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In: Jan Chleboun and Jan Papež and Karel Segeth and Jakub Šístek and Tomáš Vejchodský (eds.): Programs and Algorithms of Numerical Mathematics, Proceedings of Seminar. Hejnice, June 23-28, 2024. , Prague, 2025.
pp. 61–71.

Persistent URL: http://dml.cz/dmlcz/703211

Programs and Algorithms of Numerical Mathematics 22
J. Chleboun, J. Papež, K. Segeth, J. Šístek, T. Vejchodský (Eds.)
Institute of Mathematics CAS, Prague 2025

NUMERICAL STUDY OF TWO-LEVEL ADDITIVE SCHWARZ PRECONDITIONER FOR DISCONTINUOUS GALERKIN METHOD SOLVING ELLIPTIC PROBLEMS

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Abstract: The paper deals with the analysis and numerical study of the domain decomposition based preconditioner for algebraic systems arising from the discontinuous Galerkin (DG) discretization of the linear elliptic problems. We introduce the DG discretization of the model problem and present the spectral *hp*-bound of the corresponding linear algebraic systems. Moreover, we present the two-level additive Schwarz preconditioner together with the theoretical result related to the estimate of the condition number. Finally, we present the numerical experiments supporting the theoretical results and demonstrate the efficiency of this approach for the solution of nonlinear problems.

Keywords: domain decomposition, elliptic partial differential equation, two-level additive Schwarz preconditioner

MSC: 65N15, 65M15, 65F08

1. Introduction

Discontinuous Galerkin method (DGM) became a very popular method for solving partial differential equations, cf. [5]. DGM is based on a piecewise polynomial but discontinuous approximation where the inter-element continuity is replaced by special terms. The DGM exhibits a very robust, accurate, and efficient technique for various problems. On the other side the DG discretization leads to large sparse algebraic systems, whose solution usually exhibits the most time-consuming part of the whole computational process.

The domain decomposition techniques exhibit a powerful strategy, which allows to split the computational work and employ the parallel power of modern supercomputers. One possibility is to split the given problem in several smaller sub-problems with suitably chosen interface conditions, and solve them iteratively to coordinate the solution between neighbouring subdomains, cf. monographs [3, 11]. However,

DOI: 10.21136/panm.2024.06

more frequent is to use the domain decomposition methods as preconditioners for Krylov subspace iterative methods, such as the conjugate gradient method. In this paper, we focus on the two-level additive Schwarz (AS) preconditioner (cf. [2, 9]). In particular, we present theoretical results related to the condition number of the preconditioned system and several numerical examples demonstrating the efficiency of this technique.

In Section 2, we introduce the discretization of the linear model problem by *hp*-variant of the discontinuous Galerkin method (DGM), and present the *hp*-bound of the condition number of the corresponding algebraic system. In Section 3, we formulate the two-level additive Schwarz preconditioner and present the bounds of the condition number of the preconditioned system arising from DGM. In Section 4, we introduce the results of numerical experiments performed to support the analysis. These experiments are the main contribution since they show that the approach also works for nonlinear cases and leads to improved computational time when used with the parallel computations. Several concluding remarks are given in Section 5.

2. Discontinuous Galerkin method

We are dealing with the following symmetric linear elliptic problem

$$\begin{aligned} -\mathrm{div}(\boldsymbol{K}\nabla u) &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where $\Omega \in \mathbb{R}^d$, $d = 2, 3$ is a bounded domain with polygonal Lipschitz boundary $\partial\Omega$ and $\boldsymbol{K} = \boldsymbol{K}(x)$ is a symmetric positive definite matrix in $\mathbb{R}^{d\times d}$. We assume that $\exists k_0, k_1 > 0$, independent of $x \in \Omega$, such that $k_0|\xi| \leq |\boldsymbol{K}\xi| \leq k_1|\xi|$ $\forall \xi \in \mathbb{R}^d$. For simplicity, we assume the homogeneous Dirichlet boundary condition, however the results can be easily extended to other boundary conditions. Finally, we use the notation $L^2(M)$ for the Lebesgue space of square-integrable functions over $M \subset \mathbb{R}^d$, $d = 2, 3$ and we denote by $(\cdot, \cdot)_\Omega$ the standard inner product in $L^2(\Omega)$.

