value embedded in a project. As a result, this methodology enables to recognize the important qualitative and quantitative characteristic of some of the intrinsic attributes of the investment opportunities, namely, irreversibility of investments, choice of timing and last but not least uncertainty of the future rewards from investments, see [3]. The foundations of this modern investment theory were laid more than four decades ago by linking valuation of investment opportunities as pricing of financial options on real assets, see the pioneering paper by Myers [12]. Due to

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the analogy with an option on financial asset, the methodology has become known as real options approach that interprets the flexibility value as the option premium. Since then, a large number of various solution techniques have been developed, from a simulation approach, over dynamic programming to contingent claims analysis, see [11] for a brief overview.

In this contribution we deal with real options valuation arising in natural resources industry, especially options to change operating scale. Following a contingent claim analysis [3] the values of both the project and the embedded flexibility, expressed as functions of time and underlying output price (following a stochastic process), can be identified as solutions of relevant partial differential equations (PDEs) of the Black-Scholes type. More precisely, the link between project and flexibility values is realized through a payoff function, which is enforced with respect to the flexibility type at any time prior to or at expiration date. Taking into account our recent results on pricing of conventional financial options, see, e.g., [5] and [6], a discontinuous Galerkin method (DGM) with an implicit time stepping scheme is applied to solve the relevant governing equations and to improve the numerical pricing valuation as a whole.

The concept of the paper is based on the contributions in proceedings [7] and [8], where options to expand and options to contract were studied in a separate way. The aim is to provide readers the methodological insight to real options pricing issues, documented on simplified case studies. First, the relevant PDE models are formulated, describing a value of the project as well as the option as the solution of the terminal-boundary value problem. Next, a numerical valuation scheme is presented. Finally, a simple numerical experiment, arising from an iron ore mining industry and related to reference data [9] and [10], is provided.

2. PDE models

Consider a one-stage investment project to change (i.e., expand or contract) the production of some output commodity. More precisely, such an investment project has an embedded option to expand the production rate or an embedded option to contract the production rate, exercisable any time prior to or at prespecified time $T > 0$ and requiring the additional implementation cost $\mathcal{K} > 0$. In terms of conventional financial options, the situation is described by a call option (on expansion) or a put option (on contraction) under American exercise right with strike $\mathcal{K}$ and maturity date T.

Next, we recall valuation models from [9] and [10] to price the embedded option as well as the project itself. We assume that project/option values can be expressed as functions of the actual time t and the output commodity price P following a geometric Brownian motion (proposed in [2]):

$$\mathrm{d}P(t) = (r - \delta)P(t)\mathrm{d}t + \sigma P(t)\mathrm{d}W(t), \quad P(0) > 0, \tag{1}$$

where $r > 0$ is the risk-free interest rate, $\delta > 0$ is the mean convenience yield on holding one unit of the commodity, $W(t)$ is a standard Brownian motion and $\sigma > 0$ is the volatility of the commodity price.

Further, we denote by $V_0(P,t)$ the value of the project, which does not have any options to change operating scale. In contrast, the function $V_1(P,t)$ stands for the value of an investment project with the embedded option to expand (or contract) the production rate. Let $T^* > T$ be the maximum lifetime of both projects and $\varphi_0(P,t)$ and $\varphi_1(P,t)$ represent (after-tax) cash flow rates associated with the given project. Intuitively, from the definitions above we expect that

$$\begin{aligned} V_1(P,T^*) &= V_0(P,T^*) = 0, \quad P \geq 0, & (2) \\ \varphi_1(P,t) &= \varphi_0(P,t), \quad P \geq 0, \ t \in [0,T). & (3) \end{aligned}$$

Following [1] one can characterize value functions V_0 and V_1 between expiry date T and project lifetime T^* as solutions of a couple of deterministic backward PDEs:

$$\frac{\partial V_i}{\partial t} + \underbrace{\frac{1}{2}\sigma^2 P^2 \frac{\partial^2 V_i}{\partial P^2} + (r-\delta)P\frac{\partial V_i}{\partial P} - rV_i}_{\mathcal{L}_{\mathrm{BS}}(V_i)} = -\varphi_i, \tag{4}$$

for $P \in (0,\infty)$, $t \in [T,T^*)$ with the terminal conditions (2).

