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GODUNOV-LIKE NUMERICAL FLUXES FOR CONSERVATION LAWS ON NETWORKS

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Abstract: We describe a numerical technique for the solution of macroscopic traffic flow models on networks of roads. On individual roads, we consider the standard Lighthill-Whitham-Richards model which is discretized using the discontinuous Galerkin method along with suitable limiters. In order to solve traffic flows on networks, we construct suitable numerical fluxes at junctions based on preferences of the drivers. Numerical experiment comparing different approaches is presented.

Keywords: traffic flow, discontinuous Galerkin method, junctions, numerical flux

MSC: 65M60, 76A30, 90B20

1. Introduction

Let us have a road and an arbitrary number of cars. We would like to model the movement of cars on our road. We call this model a traffic flow model. We use *macroscopic models*, where we view our traffic situation as a continuum and study the density of cars in every point of the road. This model is described by partial differential equations.

Our aim is to numerically solve macroscopic models of traffic flow. Our unknown is density at point x and time t. As we shall see later, the solution can be discontinuous. Due to the need for discontinuous approximation of density, we use the *discontinuous Galerkin method*. The aim of modelling is understanding traffic dynamics and deriving possible control mechanisms for traffic.

1.1. Macroscopic traffic flow models

We begin with the mathematical description of macroscopic vehicular traffic, cf. [4] and [6] for details. First, we consider a single road described mathematically as a one-dimensional interval. In the basic macroscopic models, traffic flow is described by two basic fundamental quantities – *traffic flow* Q and *traffic density* ρ.

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In article [3], Greenshields realized that traffic flow is a function which depends only on traffic density in homogeneous traffic. The relationship between the ρ and Q is described by the *fundamental diagram*. There are many different proposals for the traffic flow Q derived from real traffic data, cf. [4]. Here we present only *Greenshields model*, which define traffic flow as $Q(\rho) = v_{\max}\rho(1-\frac{\rho}{\rho_{\max}})$, where $v_{\max}$ is the maximal velocity and $\rho_{\max}$ is the maximal density.

Since the number of cars is conserved, the basic governing equation is a *nonlinear first order hyperbolic partial differential equation*, cf.

$$\rho_t + (Q(\rho))_x = 0, \qquad x \in \mathbb{R},\ t > 0. \tag{1}$$

Equation (1) must be supplemented by the initial condition $\rho(x, 0) = \rho_0(x)$, $x \in \mathbb{R}$ and an inflow boundary condition.

Following [2], we consider a complex *network* represented by a directed graph. Each vertex (junction) has a finite set of incoming and outgoing edges (roads). In our case it is sufficient to study our problem only on a *simple network* with one vertex J and its n incoming and m outgoing adjacent edges. On each road we consider equation (1), while at the vertex we consider a *Riemann solver*.

It is also necessary to take into account the preferences of drivers how the traffic from incoming roads is distributed to outgoing roads according to some predetermined coefficients. There is a *traffic–distribution matrix* A describing the distribution of traffic among outgoing roads, i.e.

$$A = \begin{bmatrix} \alpha_{n+1,1} & \cdots & \alpha_{n+1,n} \\ \vdots & \vdots & \vdots \\ \alpha_{n+m,1} & \cdots & \alpha_{n+m,n} \end{bmatrix}, \tag{2}$$

where for all $i \in \{1, \dots, n\}$, $j \in \{n+1, \dots, n+m\}$: $\alpha_{j,i} \in [0, 1]$ and for all $i \in \{1, \dots, n\}$: $\sum_{j=n+1}^{n+m} \alpha_{j,i} = 1$. The i^{th} column of A describes how the traffic from the incoming road I_i distributes to the outgoing roads at J.

We denote the endpoints of road I_i as a_i, b_i. We introduce the notation of spatial limits $\rho_i^{(L)}(b_i, t) := \lim_{x\to b_{i-}} \rho_i(x, t)$ and $\rho_i^{(R)}(a_i, t) := \lim_{x\to a_{i+}} \rho_i(x, t)$.