2.1. Discretization of domain Ω

Let $\mathcal{T}_h$, $h > 0$ be a partition of the domain $\bar{\Omega}$ into non-overlapping triangles K such that $\bigcup_{K\in\mathcal{T}_h} \overline{K} = \overline{\Omega}$. We set $h = \max_{K\in\mathcal{T}_h} h_K$, where h_K is the diameter of the element K, $K \in \mathcal{T}_h$, and we denote by ∂K the boundary of $K \in \mathcal{T}_h$.

In addition, let $\mathcal{F}_h$ be the set of all faces γ of $\mathcal{T}_h$ and we put

$$\mathcal{F}_h^B = \{\gamma \in \mathcal{F}_h \colon \gamma \subset \partial\Omega\} \qquad \text{and} \qquad \mathcal{F}_h^I = \mathcal{F}_h \setminus \mathcal{F}_h^B$$

for boundary and interior edges, respectively. For each $\gamma \in \mathcal{F}_h^I$ we consider a unit normal vector $\boldsymbol{n}_\gamma$ whose orientation can be arbitrarily chosen. If $\gamma \in \mathcal{F}_h^B$, the unit normal $\boldsymbol{n}_\gamma$ is outer to $\partial\Omega$.

Let $\boldsymbol{p} := \{p_K \colon K \in \mathcal{T}_h\}$ be a set of integers that assigns to each triangular element its polynomial degree of approximation. We assume that the ratio of polynomial approximation degrees of any two neighboring elements is bounded.

The approximate solution is sought in the space of discontinuous piecewise polynomial functions

$$S_{hp} := \{v \in L^2(\Omega) \colon v|_K \in P_{p_K}(K)\, \forall K \in \mathcal{T}_h\},$$

where $P_{p_K}(K)$ denotes the space of polynomials of degree less or equal than p_K on K.

By $v|^+$ and $v|^-$ we denote the traces of function $v \in S_{hp}$ on $\gamma \in \mathcal{F}_h^I$ in the direction of $\boldsymbol{n}_\gamma$ and opposite the direction of $\boldsymbol{n}_\gamma$, respectively. Using this notation we define the jump $[\cdot]_\gamma$ and the mean value $\langle\cdot\rangle_\gamma$ of $v \in S_{hp}$ by

$$[v]_\gamma = v|^+ \boldsymbol{n}_\gamma - v|^- \boldsymbol{n}_\gamma \quad \text{and} \quad \langle v\rangle_\gamma = \frac{1}{2}(v|^+ + v|^-), \quad \gamma \in \mathcal{F}_h^I, \tag{2}$$

respectively. For $\gamma \in \mathcal{F}_h^B$, we set $[v]_\gamma = v\boldsymbol{n}_\gamma$ and $\langle v\rangle_\gamma = v$. Usually, we drop the subscript γ.

Finally, we assume that the mesh is *shape-regular* and *quasi-uniform*. Then we set the edge size by

$$h_\gamma := \max(h_K, h_{K'}) \qquad \gamma \subset \partial K \cap \partial K'.$$

More details can be found in [5, Chapter 2.3].

2.2. Primal formulation of DGM

We introduce the approximate solution of our problem, more details can be found, e.g., in [5, Chapter 2.4]. Using (2), we define the billinear form $\mathcal{A}_h(u, v)$ by

$$\begin{aligned}\mathcal{A}_h(u,v) := &\sum_{K\in\mathcal{T}_h} \int_K \boldsymbol{K}\nabla u \cdot \nabla v \,\mathrm{dx} - \sum_{\gamma\in\mathcal{F}_h^I} \int_\gamma (\langle \boldsymbol{K}\nabla u\rangle \cdot [v] + \langle \boldsymbol{K}\nabla v\rangle \cdot [u]) \,\mathrm{dS} \\ &+ \sum_{\gamma\in\mathcal{F}_h^I} \int_\gamma \sigma\,[u]\,[v] \,\mathrm{dS}, \qquad u, v \in S_{hp}.\end{aligned}$$

The last term is called the interior penalty term and is supposed to mimic the continuity of the approximate solution at the interior edges. The penalty parameter σ is given by

$$\sigma|_\gamma = \sigma_\gamma = \alpha \frac{k_0 p_\gamma^2}{h_\gamma}, \quad \gamma \in \mathcal{F}_h^I,$$

where the constant α is chosen such that we have guaranteed the coercivity of the form $\mathcal{A}_h$, see [5, Chapter 2.6.3].

