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NEW METHODS IN COLLISION OF BODIES ANALYSIS

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Abstract: The widely used method for solution of impacts of bodies, called the penalty method, is based on the contact force proportional to the length of the interpenetration of bodies. This method is regarded as unsatisfactory by the authors of this contribution, because of an inaccurate fulfillment of the energy conservation law and violation of the natural demand of impenetrability of bodies. Two non-traditional methods for the solution of impacts of bodies satisfy these demands exactly, or approximately, but much better than the penalty method. Namely the energy method exactly satisfies the conservation of energy law, whereas the kinematic method exactly satisfies the condition of impenetrability of bodies. Both these methods are superior in comparison with the penalty method, which is demonstrated by the results of several numerical examples.

Keywords: contact / impact of elastic bodies, finite element method, method of dicretization in time, energy and kinematic approaches.

MSC: 74M15, 74S05, 74S20

1. Introduction

Robust, reliable and effective computational analysis of collision of deformable bodies belongs to the important tasks of engineering mechanics, conditioned by the successful cooperation in formulation of physical models with reasonable parameters, evaluable from rather simple experiments, in mathematical and numerical analysis and in software development. Models based on the conservation principles of classical thermomechanics by [4], supplied by appropriate constitutive relations, lead

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to partial differential equations or even to their systems of hyperbolic type, supplied with the pair of Cauchy initial conditions for displacements and their rates, with the Dirichlet boundary conditions for prescribed supports and with the Neumann boundary conditions for exterior loads, together with the contacts / impacts of colliding deformable bodies. This brings substantial nonlinearities to the system, even under the hypothetical, not very realistic assumptions on both the geometrical linearity (small strains) and the physical one (linear reversible strain - stress relations). Theoretical formulations containing variational inequalities, after the discretisation both in the time and in the Euclidean space, 3-dimensional in general, replace their exact fulfilment by the introduction of some additional penalty terms, as introduced by [23]. Other serious problems are the incorporation of contact friction, non-expensive search for potential contacts – cf. the distributed and parallel computations required by [6] and [17], as well as the description of contact geometry, characterized as node-to-node, node-to-segment or segment-to-segment approaches.

The progress in this research area in more than last 3 decades can be traced from the review articles [2], [7], [10] and [18]. When treating contact problems within the finite element method, 7 steps of analysis should be followed by [22]: i) continuum based contact kinematics, ii) constitutive equations for contact interfaces, iii) weak form of contact contributions and overall solution strategies for contact problems, iv) discretisation of contact surfaces, v) algorithms for the integration of constitutive equations in the contact area, vi) contact search algorithms, vii) adaptive methods for contact problems. The detailed primal and dual variational formulations of contact problems are demonstrated by [15]. The comparison of classical Lagrange multiplier and penalty computational approaches is presented by [20]. The classical recommendations for the choice of penalty stiffness, needed for the evaluation of the contact force proportional to the length of the interpenetration of bodies, are presented in [3]; the so-called exact penalty improvement, working with the updated penalty stiffness, was suggested by [13].

Other alternatives can be found in literature, too, as i) energy conserving algorithms, introduced by [9], revisited by [24], applying certain penalty-based regularization, or ii) perturbed Lagrangian formulations, stemming from [14], working with certain kinematic conditions, developed by [12]. The implementation of b) by [1] utilizes augmented Lagrange multipliers to force all prescribed kinematic conditions, which requires an additional iterative solutions of systems of algebraic equations, unwelcome for explicit time discretisation. Another implementation of b) by [21] avoids such iterative process, but leads to a non-physical increase of energy at contact / impact interfaces typically, which must be suppressed by some artificial computational reduction of contact forces.

Two promising computational methods, presented in this paper, can be seen as certain variants of i) and ii). We shall refer to them as to i) the energy method and to ii) the kinematic method, although such nomenclature is not quite unified in the literature, to highlight i) the exact energy conservation, or ii) the exact fulfillment of kinematic conditions, involved in any space- and time-discretised computational

scheme. Assuming the space discretisation using the finite element technique, we shall work with the explicit time integration, due to the need of short time steps, forced by collision phenomena. The approach i) generalizes the 2-dimensional formulation of [16] naturally. The approach ii) here does not evaluate any contact forces as separate variables, but its specific use in the explicit time stepping forces the correction of nodal displacement at all potential interfaces under the assumptions of a) impenetrability of colliding bodies, b) evaluation of exact collision time t_c, c) decomposition of any time step of length $\mathcal{D}t$, considered as $[0, \mathcal{D}t]$ for simplicity, to $[0, t_c]$ and $[t_c, \mathcal{D}t]$, d) conservation of momentum of contact entities and e) perfectly inelastic collision.

After this introductory remarks (*Section* 1) we shall come to the general discussion of collision of bodies (*Section* 2), to the energy method (*Section* 3) and to the kinematic method (*Section* 4), supported by some illustrative examples (*Section* 5). The brief concluding remarks (*Section* 6) will be oriented to the need and priorities of further research.

2. Collisions of bodies

To demonstrate the advantages of 2 announced methods, we shall consider a finite number of deformable bodies discretised into finite elements, with the surfaces consisting of flat triangles, whereas mass is assigned to nodes. These bodies can arbitrarily collide.

2.1. Finding the time and space coordinates of the collision

For simplicity, we shall suppose that each line consists only of straight elements and each surface, or a boundary of a solid, is decomposed to triangles. That being the case, following the node-to-segment approach, just two kinds of collision can occur: collision of two line segments (element edges), or collision of a node and a triangle surface segment, as shown by Fig. 1 schematically. All parameters of collision will be evaluated using the explicit method, with sufficiently small time steps. In each such time step, constant velocities and geometric linearity are assumed for finding the time and position of the contact of discretised bodies.