Let $\rho = (\rho_1, \dots, \rho_{n+m})^T$ be a *weak solution at* J, see [2, Definition 5.1.8], where ρ has bounded variation in space. Then ρ satisfies the *Rankine–Hugoniot condition*, which represents the conservation of cars at J:

$$\sum_{i=1}^{n} Q(\rho_i^{(L)}(b_i, t)) = \sum_{j=n+1}^{n+m} Q(\rho_j^{(R)}(a_j, t)) \tag{3}$$

for almost every $t > 0$, cf. [2, Lemma 5.1.9].

1.2. Discontinuous Galerkin method

As an appropriate method for the numerical solution of (1), we choose the *discontinuous Galerkin* (DG) method, which is essentially a combination of finite volume and finite element techniques, cf. [1].

Consider an interval $\Omega = (a, b)$. Let $\mathcal{T}_h$ be a partition of $\overline{\Omega}$ into a finite number of intervals (elements) $K = [a_K,\ b_K]$. We denote the set of all boundary points of all elements by $\mathcal{F}_h$. We seek the numerical solution in the space of discontinuous piecewise polynomial functions $S_h = \{v;\ v|_K \in P^p(K),\ \forall K \in \mathcal{T}_h\}$, where $p \in \mathbb{N}_0$ and $P^p(K)$ denotes the space of all polynomials on K of degree at most p. For a function $v \in S_h$ we denote the *jump* in the point $s \in \mathcal{F}_h$ as $[v]_s = v^{(L)}(s) - v^{(R)}(s)$.

We formulate the DG method for the general first order hyperbolic problem $u_t + f(u)_x = g$, $x \in \Omega$, $t \in (0, T)$, which is supplemented by the initial and boundary condition. The DG formulation then reads, cf. [1]: Find $u_h \colon [0, T] \to S_h$ such that

$$\int_\Omega (u_h)_t \varphi \, \mathrm{d}x - \sum_{K \in \mathcal{T}_h} \int_K f(u_h)\varphi_x \, \mathrm{d}x + \sum_{s \in \mathcal{F}_h} H(u_h^{(L)}, u_h^{(R)})\, [\varphi]_s = \int_\Omega g\varphi \, \mathrm{d}x,$$

for all $\varphi \in S_h$. On $\mathcal{F}_h$ we use the approximation $f(u_h) \approx H(u_h^{(L)}, u_h^{(R)})$, where H is a *numerical flux*. We use the *Godunov* numerical flux, which is defined as the flux at the exact solution of the Riemann problem with $u_i^{(L)}$ and $u_i^{(R)}$, cf. [5]. It can be expressed as

$$H^{\mathrm{God}}_{\mathrm{orig}}\left(u^{(L)}, u^{(R)}\right) = \begin{cases} \min_{u^{(L)} \le u \le u^{(R)}} f(u), & \text{if } u^{(L)} < u^{(R)}, \\ \max_{u^{(R)} \le u \le u^{(L)}} f(u), & \text{if } u^{(L)} \ge u^{(R)}. \end{cases} \tag{4}$$

For our purpose, we derive alternative form, which is inspired by maximum possible traffic flow (see Section 2) in case with one incoming and one outgoing road.

Definition 1 (Alternative form of Godunov numerical flux)**.** *Let the Greenshields traffic flow f have global maximum at u_*. Then the Godunov numerical flux is defined as*

$$H^{\mathrm{God}}\left(u^{(L)}, u^{(R)}\right) = \min\left\{f_{\mathrm{in}}(u^{(L)}), f_{\mathrm{out}}(u^{(R)})\right\}, \tag{5}$$

where

$$f_{\mathrm{in}}(u^{(L)}) = \begin{cases} f(u^{(L)}), & \text{if } u^{(L)} < u_*, \\ f(u_*), & \text{if } u^{(L)} \ge u_*, \end{cases} \quad f_{\mathrm{out}}(u^{(R)}) = \begin{cases} f(u_*), & \text{if } u^{(R)} \le u_*, \\ f(u^{(R)}), & \text{if } u^{(R)} > u_*. \end{cases}$$

Definition 1 can be interpreted as the maximum possible flow through the boundary, where f_{in} is the maximum possible inflow from the left element and f_{out} is maximum possible outflow to the right element. The expressions (4) and (5) are equivalent in case of Greenshields traffic flow. For simplicity, by $H(\cdot\,,\cdot)$ we mean the Godunov numerical flux in the alternative form (5) in the rest of this paper.