Definition 1. The function $u_h \in S_{hp}$ is called the *approximate solution* of (1) if

$$\mathcal{A}_h(u_h, v) = (f, v)_\Omega \quad \forall v \in S_{hp}. \tag{3}$$

This scheme is called the *symmetric interior penalty Galerkin* (SIPG) method.

The discrete problem (3) is equivalent to the system of linear algebraic equations

$$\boldsymbol{A}\boldsymbol{u} = \boldsymbol{f}, \tag{4}$$

where $\boldsymbol{A}$ is the matrix having the size n equal to dimension of S_{hp} and the entries of $\boldsymbol{A}$ are given by $\mathcal{A}_h(\phi_j, \phi_i)$, where $\{\phi_i,\ i = 1, \ldots, n\}$ is a basis of S_{hp}. If the size of $\boldsymbol{A}$ is large, the use of iterative solvers is advantageous. Very efficient are methods based on Krylov subspaces, among them the *conjugate gradient* (CG) method is very popular for symmetric problems. The rate of convergence of CG can be estimated by the condition number, cf. [10, Chapter 6.11]. In [2, Section 2.4] and [8, Section 2], the following estimate of the condition number of $\boldsymbol{A}$ from (4) was derived

$$\kappa(\boldsymbol{A}) \leq C \frac{k_1}{k_0} p^4 h^{-2} \tag{5}$$

for uniform grids having mesh step h and constant polynomial approximation degree p. We aim to use the domain decomposition to construct suitable preconditioner for the algebraic system (4), such that it decreases its condition number and can be performed in parallel setting.

3. Additive Schwarz preconditioner

We start with the partition of the computational domain Ω into smaller non-overlapping subdomains Ω_i such that $\overline{\Omega} = \bigcup_{i=1}^{N} \overline{\Omega_i}$. We assume that the subdomains Ω_i are the union of elements of $\mathcal{T}_h$. We employ two-level method, hence we define a coarse mesh $\mathcal{T}_H$ such that $\mathcal{K} \in \mathcal{T}_H$ lies in one subdomain Ω_i. We assume that the partitions are *nested*, i.e. the elements from a coarser mesh are the union of elements of finer mesh, these elements can be non-convex, see Figure 1 for two examples.

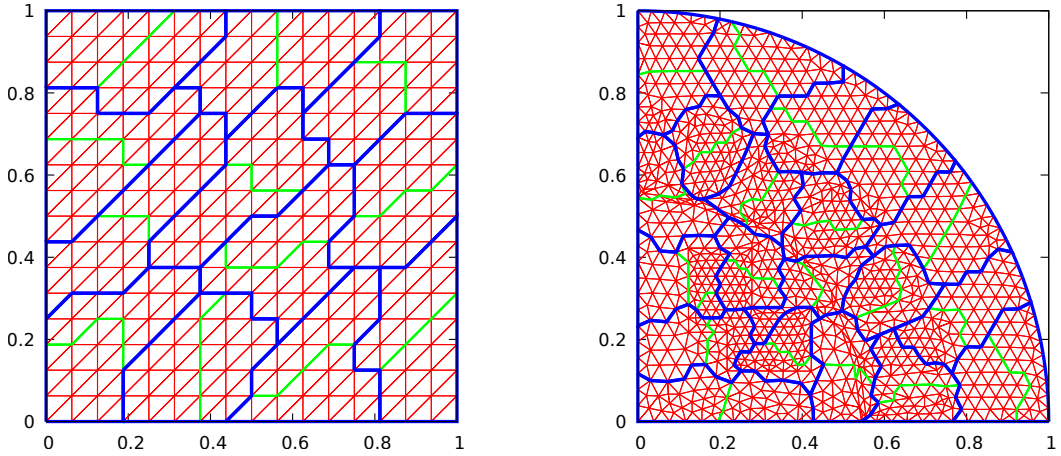

Figure 1: Examples of two fine meshes $\mathcal{T}_h$ (red, thin), subdomains Ω_i (blue, thick) and coarse meshes $\mathcal{T}_H$ (green, thin).