Let us have a line segment with its end points A, B and another line segment with

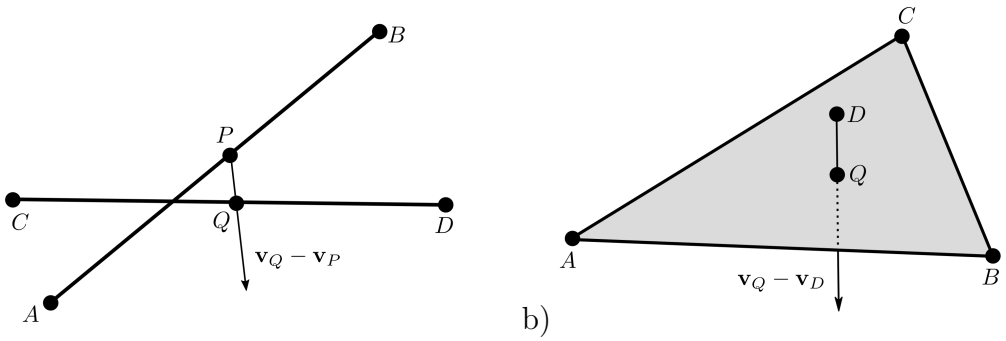


Figure 1: Two possibilities of collision of discretised bodies: a) edge to edge and b) node to surface.

its end points C, D, or, alternatively, a triangle surface element with its nodes A, B, C and another node D. At the beginning of present time step, $t = 0$ for simplicity, all these 4 points have their initial positions given by vectors $\mathbf{x}_i(0)$, and in such time step they move by velocities $\mathbf{v}_i$, assumed as constant during the whole time step. Thus, in any positive time t we come to positions

$$\mathbf{x}_i(t) = \mathbf{x}_i(0) + t\mathbf{v}_i \text{ for all } i \in \{A, B, C, D\}\,. \tag{1}$$

We need to find out, whether a) the line segments AB and CD, or b) the node D and the surface triangle ABC, will collide in the considered time step. Let P be the point of collision in the case a) and Q such point in the case b). For b) Q will be the point of the triangle ABC hit by the point D. At the collision time t_c all nodes A, B, C, D must lie in the same plane, thus for a known collision time their position can be determined, It can be also detected whether the points of the collision lie inside the pertinent segments, i. e. a) if the points P, Q lie inside the line segments AB and CD, or b) the point Q lies inside the triangle ABC. For the sake of brevity of the following formulae (2), (3) and (4), we shall write x_i instead of $x_i(t_c)$ now.

At first let us investigate whether the point D lies in the plane given by the points A, B, C. The symbols $\times$ and $\cdot$ will be reserved for the vector and scalar products in the 3-dimensional real Euclidean space. The normal vector to this plane can be then defined as $(\mathbf{x}_B(t_c) - \mathbf{x}_A(t_c)) \times (\mathbf{x}_C(t_c) - \mathbf{x}_A(t_c))$. If the point D lies in the plane ABC, then the vector connecting him with an arbitrary point of this plane, as with A in particular, must be perpendicular to the above introduced normal one; this can be written as

$$(\mathbf{x}_D - \mathbf{x}_A) \cdot ((\mathbf{x}_B - \mathbf{x}_A) \times (\mathbf{x}_C - \mathbf{x}_A)) = 0\,; \tag{2}$$

this cannot hold for any point D not belonging to the plane ABC for any non-degenerated triangle ABC, i. e. a triangle with non-zero area. Rearranging (2) formally, we obtain

$$\mathbf{x}_D \cdot (\mathbf{x}_A \times \mathbf{x}_B + \mathbf{x}_B \times \mathbf{x}_C + \mathbf{x}_C \times \mathbf{x}_A) = \mathbf{x}_A \cdot (\mathbf{x}_B \times \mathbf{x}_C)\,. \tag{3}$$

Substituting (1) with $t = t_c$ into (3), we come to the cubic equation

$$\begin{aligned}
&\mathcal{C}_3 t_c^3 + \mathcal{C}_2 t_c^2 + \mathcal{C}_1 t_c + \mathcal{C}_0 = 0\,, \qquad (4)\\
\mathcal{C}_0 \;&=\; \mathbf{x}_D \cdot (\mathbf{x}_A \times \mathbf{x}_B) + \mathbf{x}_D \cdot (\mathbf{x}_B \times \mathbf{x}_C) + \mathbf{x}_D \cdot (\mathbf{x}_C \times \mathbf{x}_A) - \mathbf{x}_A \cdot (\mathbf{x}_B \times \mathbf{x}_C)\,,\\
\mathcal{C}_1 \;&=\; \mathbf{x}_D \cdot (\mathbf{x}_A \times \mathbf{v}_B) + \mathbf{x}_D \cdot (\mathbf{v}_A \times \mathbf{x}_B) + \mathbf{v}_D \cdot (\mathbf{x}_A \times \mathbf{x}_B) + \mathbf{x}_D \cdot (\mathbf{x}_B \times \mathbf{v}_C)\\
&\quad+\; \mathbf{x}_D \cdot (\mathbf{v}_A \times \mathbf{x}_C) + \mathbf{v}_D \cdot (\mathbf{x}_B \times \mathbf{x}_C) + \mathbf{x}_D \cdot (\mathbf{x}_C \times \mathbf{v}_A) + \mathbf{x}_D \cdot (\mathbf{v}_C \times \mathbf{x}_A)\\
&\quad+\; \mathbf{v}_D \cdot (\mathbf{x}_C \times \mathbf{x}_A) - \mathbf{x}_A \cdot (\mathbf{x}_B \times \mathbf{v}_C) - \mathbf{x}_A \cdot (\mathbf{v}_B \times \mathbf{x}_C) - \mathbf{v}_A \cdot (\mathbf{x}_B \times \mathbf{x}_C)\,,\\
\mathcal{C}_2 \;&=\; \mathbf{v}_D \cdot (\mathbf{v}_A \times \mathbf{x}_B) + \mathbf{v}_D \cdot (\mathbf{x}_A \times \mathbf{v}_B) + \mathbf{x}_D \cdot (\mathbf{v}_A \times \mathbf{v}_B) + \mathbf{v}_D \cdot (\mathbf{v}_B \times \mathbf{x}_C)\\
&\quad+\; \mathbf{v}_D \cdot (\mathbf{x}_A \times \mathbf{v}_C) + \mathbf{x}_D \cdot (\mathbf{v}_B \times \mathbf{v}_C) + \mathbf{v}_D \cdot (\mathbf{v}_C \times \mathbf{x}_A) + \mathbf{v}_D \cdot (\mathbf{x}_C \times \mathbf{v}_A)\\
&\quad+\; \mathbf{x}_D \cdot (\mathbf{v}_C \times \mathbf{v}_A) - \mathbf{v}_A \cdot (\mathbf{v}_B \times \mathbf{x}_C) - \mathbf{v}_A \cdot (\mathbf{x}_B \times \mathbf{v}_C) - \mathbf{x}_A \cdot (\mathbf{v}_B \times \mathbf{v}_C)\,,\\
\mathcal{C}_3 \;&=\; \mathbf{v}_D \cdot (\mathbf{v}_A \times \mathbf{v}_B) + \mathbf{v}_D \cdot (\mathbf{v}_B \times \mathbf{v}_C) + \mathbf{v}_D \cdot (\mathbf{v}_C \times \mathbf{v}_A) - \mathbf{v}_A \cdot (\mathbf{v}_B \times \mathbf{v}_C)\,.
\end{aligned}$$

Clearly (4) can be solved analytically by the Cardano formulae, or iteratively, using e. g. the Newton method. If its positive solution t_c exists, not exceeding the time step $\mathcal{D}t$, it refers to the collision time; in the case of multiple solutions the smallest one corresponds to the needed first collision time.