For time discretization of the DG method we use the *explicit Euler method*. As a basis for S_h, we use *Legendre polynomials*. We use *Gauss–Legendre quadrature* to evaluate integrals over elements. Because we calculate physical quantity, the result must be in some interval, e.g. $\rho \in [0, \rho_{\max}]$. Thus, we use *limiters* in each time step to obtain the solution in the admissible interval. Following [5], we also apply limiting to treat spurious oscillations near discontinuities. From the definition of limiters, the average value of the solution doesn't change, i.e. the number of vehicle is conserved. Limiters are necessary in the case of an oscillating solution in a sufficiently small neighborhood of one of the limit values.

2. Maximum possible traffic flow

Based on the traffic distribution matrix, the authors of [2] define an *admissible weak solution of (1)* at the junction J as $\rho = (\rho_1, \dots, \rho_{n+m})^T$ satisfying

1) ρ is a weak solution at J such that ρ has bounded variation in space, i.e. the Rankine–Hugoniot condition holds.
2) $Q(\rho_j^{(R)}(a_j, \cdot)) = \sum_{i=1}^{n} \alpha_{j,i} Q(\rho_i^{(L)}(b_i, \cdot)),\ \forall j = n+1, \dots, n+m.$
3) $\sum_{i=1}^{n} Q(\rho_i^{(L)}(b_i, \cdot))$ is a maximum subject to 1) and 2).

Assumption 1) is the conservation of cars at the junction. Assumption 2) takes into account the prescribed preferences of drivers. Assumption 3) postulates that drivers choose to maximize the total flux through the junction.

One problem with the approach of [2] is that explicitly constructing the fluxes requires the solution of a Linear Programming problem on the incoming fluxes. This is done in [2] for the purposes of constructing a Riemann solver at the junction and in [7] for the purposes of obtaining numerical fluxes at the junction in order to formulate the DG scheme. Closed-form solutions are provided in [7] in the special cases $n = 1$, $m = 2$ and $n = 2$, $m = 1$ and $n = 2$, $m = 2$.

Now, we will study the case with one incoming and two outgoing roads. This example is important for us, because it inspires us in the construction of α-inside Godunov flux (see Section 3.2). We use the method described in [7, Section 2.2] with our notation. In this case, we have distribution coefficient $\alpha_{2,1} = \alpha$ and $\alpha_{3,1} = 1 - \alpha$. Then we calculate maximum possible inflow to the junction from incoming road as

$$H_1(t) = \min \left\{ f_{\text{in}}(\rho_1^{(L)}(b_1, t)), \frac{f_{\text{out}}(\rho_2^{(R)}(a_2, t))}{\alpha}, \frac{f_{\text{out}}(\rho_3^{(R)}(a_3, t))}{1 - \alpha} \right\}. \tag{6}$$

The outflow from the junction to outgoing road is calculated as H_1 multiplied by the distribution coefficient, i.e. $H_2(t) = \alpha H_1(t)$ and $H_3(t) = (1 - \alpha) H_1(t)$.

Remark. We can notice, that traffic congestion on one of the outgoing road influences the traffic flow to the second outgoing road. For example, when $f_{\text{out}}(\rho_2^{(R)}) = 0$, then $H_1 = H_2 = H_3 = 0$.

3. Numerical fluxes at junctions

We take a different approach from that of [7] and [2]. Our approach has the advantage that it is simple and explicitly constructed for all junction types. We shall prove the basic properties of this construction and discuss the differences with the approach of [7] and [2].

In our previous paper [6], we used Lax-Friedrichs numerical flux. When we calculate traffic distribution error, it was nearly impossible to obtain distribution error equal to zero. This phenomenon is hard to justify in cases with low traffic. That is the reason, why we choose Godunov numerical flux. As we show later in Section 3.3, distribution error makes much more sense and is more justified.