In the following, we introduce the local bilinear forms corresponding to the restriction of $\mathcal{A}_h$ on the subdomains Ω_i, $i = 1, \dots, N$ and the coarse (global) form corresponding to the restriction of $\mathcal{A}_h$ on the coarse mesh $\mathcal{T}_H$. The forms build the projection operators which are used for the definition of the two-level additive Schwarz preconditioner. For more details, we refer to, e.g, [1, 2].

3.1. Local forms

We consider a restriction of the space S_{hp} onto each sub-domain Ω_i, $i = 1, \dots, N$, i.e.

$$S_{hp}^i = \{u \in L^2(\Omega_i) \colon u|_K \in P_{p_K},\ K \in \mathcal{T}_h,\ K \subset \Omega_i\}, \qquad i = 1, \dots, N.$$

We define the *prolongation operators* $R_i^T \colon S_{hp}^i \to S_{hp}$ by

$$R_i^T u_i = \begin{cases} u_i & \text{on } \Omega_i, \\ 0 & \text{on } \Omega \setminus \Omega_i, \end{cases} \qquad u_i \in S_{hp}^i.$$

The corresponding (dual) *restriction operators* $R_i \colon S_{hp} \to S_{hp}^i$ are given by $R_i u = u|_{\Omega_i}$, $i = 1, \dots, N$. Then, we introduce the local bilinear forms $\mathcal{A}_{h,i}$

$$\mathcal{A}_{h,i}(u_i, v_i) := \mathcal{A}_h(R_i^T u_i, R_i^T v_i), \qquad u_i, v_i \in S_{hp}^i,\ i = 1, \dots N.$$

Using the prolongation operators, we can express functions from the space S_{hp} as a linear combination of functions from the local spaces.

3.2. Coarse form

In order to increase the speed of the transfer of the information among the subdomains, we formulate the problem on the coarse space S_{hp}^0 corresponding to the mesh $\mathcal{T}_H$. To deal with the inconsistency of the polynomial degree, we introduce the quantity $q_{\mathcal{K}}$, $\mathcal{K} \in \mathcal{T}_H$ defined by

$$0 \le q_{\mathcal{K}} \le \min_{K \subset \mathcal{K}} p_K.$$

The definition of the coarse space S_{hp}^0 is done similary as in the local space case, i.e.

$$S_{Hp}^0 := \{v \in L^2(\Omega) \colon v|_{\mathcal{K}} \in P_{q_{\mathcal{K}}}(\mathcal{K}),\ \mathcal{K} \in \mathcal{T}_H\}$$

Moreover, we define the prolongation operator $R_0^T \colon S_{Hp}^0 \to S_{hp}$ as a classical injection of the space S_{Hp}^0 in S_{hp}, and *restriction operator* $R_0 \colon S_{Hp} \to S_{hp}^0$ as its dual. Then, we set

$$\mathcal{A}_{h,0}(u_0, v_0) := \mathcal{A}_h(R_0^T u_0, R_0^T v_0), \qquad u_0, v_0 \in S_{Hp}^0.$$

3.3. Projection and preconditioned operators

Finally we define the local projection operators $\tilde{P}_i$, $i = 0, \dots, N$ which project the function onto the space S^i_{hp} using the local forms $\mathcal{A}_{h,i}$. Namely,

$$\tilde{P}_i \colon S_{hp} \to S^i_{hp} \qquad \mathcal{A}_{h,i}(\tilde{P}_i u, v_i) = \mathcal{A}_h(u, R_i^T v_i) \quad \forall v_i \in S^i_{hp}, \quad i = 0, \dots, N.$$