In what follows we present the governing equation for the embedded flexibility representing the value added to the project function, i.e., $V_1(P,t) \geq V_0(P,t)$ for all $P \geq 0$ and $t \in [0,T)$. More precisely, we set $F(P,t) = V_1(P,t) - V_0(P,t)$ as the option value at the current price P and actual time $t \in [0,T)$. In view of the notation above, it is possible to track values of both projects and the embedded option value simultaneously within one timeline on $[0,T)$, that are linked at the expiry date T through the function

$$\Pi(P) \equiv \max(V_1(P,T) - V_0(P,T) - \mathcal{K}, 0) = \Pi(V_1(P,T), V_0(P,T)), \qquad P \geq 0, \tag{5}$$

which plays the role equivalent to a payoff function with strike $\mathcal{K}$, well-known from financial options pricing.

Further, taking into account an equivalence of cash flow rates (3) and encompassing the early exercise constraint of American options, i.e.,

$$F(P,t) \geq \Pi(V_1(P,T), V_0(P,T)), \quad P \geq 0, \ t \in [0,T), \tag{6}$$

the value function F satisfies the so-called moving-boundary problem, where it is also necessary to determine exercise and continuation regions separated by a free boundary driven by the optimal exercise price $P^*(t)$, see [13].

There are several approaches how to handle the early exercise feature, among the widely used ones, just penalty techniques [14] allow us to reformulate moving-boundary problem as follows

$$\frac{\partial F}{\partial t} + \mathcal{L}_{\mathrm{BS}}(F) + q_F = 0, \quad P \in (0,\infty), \ t \in [0,T), \tag{7}$$

where an additional nonlinear source term q_F is defined to ensure American constraint (6) and satisfy the conditions:

$$q_F(P,t) = 0, \text{ if } F(P,t) > \Pi(P), \quad q_F(P,t) > 0, \text{ if } F(P,t) = \Pi(P). \tag{8}$$

Note that the penalty approach can be unified for both European and American exercise features, if we put $q_F(P,t) = 0$ in (7) for all $P > 0$ and $t \in [0,T)$ in the case of a European exercise right.

3. DG approach

In order to determine the present value of flexibility to expand/contract the production rate, it is necessary to proceed in backward induction, from a pair of project value functions V_0 and V_1, over a construction of the payoff function Π, to the real option value function F. Since there are no analytical formulae for finite maturity American options in general, the valuation should rely on numerical approaches. The proposed valuation methodology is based on DGM, successfully used in the field of financial option pricing, see, e.g., [5] and [6].

At first, we localize the governing equations to a bounded interval $\Omega = (0, P_{\max})$, where maximal commodity price satisfies $P_{\max} > P^*(t)$ for all $t \in [0,T)$. Then, we have to impose project as well as option values at both endpoints $P = 0$ and $P = P_{\max}$. The project values are estimated by the net present value approach for the given cash flow rates as follows

$$V_i(z,t) = \int_t^{T^*} \varphi_i(z,\xi) e^{-r(\xi - t)} \mathrm{d}\xi, \quad z \in \{0, P_{\max}\}, t \in [T, T^*), \ i = 0,1. \tag{9}$$

The real option value has to reflect the type of flexibility that this option provides. In accordance with the European exercise right, we prescribe a couple of Dirichlet boundary conditions in the form

$$\begin{aligned} &F(0,t) = 0, \quad F(P_{\max},t) = e^{-r(T-t)}\Pi(P_{\max}), && \text{(expansion)} \\ &F(0,t) = e^{-r(T-t)}\Pi(0), \quad F(P_{\max},t) = 0, \ \ t \in [0,T). && \text{(contraction)} \end{aligned} \tag{10}$$

Moreover, in the case of American options, boundary conditions (10) have to be set in the accordance with the early exercise feature which leads to the elimination of the discounted factor $e^{-r(T-t)}$ in (10).

Secondly, to handle the American early exercise feature and force the solution of (7) not to fall below its payoff function at any time $t \in [0,T)$, we introduce (as in [6]), for a sufficiently regular function v, the variational form of penalty term q_F as

$$(q_F(t), v) = c_{\mathrm{p}} \int_\Omega \chi_{\mathrm{exe}}(t) \Big(\Pi(P) - F(P,t) \Big) v \, \mathrm{d}P, \tag{11}$$

where $(\cdot,\cdot)$ denotes in fact the inner product in $L^2(\Omega)$. The function $\chi_{\mathrm{exe}}(t)$ in (11) is defined as an indicator function of the exercise region at time instant t and $c_{\mathrm{p}} > 0$ represents a weight to enforce the early exercise.