3.1. α-outside Godunov flux

At the junction, we consider an incoming road I_i and an outgoing road I_j. If these roads were the only roads at the junction, i.e. if they were directly connected to each other, the (numerical) flux of traffic from I_i to I_j would simply be $H\big(\rho_{hi}^{(L)}(b_i,t), \rho_{hj}^{(R)}(a_j,t)\big)$, where ρ_{hi} and ρ_{hj} are the DG solutions on I_i and I_j, respectively. From the traffic distribution matrix, we know the ratios of the traffic flow distribution to the outgoing roads. Thus, we take the numerical flux $H_j(t)$ at the left point of the outgoing road I_j, i.e. at the junction, at time t as

$$H_j(t) := \sum_{i=1}^{n} \alpha_{j,i} H\big(\rho_{hi}^{(L)}(b_i,t), \rho_{hj}^{(R)}(a_j,t)\big), \tag{7}$$

for $j = n+1, \dots, n+m$. The numerical flux $H_j(t)$ can be viewed as the DG analogue of taking the combined traffic outflow $\sum_{i=1}^{n} \alpha_{j,i} Q\big(\rho_i^{(L)}(b_i,t)\big)$ from all incoming roads and prescribing it as the inflow of traffic to the road I_j.

Similarly, we take the numerical flux $H_i(t)$ at the right point of the incoming road I_i, i.e. at the junction, at time t as

$$H_i(t) := \sum_{j=n+1}^{n+m} \alpha_{j,i} H\big(\rho_{hi}^{(L)}(b_i,t), \rho_{hj}^{(R)}(a_j,t)\big), \tag{8}$$

for $i = 1, \dots, n$. Again, this can be viewed as an approximation of the traffic flow $\sum_{j=n+1}^{n+m} \alpha_{j,i} Q\big(\rho_j^{(R)}(a_j,t)\big)$ being prescribed as the outflow of traffic from I_i.

3.2. α-inside Godunov flux

We find the main difference between maximum possible traffic flow and α-outside Godunov flux is in the position of the distribution coefficient, cf. (6) and (8). That is the reason, why we decide to insert distribution coefficient into Godunov numerical flux.

Definition 2 (Godunov numerical flux with parameter)**.** *Let Greenshields traffic flow f has global maximum at u_*. Then Godunov numerical flux with parameter is defined as*

$$H^{\text{God}}\left(u^{(L)}, u^{(R)}, \alpha\right) = \min\left\{\alpha f_{\text{in}}(u^{(L)}), f_{\text{out}}(u^{(R)})\right\}, \tag{9}$$

where $f_{\text{in}}(u^{(L)})$ and $f_{\text{out}}(u^{(R)})$ are defined as in Definition 1.

The reason, why we put distribution coefficient in front of the f_{in} term, is the representation of the real supply from the incoming road. Only $\alpha_{j,i} f_{\text{in}}(\rho_i^{(L)}(b_i,t))$ cars per time want to go from incoming road i to outgoing road j. In case of $\alpha = 1$, the flux (9) is equivalent to the alternative form of Godunov numerical flux (5). For simplicity, by $H(\cdot\,,\cdot\,,\cdot\,)$ we mean the Godunov numerical flux with parameter in the rest of this paper.

Now we are able to take numerical flux with α-inside $H_j(t)$ at the left point of the outgoing road I_j at time t as

$$H_j(t) := \sum_{i=1}^{n} H\big(\rho_{hi}^{(L)}(b_i,t), \rho_{hj}^{(R)}(a_j,t), \alpha_{j,i}\big), \tag{10}$$

for $j = n+1, \dots, n+m$. Similarly, we take the numerical flux with α-inside $H_i(t)$ at the right point of the incoming road I_i at time t as

$$H_i(t) := \sum_{j=n+1}^{n+m} H\big(\rho_{hi}^{(L)}(b_i,t), \rho_{hj}^{(R)}(a_j,t), \alpha_{j,i}\big), \tag{11}$$

for $i = 1, \dots, n$.

3.3. Properties

It can be shown, that our choice of numerical fluxes conserves the number of cars at junctions, similarly as in (3), see Theorem 1. However, this choice does not distribute the traffic according to the traffic–distribution matrix (2) exactly, only approximately, see Theorem 2.