Its smallest positive solution t_c (if exists, not exceeding the time step $\mathcal{D}t$) refers to the first collision time. Consequently, all position vectors $\mathbf{x}_i(t_c)$ for the points $i \in \{A, B, C, D\}$ can be evaluated by (1).

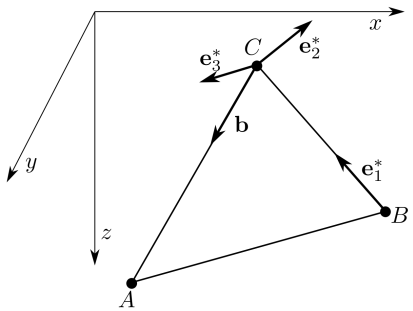


Figure 2: Definition of the contact plane by the nodes A, B, C.

2.2. Determination of the contact plane and its properties

Let us notice that the determination of contact time and location is based on the existence of a plane containing all points A, B, C, D. Such plane, as sketched by Fig. 2, is determined by an arbitrary triple selected from these 3 points, e. g. A, B, C for simplicity. Let us introduce a local coordinate system, whose 2 basis vectors $\mathbf{e}_1, \mathbf{e}_2$ can be chosen as arbitrary orthogonal vectors in this plane, whereas the remaining basis vector $\mathbf{e}_3$ is normal to this plane, with an appropriate orientation to satisfy $\mathbf{e}_1 \cdot (\mathbf{e}_2 \times \mathbf{e}_3) > 0$; thus we have a new coordinate system $\mathbf{x}^* = x_1^*\mathbf{e}_1 + x_2^*\mathbf{e}_2 + x_3^*\mathbf{e}_3$. In particular, we can introduce the unit vectors $\mathbf{e}_1^* = (\mathbf{x}_C - \mathbf{x}_B)/|\mathbf{x}_C - \mathbf{x}_B|$, $\mathbf{b} = (\mathbf{x}_A - \mathbf{x}_C)/|\mathbf{x}_A - \mathbf{x}_C|$, $\mathbf{e}_3 = \mathbf{b} \times \mathbf{e}_1^*$ and $\mathbf{e}_2^* = \mathbf{e}_3^* \times \mathbf{e}_1^*$. Thus we can work the local transform of coordinates $\mathbf{x}^* = \mathbf{R}\mathbf{x}$, containing certain rotation matrix $\mathbf{R} = (\mathbf{e}_1^{*\mathrm{T}}, \mathbf{e}_2^{*\mathrm{T}}, \mathbf{e}_3^{*\mathrm{T}})^{\mathrm{T}}$. Clearly $x_3^* = 0$ only for all points lying in the contact plane, whereas 2 remaining axes create a contact plane coordinate system, needed in our following considerations.

The contact plane can be used for definition of arbitrary friction models. Here, due to the limited extent of this paper, let us introduce a very simple property of such contact plane, which can be characterized as "the elastic friction", based on the introduction of 2 limit cases. The 1st one can be called "the zero friction", which means that both surfaces of elastic bodies are perfectly slippery, so no in-plane contact force and no friction dissipation can occur. The 2nd one can be called "the absolute friction", which means that no mutual sliding can occur during the collision, thus, no dissipation occurs in this case as well. More general cases of the elastic friction can be received as linear combinations of these 2 limit cases. Potential dissipation could be easily put into the energy balance in the energy method.

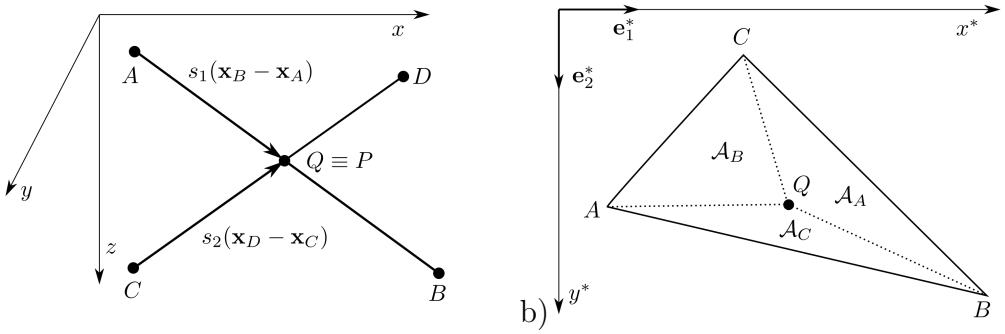


Figure 3: Collision of: a) two line segments or edges, or b) a node and a triangle surface.

2.3. Determination of the positions and velocities of the colliding points

To be able to determine the positions and velocities of all colliding points, we shall discuss only collisions of a) two segments or edges and b) a node and a triangle surface, as sketched by Fig. 3, in details. Collisions of the types node-to-node, or segment-to-segment (rarely exact) can be derived from a), b) using the limit passage. For the evaluation of the contact force direction, it is necessary to accept a suitable hypothesis. Let P, Q be the colliding points; thus, for the case of a collision of a node and a general point of a surface segment let us assume $Q \equiv D$.

2.4. Collision of two segments or edges

In the case a) positions of the colliding points P, Q can be determined as the intersection points of 2 lines, whose equations for real parameters s_1, s_2 are $\mathbf{x}_P(s_1) = \mathbf{x}_A + s_1(\mathbf{x}_B - \mathbf{x}_A)$, $\mathbf{x}_Q(s_2) = \mathbf{x}_C + s_2(\mathbf{x}_D - \mathbf{x}_C)$, and, moreover, $\mathbf{x}_P(s_1) = \mathbf{x}_Q(s_2)$ is required, thus

$$\mathbf{x}_A + s_1(\mathbf{x}_B - \mathbf{x}_A) = \mathbf{x}_C + s_2(\mathbf{x}_D - \mathbf{x}_C)\,. \tag{5}$$

For the evaluation of 2 parameters s_1, s_2 we have 3 equations (5) now; arbitrary 2 of them are sufficient for practical computations. Taking only the line segments AB, CD into account, unlike the whole lines, we have $s_1 \in [0,1] \Rightarrow P \in AB$, $s_2 \in [0,1] \Rightarrow Q \in CD$ evidently. Consequently, we can write $\mathbf{x}_P(t_c) = \mathbf{x}_Q(t_c)$ and

$$\begin{aligned} \mathbf{x}_P &= \mathbf{x}_A N_A + \mathbf{x}_B N_B\,, & \mathbf{x}_Q &= \mathbf{x}_C N_C + \mathbf{x}_D N_D\,, \\ \mathbf{v}_P &= \mathbf{v}_A N_A + \mathbf{v}_B N_B\,, & \mathbf{v}_Q &= \mathbf{v}_C N_C + \mathbf{v}_D N_D\,, \end{aligned} \tag{6}$$

taking $N_A = s_1$, $N_B = 1 - s_1$, $N_C = s_2$, $N_D = 1 - s_2$.