The cornerstone of the method applied is to construct a numerical solution as a composition of piecewise polynomial, generally discontinuous, functions on a spatial mesh without any requirements on the continuity of the solution across the partition nodes. We introduce the finite dimensional space

$$S_h^p = \{v_h \in L^2(\Omega) \colon v_h|_{(P_l,P_{l+1})} \in P^p((P_l, P_{l+1})),\ 0 \le l < N\}, \tag{12}$$

defined over the partition $0 = P_0 < P_1 < \ldots < P_N = P_{\max}$ of the domain Ω with the assigned mesh size h. Similarly as in [6], we carried out the DG spatial semi-discretization and temporal time discretization using an implicit Euler scheme. As a result, we obtain a sequence of linear algebraic problems related to a time partition $T^* = t_0 > t_1 > \cdots > t_R = T > t_{R+1} > \cdots > t_M = 0$ with fixed time step $\tau = T^*/M$. Further, denote $u_{h,m}^{(i)} \in S_h^p$, $i = 0, 1$, the approximation of the corresponding project value functions V_i from (4) at time level $t_m \in [T, T^*]$, $m = 0, \ldots, R$. Similarly, we define the DG approximate solution of problem (7) as functions $w_h^m \approx F(\cdot, t_m)$, $t_m \in [0, T]$, $m = R, \ldots, M$. Starting from zero initial project values $u_{h,0}^{(0)}$ and $u_{h,0}^{(1)}$, the desired value of flexibility $w_h^M \approx F(\cdot, 0)$ is computed in the following three steps

$$\begin{aligned}\left(u_{h,m+1}^{(i)}, v_h\right) - \tau\mathcal{A}_h\left(u_{h,m+1}^{(i)}, v_h\right) &= \left(u_{h,m}^{(i)}, v_h\right) - \tau\ell_h^{(i)}(v_h)(t_{m+1}) \\ &+\tau\left(\varphi_i(t_{m+1}), v_h\right) \quad \forall\, v_h \in S_h^p,\ m = 0, 1, \ldots, R-1,\ i = 0, 1,\end{aligned} \tag{13}$$

$$\left(w_h^R, v_h\right) = \left(\Pi\left(u_{h,R}^{(1)}, u_{h,R}^{(0)}\right), v_h\right) \quad \forall\, v_h \in S_h^p, \tag{14}$$

$$\begin{aligned}\left(w_h^{m+1}, v_h\right) - \tau\mathcal{A}_h\left(w_h^{m+1}, v_h\right) + \tau\mathcal{Q}_h\left(w_h^{m+1}, v_h\right) &= \left(w_h^m, v_h\right) \\ -\tau\ell_h(v_h)(t_{m+1}) + \tau q_h(v_h)(t_{m+1}) \quad &\forall\, v_h \in S_h^p,\ m = R, \ldots, M-1,\end{aligned} \tag{15}$$

where the bilinear form $\mathcal{A}_h(\cdot,\cdot)$ stands for the discrete variant of the operator $\mathcal{L}_{\mathrm{BS}}$ from (4). The linear forms $\ell_h^{(i)}(\cdot)(t)$ and $\ell_h(\cdot)(t)$ are associated with boundary conditions (9) and (10), related to the particular project value V_i and the option value F, respectively. Further, the treatment of the American constraint leads to new forms $\mathcal{Q}_h(\cdot,\cdot)$ and $q_h(\cdot)(t)$ in scheme (15), defined as discrete variants of the bilinear and linear part of (11), respectively. For the detailed derivation of the above-mentioned forms we refer the interested reader to [5].

Moreover, for practical purpose, to evaluate forms $\mathcal{Q}_h$ and q_h we use

$$\chi_{\mathrm{exe}}(t_m)\big|_{[P_l,P_{l+1}]} \approx \widetilde{\chi_{\mathrm{exe}}}(t_m)\big|_{[P_l,P_{l+1}]} := \begin{cases} 1, & \text{if } w_h^{m-1}\left(P_{\mathrm{c}}^{l+1}\right) < w_h^R\left(P_{\mathrm{c}}^{l+1}\right) \\ 0, & \text{if } w_h^{m-1}\left(P_{\mathrm{c}}^{l+1}\right) \ge w_h^R\left(P_{\mathrm{c}}^{l+1}\right) \end{cases} \tag{16}$$

for $t_m \in [0, T)$, $0 \le l \le N-1$, where P_{c}^{l+1} is the midpoint of the interval $[P_l, P_{l+1}]$ and w_h^R is given as S_h^p-approximation of the payoff function Π depending on states $u_{h,R}^{(i)}$, $i = 0, 1$, see (14).