Firstly, we show the discrete version of Rankine–Hugoniot condition.

Theorem 1 (Discrete Rankine–Hugoniot condition)**.** *The numerical flux at junction J satisfies the discrete version of the Rankine–Hugoniot condition (3):*

$$\sum_{i=1}^{n} H_i(t) = \sum_{j=n+1}^{n+m} H_j(t) \tag{12}$$

whether

a) we use (7) and (8) with α-outside or

b) we use (10) and (11) with α-inside.

Proof. From the definition of H_i and H_j with α-outside, we immediately obtain:

$$\begin{aligned}\sum_{i=1}^{n} H_i(t) &= \sum_{i=1}^{n}\sum_{j=n+1}^{n+m} \alpha_{j,i} H\big(\rho_{hi}^{(L)}(b_i,t), \rho_{hj}^{(R)}(a_j,t)\big) \\ &= \sum_{j=n+1}^{n+m}\sum_{i=1}^{n} \alpha_{j,i} H\big(\rho_{hi}^{(L)}(b_i,t), \rho_{hj}^{(R)}(a_j,t)\big) = \sum_{j=n+1}^{n+m} H_j(t).\end{aligned}$$

Proof of the case b) is similar with the corresponding definition of H_i and H_j with α-inside. □

The second theorem is important for identifying the difference between maximum possible traffic flow described in Section 2 and our numerical fluxes at junction.

Theorem 2 (Traffic distribution error)**.** *The numerical flux at junction satisfies*

$$H_j(t) = \sum_{i=1}^{n} \alpha_{j,i} H_i(t) + E_j(t) \tag{13}$$

for all $j = n+1, \ldots, n+m$, where

a) in case of (7) and (8) with α-outside, the error term is

$$E_j(t) = \sum_{i=1}^{n} \sum_{\substack{l=n+1 \\ l \neq j}}^{n+m} \alpha_{j,i}\alpha_{l,i}\big(H_{i,j}(t) - H_{i,l}(t)\big), \tag{14}$$

where $H_{i,j}(t) := H\big(\rho_{hi}^{(L)}(b_i,t), \rho_{hj}^{(R)}(a_j,t)\big)$.

b) in case of (10) and (11) with α-inside, the error term is

$$E_j(t) = \sum_{i=1}^{n} \sum_{\substack{l=n+1 \\ l \neq j}}^{n+m} \left(\alpha_{l,i} H_{i,j}(t) - \alpha_{j,i} H_{i,l}(t)\right), \tag{15}$$

where $H_{i,j}(t) := H\big(\rho_{hi}^{(L)}(b_i,t), \rho_{hj}^{(R)}(a_j,t), \alpha_{j,i}\big)$.

Proof. We prove only the case a). By definition (7),

$$H_j(t) = \sum_{i=1}^{n} \alpha_{j,i} H_{i,j}(t) = \sum_{i=1}^{n} \alpha_{j,i} H_i(t) + \underbrace{\sum_{i=1}^{n} \alpha_{j,i}\big(H_{i,j}(t) - H_i(t)\big)}_{E_j(t)},$$

where $E_j(t)$ is the error term which we will show has the form (14): by definition (8), we have

$$\begin{aligned} E_j(t) &= \sum_{i=1}^{n} \alpha_{j,i}\Big(H_{i,j}(t) - \sum_{l=n+1}^{n+m} \alpha_{l,i} H_{i,l}(t)\Big) \\ &= \sum_{i=1}^{n} \alpha_{j,i} \sum_{l=n+1}^{n+m} \alpha_{l,i}\big(H_{i,j}(t) - H_{i,l}(t)\big) \\ &= \sum_{i=1}^{n} \sum_{\substack{l=n+1 \\ l \neq j}}^{n+m} \alpha_{j,i}\alpha_{l,i}\big(H_{i,j}(t) - H_{i,l}(t)\big), \end{aligned}$$

since $\sum_{l=n+1}^{n+m} \alpha_{l,i} = 1$. The proof of case b) is similar. □

An artifact of our model is that sometimes we do not satisfy the traffic–distribution coefficients exactly, cf. (13) and assumption 2) of maximum possible traffic flow (see Section 2). This corresponds to the real situation where some cars decide to use another road instead of staying in the traffic jam.