2.5. Collision of a node and a triangle surface element

Coming back to the case b), we have $Q = D$, thus $\mathbf{x}_Q(t_c) = \mathbf{x}_D(t_c)$. Here we need the usual area coordinates $N_j = \mathcal{A}_j/\mathcal{A}$ for $j \in \{A, B, C\}$ where the areas $\mathcal{A}_j$ are evident from Fig. 3 and $\mathcal{A}$ is their sum. If any of $N_j < 0$, the pertinent point does not lie in the triangle ABC. The vectors $\mathbf{x}_Q$ and $\mathbf{v}_Q$ in the plane ABC can be obtained in the terms of coordinates N_j as

$$\mathbf{x}_Q = \mathbf{x}_j N_j\,, \qquad \mathbf{v}_Q = \mathbf{v}_j N_j\,. \tag{7}$$

3. Energy method

Let us suppose that P, Q are the colliding points, as in the case a). The similar details of the case b) are left to the curious reader. For the simple implementation of friction, let us consider two limit cases, announced by *Section* 2. We shall work with the mutual velocity of the points P, Q, denoted as $\mathbf{v}_{PQ}(t) = v_Q(t) - v_P(t)$, and with the contact force $\mathbf{f}_{PQ}$ in this case, which can be interpreted as an internal force, acting by its components $\mathbf{f}_P$ and $\mathbf{f}_Q$ in sense of the 3rd Newton law. Their upper indices ^{a}, ^{z} will refer to the absolute friction, or to the zero friction, respectively.

3.1. Absolute friction

At first let us assume that the friction is absolute, without any slippage between the collision points P, Q during the contact at $t = t_c$. Thus, these points will bounce in the same relative direction $\mathbf{v}_{PQ}$ as before the collision; $\mathbf{f}_P^a = -\mathbf{f}_Q^a$ evidently. The unit vector in the direction of $\mathbf{f}_Q$, needed in the following considerations, can be introduced as $\mathbf{e}_Q^a = \mathbf{f}_Q^a/|\mathbf{f}_Q^a| = \mathbf{v}_{PQ}(t_c)/|\mathbf{v}_{PQ}(t_c)|$.

3.2. Zero friction

In this case, the direction of the contact force is in the direction $\mathbf{n}$ perpendicular to the plane given by the triangle ABC; this can be extended from b) to a) naturally, without all details here. Thus we have $\mathbf{n} = (\mathbf{x}_B - \mathbf{x}_A) \times (\mathbf{x}_D - \mathbf{x}_C)$. We are allowed to introduce $\mathbf{n}_Q$ using the relations $\mathbf{n} \cdot \mathbf{v}_{PQ} \geq 0 \Rightarrow \mathbf{n}_Q = \mathbf{n}$, $\mathbf{n} \cdot \mathbf{v}_{PQ} \leq 0 \Rightarrow \mathbf{n}_Q = -\mathbf{n}$. Finally, we can evaluate, similarly to $\mathbf{e}_Q^a$, $\mathbf{e}_Q^a = \mathbf{f}_Q^z/|\mathbf{f}_Q^z| = \mathbf{n}_Q/|\mathbf{n}_Q|$.

3.3. General friction

The simplest way for the interpolation between 2 preceding cases is to consider $\mathbf{e}_Q = \beta\mathbf{e}_Q^a + (1-\beta)\mathbf{e}_Q^z$, working with certain friction coefficient $\beta \in [0, 1]$. Consequently, we can write $\mathbf{e}_Q = \mathbf{f}_Q/|\mathbf{f}_Q| = \mathbf{e}_Q/|\mathbf{e}_Q|$, $|\mathbf{f}_Q| = |\mathbf{f}_P|$, $\mathbf{f}_Q = -\mathbf{f}_P$.

3.4. Determination of the magnitude of the contact force

The direction of the contact force is already known; its magnitude remains to be determined. We shall assume that the above introduced forces cause such accelerations of the nodes A, B, C, D that the velocities and positions of these nodes at the end of certain fictious time step $\mathcal{D}t$ conserve the total potential energy Π, decreased by dissipation caused by plasticizing or damage due to the collision, i. e. its new value can be expressed as $\Pi^{\times}(\mathbf{f}_{PQ}, \mathcal{D}t) = \Pi - \mathfrak{E}$ where $\mathfrak{E}$ denotes the dissipated energy, coming from additional considerations about irreversible plastic strains, fracture, etc. The acceleration of the point P, caused by the force $\mathbf{f}_P$, is $\mathbf{a}_P = \mathbf{f}_P/m_P$ where m_P is the mass assigned to the point P, which causes the velocity increment $\Delta\mathbf{v}_P = \mathbf{a}_P\mathcal{D}t$. For the point Q we can write $\mathbf{a}_Q = \mathbf{f}_Q/m_Q$, $\Delta\mathbf{v}_Q = \mathbf{a}_Q\mathcal{D}t$ similarly. Our distinguishing between the cases a) and b) will be useful in the following considerations. The obvious motivation is that the points P, Q are not the nodes in the original discretised system, therefore no discretised mass is assigned to them.

3.5. Collision of two line segments or edges

The substitution of $\mathbf{f}_P, \mathbf{f}_Q$ with $\mathbf{f}_A, \mathbf{f}_B, \mathbf{f}_C, \mathbf{f}_D$ can be done by the static equivalence of forces $\mathbf{f}_P = \mathbf{f}_A + \mathbf{f}_B$, $\mathbf{f}_Q = \mathbf{f}_C + \mathbf{f}_D$, together with $\mathbf{f}_A N_A + \mathbf{f}_B N_B = \mathbf{o}$, $\mathbf{f}_C N_C + \mathbf{f}_D N_D = \mathbf{o}$, coming from the equivalence of moments; $\mathbf{o}$ denotes the 3-dimensional zero vector. Solving this system of 4 linear algebraic equations, we obtain the formally simple relation

$$\mathbf{f}_i = N_i \mathbf{f}_P \text{ for any } i \in \{A, B, C, D\} \tag{8}$$

applying the coefficients N_i stemming from (6).