4. Numerical experiments

In this section, we briefly illustrate the usage of the DG approach on idealized case studies from the iron ore mining industry. The three-step valuation scheme (13)–(15) is implemented in the solver Freefem++, incorporating GMRES as a solver for non-symmetric sparse systems, for more details, see [4].

As in [9] and [10] we consider iron ore mine, having the value given by the function $V_0(P,t)$, that depends on commodity price P, expressed in USD per dry metric tonne (dmt) of iron ore. Further, we have a mining project of value $V_1(P,t)$, adopting the embedded option $F(P,t)$ to scale up (or down) the production rate any time $t \in [0,T]$. Let Q denote the total reserve of the iron ore mine (in thousands of million dmt) and $q_i(t) \geq 0$, $i = 0,1$, be the iron ore production rates (in thousands of million dmt per year) associated with projects V_i, $i = 0,1$. Depending on how the mine is operated, project lifetimes are defined as minimum admissible values T_0 and T_1 (in years) that satisfy the relationship

$$Q = \int_0^{T_0^*} q_0(\xi)\mathrm{d}\xi = \int_0^{T_1^*} q_1(\xi)\mathrm{d}\xi, \tag{17}$$

where

$$q_0(t) = \begin{cases} s(t), & \text{if } t \in [0,T_0^*), \\ 0, & \text{if } t \in [T_0^*,T^*], \end{cases} \quad q_1(t) = \begin{cases} s(t), & \text{if } t \in [0,T), \\ \kappa \cdot s(t), & \text{if } t \in [T,T_1^*), \\ 0, & \text{if } t \in [T_1^*,T^*], \end{cases} \tag{18}$$

for $s(t)$ corresponding to the production rate related to the project having no embedded options and factor $\kappa > 0$ representing the extracted ($\kappa > 1$) or contracted ($\kappa < 1$) mining production rate. Further, we define the after-tax cash flow rates of relevant projects as follows

$$\varphi_i(P,t) = q_i(t)\Big((1-D)P - c(t)\Big)(1-B), \qquad i = 0,1, \tag{19}$$

for $P \in [0,P_{\max}]$ and $t \in [0,T^*]$, where $c(t)$ is the average cash cost rate of iron ore production per dmt, D is the rate of state royalties and B is the income tax rate. The numerical experiments are performed on the following (reference) project and market data:

$$\begin{aligned} &Q = 10, \qquad s(t) = 0.1\, e^{0.007t}, \qquad D = 0.05, \qquad B = 0.3, \\ &c(t) = C_0 e^{0.005t}, C_0 > 0, \qquad r = 0.06, \qquad \delta = 0.02, \end{aligned} \tag{20}$$

which are the representatives of parameter values of practical significance.

4.1. European expansion option

Referring to [9] we price an expansion option exercisable only at maturity date $T = 2$ under discretization parameters $p = 2$, $P_{\max} = 100$, $h = 1$ and $\tau = 0.02$.