For the projector on the space S_{hp} we use the definition

$$P_i := R_i^T \tilde{P}_i \; : \; S_{hp} \to S_{hp}, \quad i = 0, \dots, N.$$

Finally, the *two-level additive Schwarz operator* reads

$$P_{ad} := \sum_{i=0}^{N} P_i. \tag{6}$$

3.4. Algebraic representation

We introduce the algebraic representation of the local bilinear forms $\mathcal{A}_{h,i}$ and the projector operators $\tilde{P}_i$ and P_i, $i = 0, \dots, N$ from previous paragraphs. Let $n = \dim(S_{hp})$, $n_i = \dim(S^i_{hp})$, $i = 1, \dots, N$, and $n_0 = \dim(S^0_{Hp})$. Let $\boldsymbol{R}_i^T \in \mathbb{R}^{n \times n_i}$, $i = 0, \dots, N$ be the matrices corresponding to the prolongation operators R_i^T with respect to the used basis of S_{hp}. Their construction is simple since $S^i_{hp} \subset S_{hp}$, $i = 1, \dots, N$ and $S^0_{Hp} \subset S_{hp}$. Then the algebraic representations of the restriction operators R_i, $i = 0, \dots, N$ are just the transposed matrices $\boldsymbol{R}_i = (\boldsymbol{R}_i^T)^T$.

Moreover, the algebraic representation of the local bilinear forms $\mathcal{A}_{h,i}$ are matrices $\boldsymbol{A}_i = \boldsymbol{R}_i \boldsymbol{A} \boldsymbol{R}_i^T \in \mathbb{R}^{n_i \times n_i}$, $i = 0, \dots, N$. Consequently, the matrix representation of projection operators $\tilde{P}_i$ and P_i reads

$$\tilde{\boldsymbol{P}}_i = \boldsymbol{A}_i^{-1} \boldsymbol{R}_i \boldsymbol{A} \quad \text{and} \quad \boldsymbol{P}_i = \boldsymbol{R}_i^T \boldsymbol{A}_i^{-1} \boldsymbol{R}_i \boldsymbol{A}, \qquad i = 0, \dots, N,$$

respectively. Finally, the matrix representation of the additive Schwarz operator is given by

$$\boldsymbol{P}_{ad} = \sum_{i=0}^{N} \boldsymbol{P}_i = \sum_{i=0}^{N} \boldsymbol{R}_i^T \boldsymbol{A}_i^{-1} \boldsymbol{R}_i \boldsymbol{A} =: \boldsymbol{M}_{ad}^{-1} \boldsymbol{A}. \tag{7}$$

Hence, the matrix $\boldsymbol{M}_{ad}^{-1}$ is a preconditioner of system (4) arising from DG discretization. Therefore, we replace (4) by the equivalent problem

$$\boldsymbol{M}_{ad}^{-1} \boldsymbol{A} \boldsymbol{u} = \boldsymbol{M}_{ad}^{-1} \boldsymbol{f}, \tag{8}$$

where the application of $\boldsymbol{M}_{ad}^{-1}$ exhibits a solution of small algebraic systems which can be done in a parallel way. For the solution of (8), we use standard Krylov iterative solver, namely the conjugate gradient (CG) method. More details on the solver can be found in [10, Chapter 6].

3.5. Analysis of the preconditioner

In this section, we present the upper bound of the condition number of the matrix $\boldsymbol{P}_{ad} = \boldsymbol{M}_{ad}^{-1}\boldsymbol{A}$ using the abstract technique from [11, Chapter 2], which is based on the following three assumptions.

Assumption 1 (Stable decomposition) There exists a constant $C_0 > 0$ such that $\forall u \in S_{hp}$ we have the decomposition $u = \sum_{i=0}^{N} R_i^T u_i$, with $u_0 \in S_{Hp}^0$, $u_i \in S_{hp}^i$, $i = 1, \ldots, N$, that satisfies $\sum_{i=0}^{N} \mathcal{A}_{h,i}(u_i, u_i) \leq C_0^2 \mathcal{A}_h(u, u)$.