For the comparison of maximum possible traffic flow described by (6) and our approach, we take a junction with one incoming and two outgoing roads. As it was mentioned in Remark 2, if there is a traffic jam in one of the outgoing roads, the maximum possible flow through the junction is 0, thus the whole junction is blocked by a traffic jam in one of the outgoing roads. On the other hand, in our approach the junction is not blocked by a traffic jam on one of the outgoing roads and the cars can still go into the second outgoing road according to the traffic–distribution coefficients. So our choice of numerical fluxes corresponds to modelling turning lanes, which allow the cars to separate before the junction according to their preferred turning direction. Since macroscopic models are intended for long (multi–lane) roads with huge numbers of cars, our model makes sense in this situation. The original approach from [2, 7] works for one–lane roads, where splitting of the traffic according to preference is not possible.

Another difference is that we can use all varieties of traffic lights. The model of [2, 7] can use only the so-called full green lights. Our approach gives us an opportunity to change the lights for each direction separately.

4. Numerical results

We consider a simple network with one incoming road (Road 1) and two outgoing roads (Road 2 and Road 3). The network will be closed at their endpoints (a_1, b_2 and b_3). Thus, we can check the total number of cars, because we have neither inflow nor outflow. We choose $\alpha_{2,1} = 0.75$ and $\alpha_{3,1} = 0.25$. The length of all roads is 1. As we mention above, we use the combination of the explicit Euler method (step size $\tau = 10^{-4}$) and DG method (number of elements $N = 150$ on each road). We calculate the piecewise linear approximations of solutions and we use two Gaussian quadrature points in each element. We use Greenshields model with $v_{\max} = 1$ and $\rho_{\max} = 1$. We have initial conditions

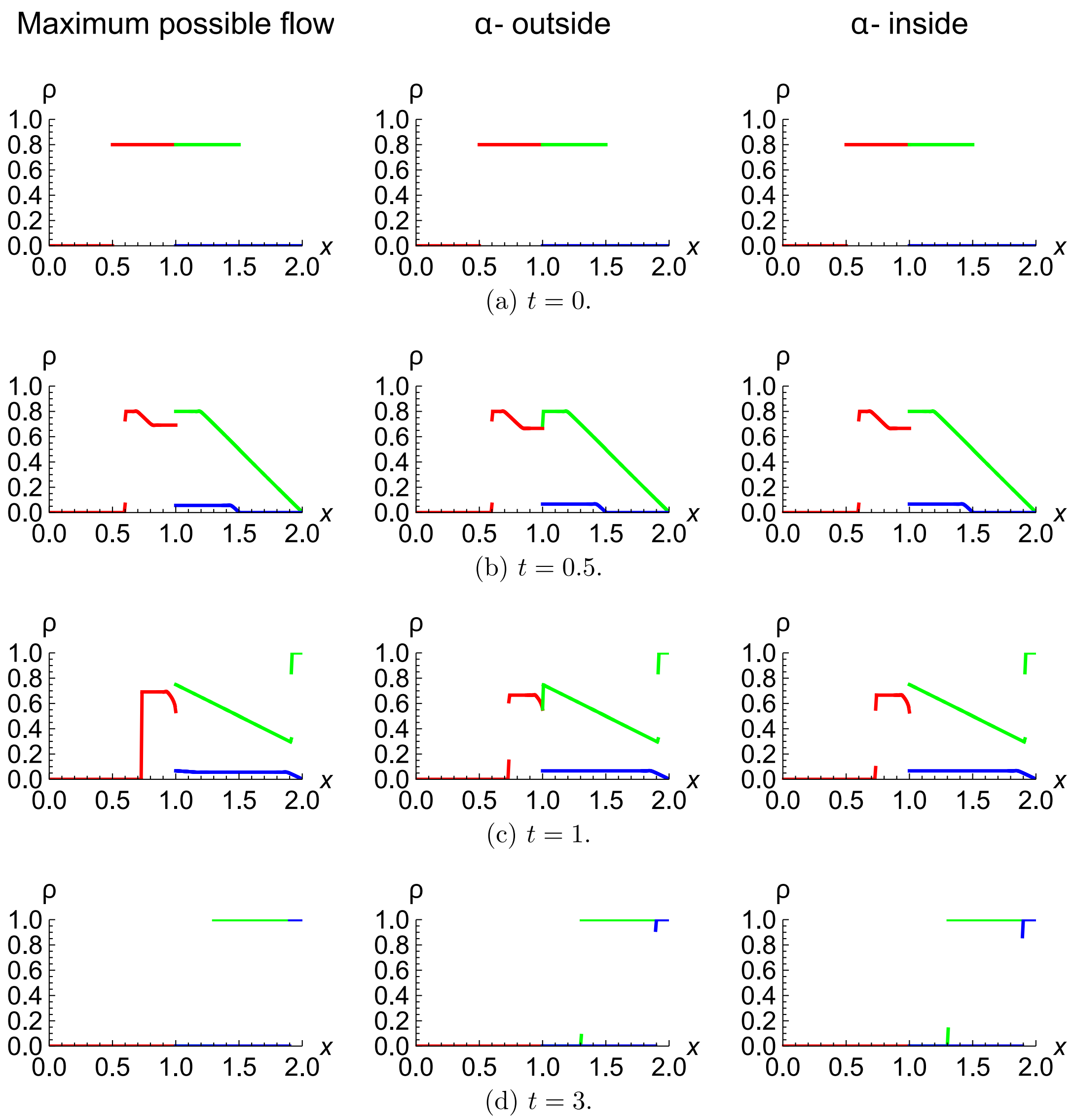
$$\rho_{0,1}(x) = \begin{cases} 0, \\ 0.8, \end{cases} \qquad \rho_{0,2}(x) = \begin{cases} 0.8, \\ 0, \end{cases} \qquad \rho_{0,3}(x) = \begin{cases} 0, & x \in [0, 0.5], \\ 0, & x \in (0.5, 1], \end{cases}$$