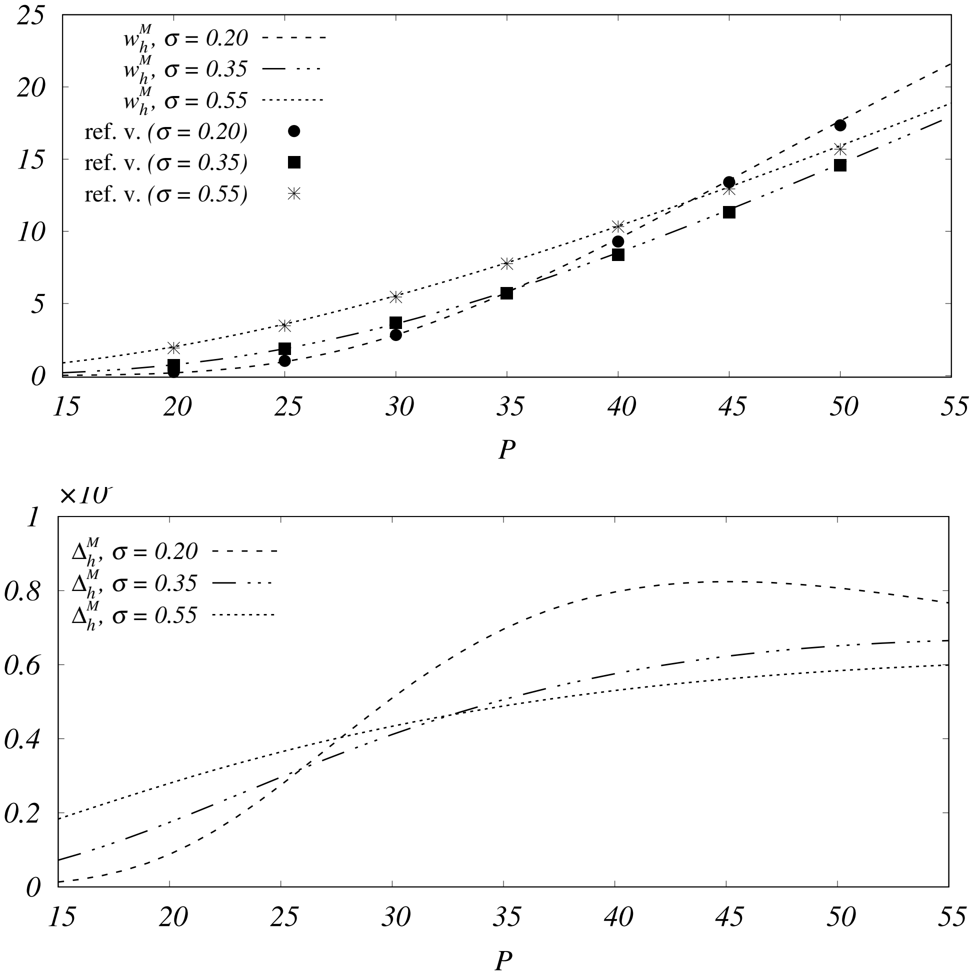

Figure 1: The approximate option values (in 10^9 USD) for different scenarios (top) and the corresponding Delta values (bottom).

Further, we take $C_0 = 35$ USD (based on prices from 2007) and the implementation cost to double production ($\kappa = 2$) is set as $\mathcal{K} = 10$, given in 10^9 USD. Using (17), (18) and (20), easy calculation leads to $T^* \doteq 75.8$ and $T_1^* \doteq 43.6$.

Consistent with the referenced experiment we investigate the behaviour of the option values for various values of volatility. Figure 1 (top) records flexibility values at present time ($t = 0$) for all scenarios considered. One can easily observe that plots are similar to the conventional financial European call options with the relevant Black-Scholes model parameters. Moreover, piecewise quadratic DG approximations match well the reference values (evaluated at underlying reference prices) and give fairly the same results as the upwind finite difference methods from [9]. More precisely, we can deduce that option values seem to be an increasing function of volatility σ in the region of low commodity prices (i.e., for less than some critical value). On the other hand, in the case of high commodity prices, the situation is quite opposite and the most valuable option is the one with the smallest volatility ($\sigma = 0.2$). This intuitive expectation is well illustrated in Figure 1 (bottom), where the corresponding Delta sensitivity measures, $\Delta_h^M \approx \frac{\partial F}{\partial P}(\cdot, 0)$, are depicted. At first glance, the most sensitive flexibility value with respect to the commodity price is related to the low volatility scenario, because in this case the commodity price has little chance to fluctuate. From this point of view, we come to the same conclusions as in the paper [9].

4.2. American contraction option

Secondly, we price a contraction option exercisable any time prior to or at $T = 1$ under discretization parameters $p = 2$, $P_{\max} = 60$, $h = 0.6$, $\tau = 0.01$ with early exercise weight $c_{\mathrm{p}} = 10/\tau$. As in [10] we set $C_0 = 25$ USD (prices from 1988) and the implementation cost $\mathcal{K} = 1 - \kappa$ (given in 10^9 USD) and investigate the behaviour of the option values with the fixed volatility $\sigma = 0.3$ for various contraction factors under American as well as European exercise rights. The lifetime T_1^* is determined in a similar way as in preceding experiment for various κ. The approximate option values at present time for selected contraction factors are depicted in Figure 2 (top). Analogously to the previous experiment, plots are similar to the conventional financial put options and illustrate an intuitive expectation that the value of flexibility to contract F is a decreasing function of the factor κ in the region of low commodity prices. Moreover, it is apparent for all cases that American options cost more than their European counterparts, i.e., early exercise feature increases value of the project flexibility. This distinctive feature of American options is also well resolved by Delta measures in Figure 2 (bottom), i.e., $|\Delta_h^M(\mathrm{Am})| \geq |\Delta_h^M(\mathrm{Eu})|$ for a particular κ. Thus, these observations are in good agreement with the expectations of practitioners.