Assumption 2 (Local stability) There exists a constant ω, $0 \leq \omega \leq 2$, such that

$$\mathcal{A}_h(R_i^T u_i, R_i^T u_i) \leq \omega \mathcal{A}_{h,i}(u_i, u_i) \quad \forall u_i \in S_{hp}^i,\ i = 1, \ldots, N,$$
$$\mathcal{A}_h(R_0^T u_0, R_0^T u_0) \leq \omega \mathcal{A}_{h,0}(u_0, u_0) \quad \forall u_0 \in S_{Hp}^0.$$

Assumption 3 (Strengthened Cauchy-Schwarz inequalities) There exist constants $0 \leq \epsilon_{ij} \leq 1$, $i, j = 1, \ldots, N$, such that

$$|\mathcal{A}_h(R_i^T u_i, R_j^T u_j)| \leq \epsilon_{ij} \mathcal{A}_h(R_i^T u_i, R_i^T u_i)^{\frac{1}{2}} \mathcal{A}_h(R_j^T u_j, R_j^T u_j)^{\frac{1}{2}}, \quad i, j = 1, \ldots N,$$

for all $u_i \in S_{hp}^i$, $u_j \in S_{hp}^j$. By $\rho(\boldsymbol{\varepsilon})$ we denote the spectral radius of $\boldsymbol{\varepsilon} = \{\epsilon_{ij}\}_{i,j=0}^{N}$

Using [11, Theorem 2.7], we have the following results.

Theorem 1. *Let Assumptions 1–3 be satisfied. Then the condition number of the two-level additive Schwarz operator can be bounded by*

$$\kappa(P_{ad}) \leq C_0^2\, \omega\, (\rho(\boldsymbol{\varepsilon}) + 1).$$

Verifying Assumptions 1–3 for the presented additive Schwarz formulation and using Theorem 1, cf. [2, 9], we get the bound

$$\kappa(\boldsymbol{P}_{ad}) \leq C\alpha \frac{p^2 H k_1}{q h k_0}, \tag{9}$$

where $\boldsymbol{P}_{ad}$ is given by (7).

4. Numerical study

The objective of this section is to numerically compute the bounds (5) and (9) and to demonstrate their accuracy. We focus on the experiments dealing with the condition number of the non-preconditioned systems and also the preconditioned systems. In the end, we show that the application of this approach can be used to solve non-linear problems.

All experiments were performed using the ADGFEM code [4] for the generation of the system matrices and then exported to MATLAB, where we used the function `eigs` to compute the approximations of the largest and smallest eigenvalues of $\boldsymbol{A}$ and $\boldsymbol{M}_{ad}^{-1}\boldsymbol{A}$. Then we set the condition number as the ratio of the largest and smallest eigenvalues. This approach is valid since we are dealing with symmetric positive definite matrices. We investigate the dependence of condition number $\kappa(\boldsymbol{A})$ and $\kappa(\boldsymbol{M}_{ad}^{-1}\boldsymbol{A})$ on the parameters h, H, p and the ratio of k_1/k_0 as we have seen.

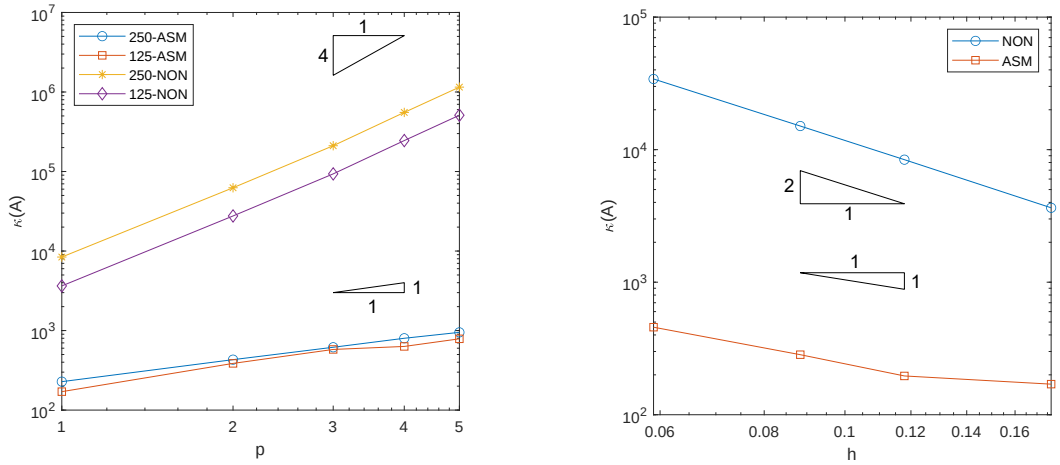

Figure 2: The dependence of $\kappa(\boldsymbol{A})$ (NON) and $\kappa(\boldsymbol{M}_{ad}^{-1}\boldsymbol{A})$ (ASM) on the polynomial degree p (left) and on the mesh size h using $p = 1$ (right).