3.6. Collision of a node and a triangle surface segment

Since $Q = D$ in this case, the calculated values for the node D can be applied to the collision point Q, too, whereas for the collision point P the needed values of force, acceleration, velocity and position need to be expresses by the nodal values of the triangle ABC. Let us assume that all forces $\mathbf{f}_i$ for $i \in \{A, B, C\}$ are parallel, in the direction of the unit vector $\mathbf{e}_Q$. The static equivalence conditions then are $\mathbf{f}_P = \mathbf{f}_A + \mathbf{f}_B + \mathbf{f}_C$, $\mathbf{x}_P \times \mathbf{f}_P = \mathbf{x}_A \times \mathbf{f}_A + \mathbf{x}_B \times \mathbf{f}_B + \mathbf{x}_C \times \mathbf{f}_C$; moreover the identity condition $\mathbf{f}_Q = \mathbf{f}_D$ is valid. Consequently, expressing $\mathbf{f}_Q$ as $\mathbf{e}_Q|\mathbf{f}_Q|$, we come to $|\mathbf{f}_P| = |\mathbf{f}_A| + |\mathbf{f}_B| + |\mathbf{f}_C|$, $\mathbf{x}_P \times \mathbf{e}_Q|\mathbf{f}_P| = \mathbf{x}_A \times \mathbf{e}_Q|\mathbf{f}_A| + \mathbf{x}_B \times \mathbf{e}_Q|\mathbf{f}_B| + \mathbf{x}_C \times \mathbf{e}_Q|\mathbf{f}_C|$, $|\mathbf{f}_Q| = |\mathbf{f}_D|$. This implies (8) again, using the coefficients N_i from the text preceding (7).

3.7. Calculation of the change of the position

For any $i \in \{A, B, C, D\}$ the increments of velocities in the considered time step can be evaluated as $\Delta\mathbf{v}_i(\mathbf{f}_P, \mathcal{D}t) = (\mathbf{f}_i/m_i)\mathcal{D}t$, thus, with respect to (8), we receive $\mathbf{v}_i^{\times}(\mathbf{f}_P, \mathcal{D}t) = \mathbf{v}_i + \Delta\mathbf{v}_i$ in the form

$$\mathbf{v}_i^{\times}(\mathbf{f}_P, \mathcal{D}t) = \mathbf{v}_i + (N_i/m_i)\,\mathbf{f}_P \mathcal{D}t\,. \tag{9}$$

Consequently, since the increments of displacements can be expressed as $\Delta\mathbf{u}_i(\mathbf{f}_P, \mathcal{D}t) = \mathbf{v}_i \mathcal{D}t$ then $\mathbf{u}_i^{\times}(\mathbf{f}_P, \mathcal{D}t) = \mathbf{u}_i + \Delta\mathbf{u}_i$ gets the form

$$\mathbf{u}_i^{\times}(\mathbf{f}_P, \mathcal{D}t) = \mathbf{u}_i + \mathbf{v}_i \mathcal{D}t + (N_i/m_i)\,\mathbf{f}_P \mathcal{D}t^2\,. \tag{10}$$

Both (9) and (10) will be needed in the calculation of the change of energy for the contact forces $\mathbf{f}_P, \mathbf{f}_Q$ during the fictious time step $\mathcal{D}t$, which consists of 3 parts: i) the change of the kinetic energy $\Delta\Pi_k$, ii) the change of the elastic potential energy Π_σ and iii) the change of the potential energy of the position $\Delta\Pi_p$.

3.8. Calculation of the change of the kinetic energy

During the fictitious time step, only the velocities of the nodes A, B, C, D are influenced by the contact force. The kinetic energy of these mass points before and after the collision, i. e. at the beginning and at the end of the fictitious time step, using $i \in \{A, B, C, D\}$ as the Einstein summation index here, is $\Pi_k = (m_i/2)\,\mathbf{v}_i \cdot \mathbf{v}_i$, at its end $\Pi_k^{\times}(\mathbf{f}_P, \mathcal{D}t) = (m_i/2)\,\mathbf{v}_i^{\times} \cdot \mathbf{v}_i^{\times}$, thus, applying (9), for $\Delta\Pi_k(\mathbf{f}_P, \mathcal{D}t) = \Pi_k^{\times}(\mathbf{f}_P, \mathcal{D}t) - \Pi_k$ we have $\Delta\Pi_k^{\times}(\mathbf{f}_P, \mathcal{D}t) = (m_i/2)(2\mathbf{v}_i \cdot \Delta\mathbf{v}_i + \Delta\mathbf{v}_i \cdot \Delta\mathbf{v}_i)$, which yields

$$\Delta\Pi_k(\mathbf{f}_P, \mathcal{D}t) = N_i \mathbf{v}_i \mathbf{f}_P N_i \mathcal{D}t + (N_i^2/(2m_i))\,\mathbf{f}_P \cdot \mathbf{f}_P \mathcal{D}t^2\,. \tag{11}$$

3.9. Calculation of the change of the elastic potential energy

Since only the positions of nodes A, B, C, D will be influenced by the contact force, only the elastic energy of such j-th elements is relevant here.

$$\Delta\Pi_\sigma(\mathbf{f}_P, \mathcal{D}t) = \mathbf{f}_j^e \Delta\mathbf{d}_k^e \,; \tag{12}$$

here $\mathbf{f}_j^e$ and $\Delta\mathbf{d}_j^e$ form the vectors of the element nodal forces and of the element deformation parameters, derived using (10), respectively; e must be understood as an element index and j as an index referring to the above introduced list, both taken as the Einstein summation indices. Let us remind that $\Delta\mathbf{u}_i = \mathbf{v}_i\delta$ is satisfied for other nodes than $i \in \{A, B, C, D\}$, too.

3.10. Calculation of the change of the elastic potential energy

Since only the positions of the nodes will change in the (very short) fictitious time step, the change of the elastic potential energy $\Delta\Pi_\sigma(\mathbf{f}_P, \mathcal{D}t) = -\Delta u_i \mathbf{f}_i^{\text{ext}}$, $\mathbf{f}_i^{\text{ext}}$ being the components of external forces, can be formulated as

$$\Delta\Pi_p(\mathbf{f}_P, \mathcal{D}t) = (N_i/m_i)\, \mathbf{f}_P \cdot \mathbf{f}_i^{\text{ext}} \mathcal{D}t^2 \,. \tag{13}$$