5. Conclusion

The real options approach and especially related valuation techniques pose a very challenging part of corporate finance. In this paper we have recalled PDE models to valuation of investment projects together with the embedded flexibility of a one-stage expansion or contraction of the production rate. The particular governing equations were solved by a numerical scheme based on DGM. The presented numerical experiments, arising from the iron ore mining industry, provides financially meaningful results and thus illustrates a suitability of DGM for real options pricing issues that take into account fluctuations in commodity prices as well as different expansion/contraction factors. One possible future research objective should be addressed to extend the DG approach to advanced combinations of options to change operating scale incorporated into a compound option that enables to properly capture changing investment strategies in a long time horizon.

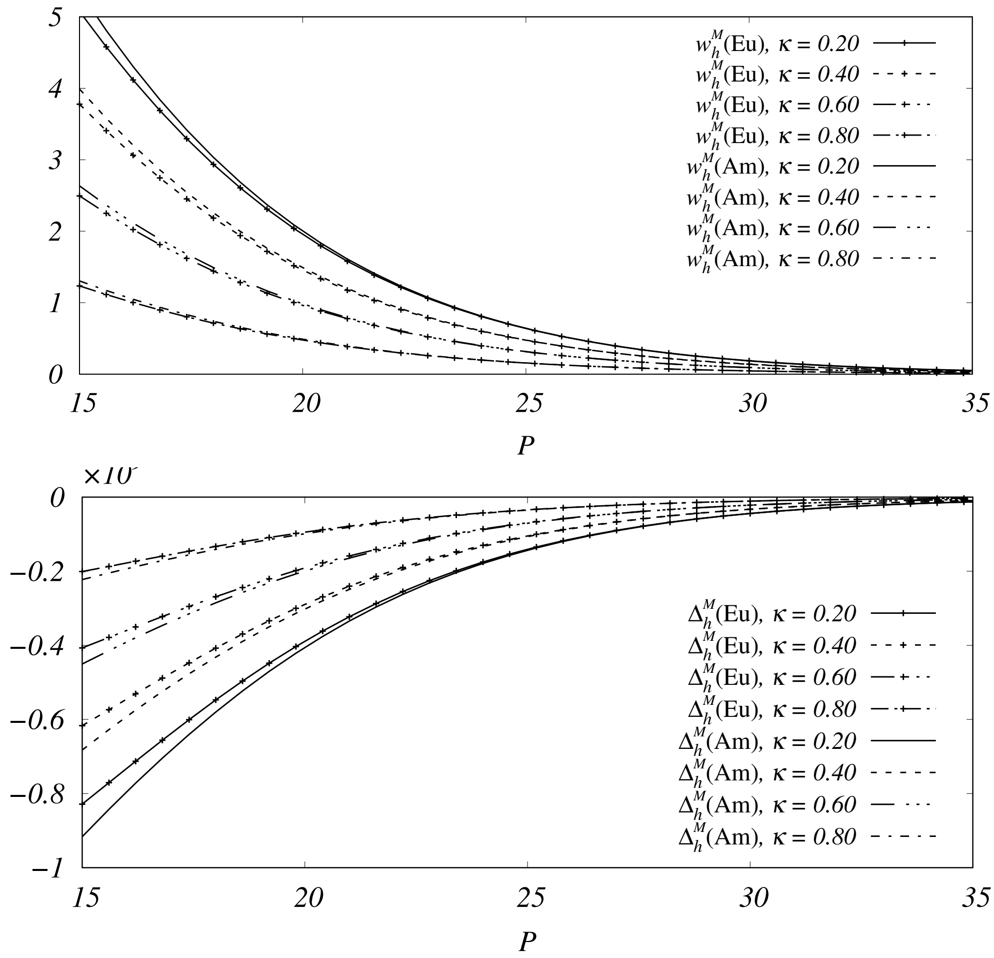

Figure 2: The approximate option values (in 10^9 USD) for different scenarios (top) and the corresponding Delta values (bottom) under European and American constraints.

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