It is important to say that similar numerical examples were performed in [8], but there we had not quite correct implementation of system matrix generation and the condition number was computed using the function `condest`, which computes different type of condition number.

4.1. Laplace equation

We consider the problem (1) with $\boldsymbol{K} = \boldsymbol{I}$, where $\boldsymbol{I}$ is the identity matrix and with $\Omega = (0, 1)^2$. The corresponding mesh is on the left of Figure 1. Since $k_0 = k_1 = 1$ the results (5) and (9) depend only on h, H and p. Similarly as in [2], we plot the dependence of $\kappa(\boldsymbol{A})$ in logarithmic scale to see the slope.

- First, we investigate the dependence of $\kappa(\boldsymbol{A})$ and $\kappa(\boldsymbol{M}_{ad}^{-1}\boldsymbol{A})$ on p for two uniform meshes having (approximately) 125 and 250 elements. We set $N = 12$, each Ω_i is one coarse element, and the coarse polynomial degree is set $q = p$.
- Moreover, we investigate the dependence of $\kappa(\boldsymbol{A})$ on h for $p = 1$, where we use meshes having 128, 288, 512 and 1152 mesh elements.

Figure 2, left shows that $\kappa(\boldsymbol{A})$ behaves as $O(p^4)$ and $\kappa(\boldsymbol{M}_{ad}^{-1}\boldsymbol{A})$ behaves as $O(p)$ which is in agreement with (5) and (9). Moreover, Figure 2, right shows that $\kappa(\boldsymbol{A})$ behaves as $O(h^2)$ and $O(h)$, which is again expected based on the result (5) and (9), respectively.

4.2. Symmetric linear elliptic equation

Furthermore, we deal with a linearization of the example from [7, Section 5.4]. This corresponds to a simulation of the magnetostatic field in the alternator. Due to symmetry, we consider only a quarter of the alternator. The domain Ω is divided into Ω_s (Stator), Ω_r (Rotor), and Ω_a (Air) (geometry can be seen in [8, Figure 3.6]).

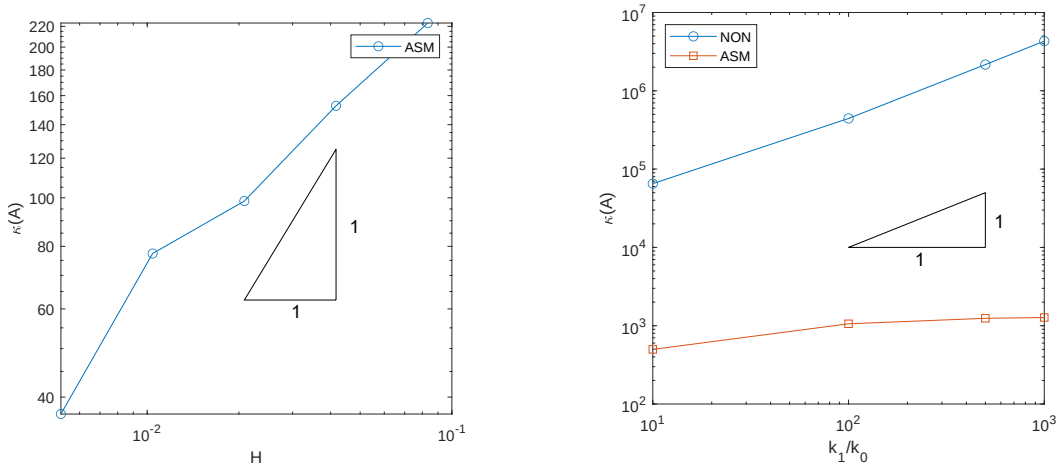

Figure 3: The dependence of the condition number of the preconditioned system (ASM) on the coarse mesh size (left) and the dependence of the condition number of the preconditioned and non-preconditioned system on the ration of k_1/k_0 for $p = 1$ and $p = 2$ (right).