3.11. Final evaluation of the magnitude of the contact force

The aim of this method is to satisfy the energy conservation law in collisions of bodies exactly. For all elastic bodies this means that the total energy after collision must remain the same as before the collision. To achieve this goal, it is necessary to adopt the equation of the energy conservation into the solution. The change of total energy during the collision must be zero. Since we have $\mathbf{f}_P = \mathbf{e}_p|\mathbf{f}_P|$ in all cases – cf. the discussion on friction, in the fictious time step we are allowed to write

$$\Delta\Pi_k(|\mathbf{f}_P|, \mathcal{D}t) + \Delta\Pi_\sigma(|\mathbf{f}_P|, \mathcal{D}t) + \Delta\Pi_p(|\mathbf{f}_P|, \mathcal{D}t) + \mathfrak{E} = 0 \,, \tag{14}$$

replacing $\mathbf{f}_P$ in all additive terms by (11), (12) and (13) by $|\mathbf{f}_P|$ only; $\mathfrak{E}$ here refers to the eventual energy dissipation by plasticizing or damage. It is clear that we are looking for a nontrivial solution $|\mathbf{f}_P|$ of (14), i. e. for its non-zero root, which can be performed e. g. using some inexact version of Newton iterations, avoiding the evaluation of the derivatives of particular additive terms of the left-hand side of (14). The 1st estimate for $|\mathbf{f}_P|$ can exploit the fact that the contribution of the elastic potential energy by (13) can be neglected for this purpose, as well as the contribution of $\mathfrak{E}$, not analyzed in more details here; therefore (14) degenerates to a quadratic equation, which can be solved analytically. Thus we have all data for the evaluation of (9) and (10), thus all positions of the nodes A, B, C, D at the end of the time step can be adjusted as

$$\mathbf{x}_i^\times = \mathbf{x}_i + w t_c \mathbf{v}_i + (1-w) t_c \mathbf{v}_i^\times \,, \tag{15}$$

using some appropriate weight $w \in [0, 1]$, e. g. $w = 1/2$ (if no better arguments for this choice are available).

4. Kinematic method

The fundamental assumption for the collision of bodies Ω_1 and Ω_2 in this method is the condition of their impenetrability, i. e. $\Omega_1 \cup \Omega_2$ must be empty. We have the discretised masses m_i related to the points $i \in \{A, B, C, D\}$, as in *Section* 3. The velocity vectors before the impact (a priori known) are $\mathbf{v}_i$, the velocity vectors after the impact (undetermined yet) are $\mathbf{v}_i^{\times}$. Altogether, four velocity vectors have to be determined, i. e. 12 scalar unknowns. The equations for determining the components of $\mathbf{v}_i^{\times *}$ can be obtained from the law of conservation of linear and angular momentum, then from kinematic condition, expressing the impossibility of change of shape of any colliding line or triangle in time of the impact, and lastly from the properties of the contact plane and the influence of friction. This approach is applicable to both cases a) and b) from *Section* 2. The obvious transformation to the local coordinate system $\mathbf{v}_i^* = \mathbf{R}\mathbf{v}_i$ is available again, as well as its inverse $\mathbf{v}_i = \mathbf{R}^{\mathrm{T}}\mathbf{v}_i^*$.

4.1. Absolute friction

We shall start with the choice $\beta = 1$, as introduced in *Section* 3.

4.2. Conservation of momentum

Generally, due to the conservation of linear momentum we can write $m_i\mathbf{v}_i^{\times *} = m_i\mathbf{v}_i^*$, $i \in \{A, B, C, D\}$ being considered as the Einstein summation index again. The conservation of angular momentum can be related to an arbitrary point, e. g. to the origin of coordinates; then it reads

$$\mathbf{x}_i^* \times m_i\mathbf{v}_i^{\times *} = \mathbf{x}_i^* \times m_i\mathbf{v}_i^* \,. \tag{16}$$

For the case a) the conservation of angular momentum can be related to the point $P \equiv Q$ and consequently, we can write two vector equations

$$\begin{aligned} m_A N_B \mathbf{v}_A^{\times} - m_B N_A \mathbf{v}_B^{\times} &= m_A N_B \mathbf{v}_A - m_B N_A \mathbf{v}_B \,, \\ m_C N_D \mathbf{v}_C^{\times} - m_D N_C \mathbf{v}_D^{\times} &= m_C N_D \mathbf{v}_C - m_D N_C \mathbf{v}_D \,. \end{aligned} \tag{17}$$

4.3. Kinematic conditions

For the case b) all in-planar velocity components must satisfy the condition of rigid body motion in the element plane. It will be useful to omit all directions $\mathbf{x}_{3i}$ for both position and velocity vectors, since they have no influence on deformation of the involved elements All upper indices * will be omitted for brevity, considering the local coordinate system in the compatible way with *Section* 3. Let $\mathfrak{I}_m$ be the mass moment of inertia, related to the axis x_3, in the center T of gravity of the total mass $m_T = m_A + m_B + m_C + m_D$, due to the absolute friction and the condition $\mathbf{v}_D = \mathbf{v}_Q$. Then the angular momentum to such axis is $\mathfrak{I}_m\omega = (x_{i1} - x_{T1})m_i v_{i2} - (x_{i2} - x_{T2})m_i v_{i1}$, where ω denotes the angular velocity to the axis x_3, introduced as $\omega = \mathfrak{I}_m/\mathfrak{I}$, $\mathfrak{I}$ being the moment of inertia to such axis. Then we have $v_{T1}^{\times} = m_i v_{i1}^{\times}/m_T$, $v_{T2}^{\times} = m_i v_{i2}^{\times}/m_T$. Taking also the impact rigidity into account, for $j \in \{A, B, C\}$ we can write finally

$$v_{j1}^{\times} = v_{T1}^{\times} - \omega(x_{j2} - x_{T2}) \,, \qquad v_{j1}^{\times} = v_{T2}^{\times} + \omega(x_{j1} - x_{T1}) \,. \tag{18}$$

4.4. Contact conditions, influence of friction

In *Section* 4 we have still considered, up to now, $\beta = 1$, thus $\mathbf{v}_P^\times = \mathbf{v}_Q^\times$, as explained in *Section* 3. Thus for the case a) we can write $N_A\mathbf{v}_A^\times + N_B\mathbf{v}_B^\times = N_C\mathbf{v}_C^\times + N_D\mathbf{v}_D^\times$, whereas for the case b) with $\mathbf{v}_D^\times = \mathbf{v}_Q^\times$ we have $N_j\mathbf{v}_j^\times = \mathbf{v}_D^\times$, using the notation compatible with (18) and (17).