cf. Figure 1a. There is 0.4 cars on Road 1. These cars are distributed into Road 2 (it has 0.4 cars already) and Road 3 by distribution coefficients. At the end, we can expect 0.7 cars on Road 2 and 0.1 cars on Road 3.

We can see the results in Figure 1. Maximum possible flow is in the left column, the numerical flux with α-outside is in the middle column and with α-inside is in the right column. If we compare inflow to the Road 3 in Figure 1b between Maximum possible flow and our numerical flux (doesn't depend on the position of α), we can see that our numerical flux allows more inflow. If we look at inflow to the Road 2, see

Figures 1b and 1c, we observe similar inflow of maximum possible flow and numerical flux with α-inside. However, the inflow in case of numerical flux with α-outside is slightly smaller. In general, the numerical flux with α-inside is the combination of the two other approaches. It allows as much as possible cars go to the Road 2 like the maximum possible flow do. On the other hand, some drivers change their minds and choose Road 3 instead of Road 2 due to the congestion on Road 2, same as in the case of numerical flux with α-outside.

The final results are in Figure 1d. Maximum possible traffic flow has 0.7 cars on Road 2 and 0.1 on Road 3. Numerical flux with α-outside has 0.6936 cars on Road 2 and 0.1064 on Road 3. Numerical flux with α-inside has 0.6938 cars on Road 2 and 0.1062 on Road 3.



(a) $t = 0$.

(b) $t = 0.5$.

(c) $t = 1$.

(d) $t = 3$.

Figure 1: Comparison of network with Road 1, Road 2 and Road 3

We choose this congested example due to the demonstration of distribution error from Theorem 2. In the non–congested cases, the traffic distribution error is zero.

5. Conclusion

We have demonstrated the numerical solution of macroscopic traffic flow models using the discontinuous Galerkin method. For traffic networks, we construct special numerical fluxes at the junctions. The use of DG methods on networks is not standard. We have described the differences between our approach and the paper [7] by Čanić, Piccoli, Qiu and Ren, where the maximum possible flow at the junction is used.

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