The corresponding mesh can be seen on the right of Figure 1. The formulation in terms of the magnetic potential u reads:

$$-\mathrm{div}(\nu(x)\nabla u(x)) = f \quad \text{in } \Omega, \tag{10}$$

where ν is in the form $\nu(x) = \begin{cases} \frac{1}{\mu_0} & \text{for } x \in \Omega_a, \\ \frac{100}{\mu_0} & \text{for } x \in \Omega_s \cup \Omega_a, \end{cases}$ where $\mu_0 = 1.256 \cdot 10^{-6}$.

We use the same technique as described above to generate system matrices and compute the condition number. We focus on the following.

- We investigate the dependence of $\kappa(\boldsymbol{M}_{ad}^{-1}\boldsymbol{A})$ on the coarse mesh size H with $p = 1$ and $N = 12$ and the division of the subdomains Ω_i into 1,2,4,8 and 12 coarse elements.
- We investigate the dependence of $\kappa(\boldsymbol{A})$ and $\kappa(\boldsymbol{M}_{ad}^{-1}\boldsymbol{A})$ on the ratio of k_1/k_0 for $p = 1$.

Figure 3, left supports the theoretical result $\kappa(\boldsymbol{M}_{ad}^{-1}\boldsymbol{A}) = O(H)$, at least asymptotically. Figure 3, right gives that the dependency on the ratio of k_1/k_0 is also in agreement with the result (5) and (9), in which we see that $\kappa(\boldsymbol{M}_{ad}^{-1}\boldsymbol{A})$ and $\kappa(\boldsymbol{A})$ behaves as $O(k_1/k_0)$. We can see that we are getting slightly better result for the preconditioned system than we expected.

4.3. Symmetric nonlinear elliptic equation

Finally we present numerical result for the nonlinear variant of the alternator equation (10), namely

$$-\mathrm{div}(\nu(x, |\nabla u(x)|^2)\nabla u(x)) = f \quad \text{in } \Omega,$$

N	non-linear iter	linear iter	time on 1 processor	theoretical $\frac{time \times 2}{\#\Omega_i}$
4	81	7049	158 s	79 s
8	71	8073	158 s	39 s
16	68	8939	156 s	19 s
32	71	10493	194 s	12 s
64	70	11649	230 s	7 s

Table 1: Number of iterations and the computational time for increasing number of subdomains.

where the function ν is a strongly nonlinear function in $|\nabla u(x)|$, see [7, Section 5.4] for the explicit form of ν. The nonlinear problem is solved as a sequence of linear ones, namely

$$-\mathrm{div}(\nu(x, |\nabla u_{k-1}|)\nabla u_k) = f \quad \text{in } \Omega, \qquad k = 1, 2, \dots$$

where k is the index of nonlinear iterations. In every non-linear iteration we compute 100 linear iterations. As the linear solver we used conjugate gradient method with the two-level additive Schwarz preconditioner (7). The stopping criterion was the ratio of algebraic residual error estimator over the space residual error estimator, cf. [6].

We investigate the speed of convergence for increasing number of subdomains $N = \#\Omega_i$, each Ω_i is just one coarse element $\mathcal{K} \in \mathcal{T}_H$. The number of conjugate gradient iterations and computational time in seconds (using one processor) is shown in Table 1. Although the computations were performed using one processor, we present in the last column of Table 1 the theoretical computational time using an ideal parallelization, i.e., one processor for one subdomain (excluding overheads). We observe an almost optimal speed up of the computation.

5. Conclusion

We presented the outline of the theory used for the condition number bounds of the two-level additive Schwarz preconditioner for the solution of partial differential equations using DGM. The main part of our work was the numerical study done on a more complex example and also the application of the method for the non-linear problem. We have shown that the method has potential for non-linear problems and can be further investigated.

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