4.5. Zero friction

For $\beta = 0$ all velocity vector components parallel to the sliding plane remain the same after impact as before, i. e. $v_1^\times = v_{i1}$, $v_2^\times = v_{i2}$, which decreases the number of unknowns from 12 to 4. Due to the conservation of linear momentum we have $m_i v_{i3}^\times = m_i v_{i3}$. Using the same notation as in the considerations related to $\beta = 1$, the conservation of angular momentum gives two scalar equations $x_{i1} m_i v_{i3}^\times = x_{i1} m_i v_{i3}$, $x_{i2} m_i v_{i3}^\times = x_{i2} m_i v_{i3}$. The velocity vector components perpendicular to the sliding plane at the colliding points P, Q after the impact are the same, i. e. $v_{Q3}^\times = v_{P3}^\times$, respecting the impenetrability assumption, which provides the last needed equation. Then for the case a) we have $N_A v_{A3}^\times + N_B v_{B3}^\times = N_C v_{C3}^\times + N_D v_{D3}^\times$ and for the case b) $N_j v_{j3}^\times = v_{D3}^\times$ analogously.

4.6. General friction

For $\beta \in [0, 1]$ the interpolation $\mathbf{v}^\times = \beta\mathbf{v}_i^{a\times *} + (1-\beta)\mathbf{v}_i^{z\times *}$ for all $i \in \{A, B, C, D\}$ can be recommended again, as in *Section* 3.

4.7. Adjustment of the coordinates of nodes

Let us remind the transformations of the type $\mathbf{v}_i^\times = \mathbf{R}^{\mathrm{T}}\mathbf{v}_i^{\times *}$, needed for the final update. Thus we come back to (15). Let us also remark that it is useful to keep the sign of the difference of velocities $\mathbf{v}_i^\times$ of the colliding points P, Q in memory until the next time step: if the sign does not change then the contact must be still handled, unlike the opposite case.

5. Illustrative examples

The first example is the problem of an elastic rod impacting a rigid barrier. The input values were taken from [8]. Fig. 4 shows a model of a rod of total length $L = 1\,\mathrm{m}$, cross section area $A = 1\,\mathrm{m}^2$, Young's modulus $E = 1\,\mathrm{MPa}$ and mass density $\rho = 1\,\mathrm{kg/m}^3$, divided into 100 elements with its mass discretised to the nodes, which is situated at distance $g_0 = 0\,\mathrm{m}$ towards the rigid barrier; its initial velocity is $v_0 = 1\,\mathrm{m/s}$. The time step for calculation $\mathcal{D}t = 10^{-8}\,\mathrm{s}$ is applied. Fig. 5 shows the comparison of results obtained by the energy, kinematic and penalty method, particularly displacement of the impacting node and the change of the components of energy.

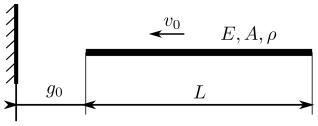


Figure 4: An elastic rod impacting a rigid barrier.

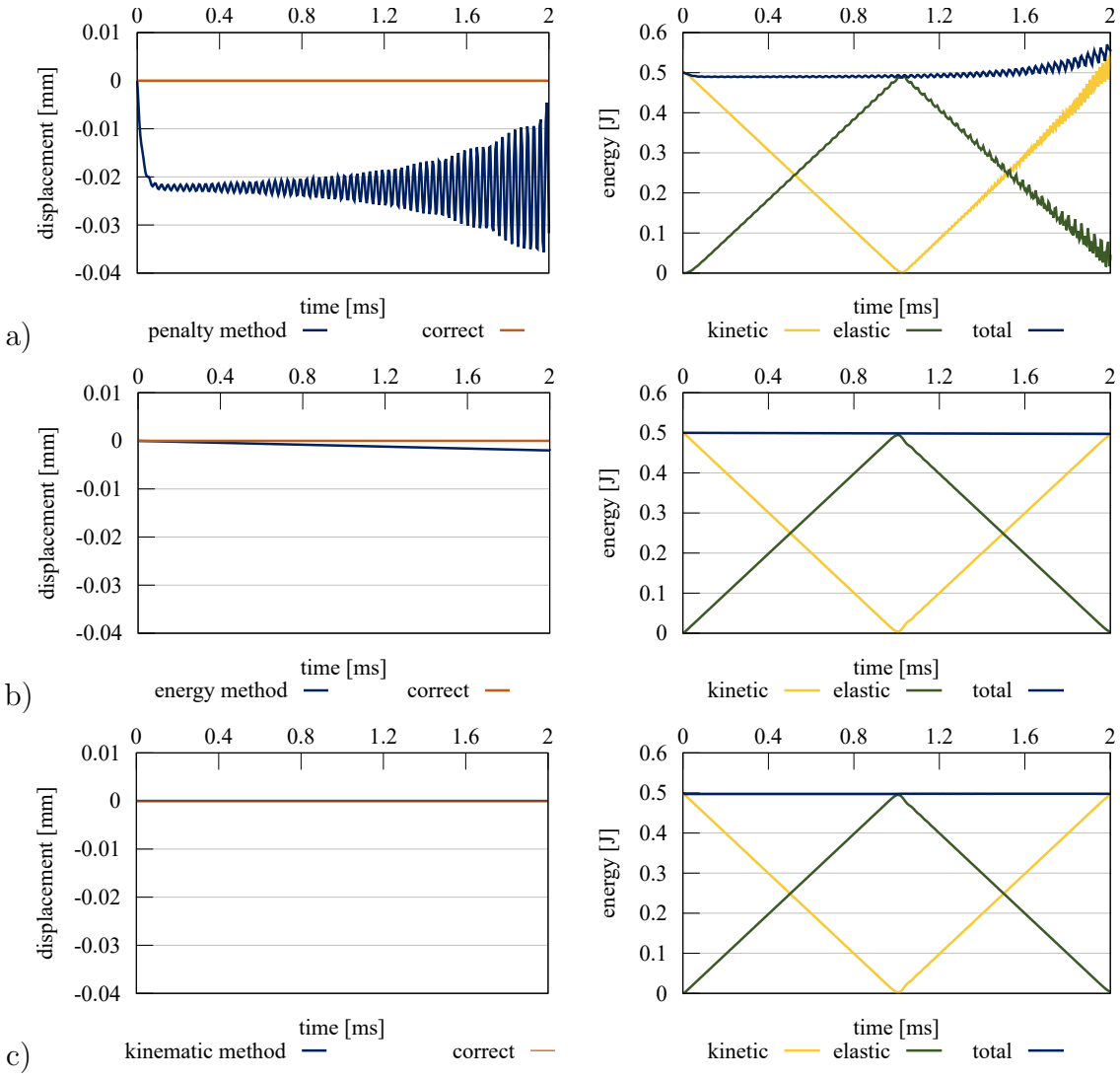


Figure 5: Time distribution of displacement and energy balance for a) the penalty method, using the penalty stiffness $P = 10^{11}\,\mathrm{Nm}^{-1}$, as introduced by [3], b) the energy method and c) the kinematic method.

The second example presents collision of two symmetrical cylinders, described in detail in [11], where the input data were also taken from. Fig. 6 shows two identical cylinders with radius $R = 4\,\mathrm{m}$, Young's modulus $E = 1000\,\mathrm{MPa}$, Poisson's ratio $\nu = 0.2$ and mass density $\rho = 1000\,\mathrm{kg/m}^3$ moving with the initial velocity $v_0 = 2\,\mathrm{m/s}$ against each other. The time step for calculation $\mathcal{D}t = 5 \cdot 10^{-6}\,\mathrm{s}$ is applied. Symmetry boundary conditions are applied. Fig. 7 demonstrates the time propagation and stress in time for the energy and kinematic methods separately, whereas Fig. 8 shows the time distribution of energy balance.

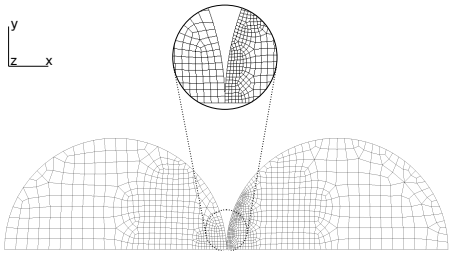
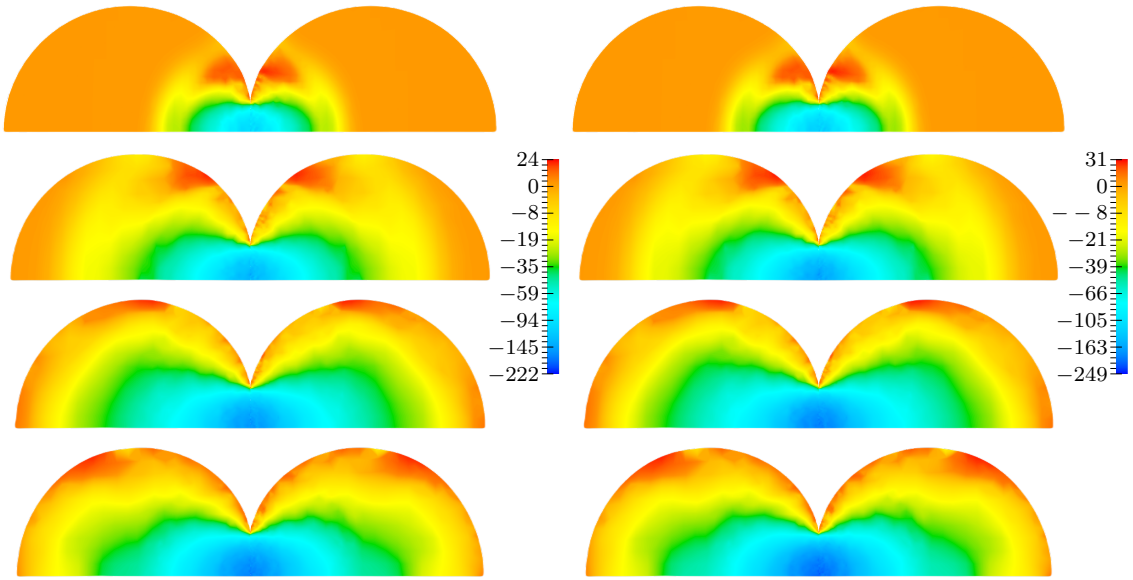
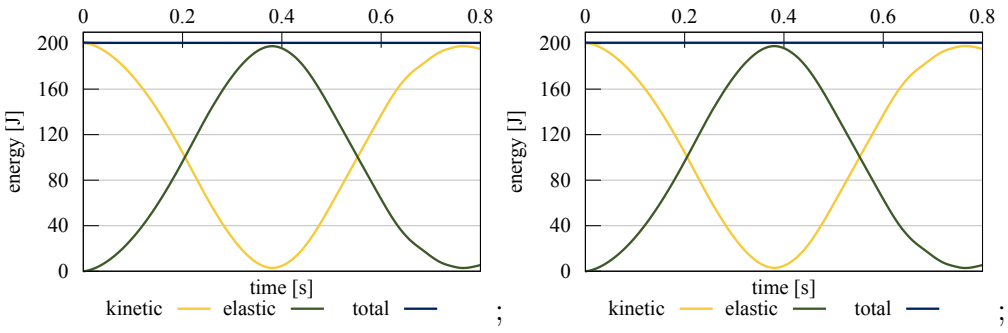


Figure 6: FE mesh, impact of two cylinders.



energy method kinematic method

Figure 7: Stress in the horizontal direction $\sigma_x[\mathrm{N/m^2}]$ during the wave propagation in times $t = 0.1, 0.2, 0.3, 0.4\,\mathrm{s}$.



Energy method Kinematic method

Figure 8: Time distribution of energy balance.

6. Conclusions

The most commonly used method for impact of bodies, called the penalty method, showed itself as unsatisfactory. This method is based on the idea that the contact force is proportional to the penetration of the colliding bodies. Therefore a violation of the principle of impenetrability of bodies is assumed and even necessary for it to work. This method also does not satisfy the conservation of energy law with sufficient precision and provides rather random results. Both methods introduced in this paper validated their superiority over the penalty method. The energy method satisfies the conservation of energy exactly, whereas the kinematic method preserves the principle of impenetrability. In the last decades several improvement of the penalty method and new approaches to the impact of bodies have been published, some of them being mentioned in References.

The authors of this paper have introduced two methods for transient analysis of impacts of bodies suitable for the explicit method. Both methods proved their good accuracy, efficiency and robustness. The energy conservation law is fulfilled very well without necessity of substantial shortage of the global time step of numerical integration and without necessity of introducing additional computational parameters, understanding them and determining their values. Both methods take the exact time of the impact for each contact into consideration. In the case of the kinematic method, all deformations, velocities and accelerations are determined with help of division of the time step into its substeps before and after the impact. The energy method introduces the equation of conservation of energy in each time step when a contact occurs, so all unwanted energy changes are eliminated.

The suggested approaches enable contacts of one surface with more nodes, as well as of one line with more lines, in one time step, as presented by the second numerical example. The methods do not demand any use of neither penalty method nor Lagrangian multipliers.

As for the problem of friction, all methods of static friction, based on the idea of pulling a burden on a surface, are problematic for the impact analysis. Unlike them, a very general model of the impact friction, assuming the theoretically clear limits of the friction, namely the zero friction and the absolute friction, is introduced in this paper. Then the real friction can be seen as a linear combination of those two limit cases. The friction coefficient, which is the relative weight factor of the absolute friction, can be determined by simple experiments.

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