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GALERKIN-TYPE SOLUTION OF NON-STATIONARY AEROELASTIC STOCHASTIC PROBLEMS

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Abstract: The assessment of vibration characteristics in slender engineering structures, influenced by both deterministic harmonic and stochastic excitation, poses a challenging problem. Due to its complexity, transverse vibration of the structure (relative to the wind direction) is typically modelled using the single-degree-of-freedom van der Pol-type equation. Determining the response probability density function comprises solving the Fokker-Planck equation, a task that generally necessitates the use of approximate numerical methods. Some of these methods rely on Galerkin-type approximation employing orthogonal polynomial or exponential-polynomial basis functions. This contribution reviews available techniques for stationary and non-stationary cases and proposes some modifications while highlighting unresolved questions in the field.

Keywords: van der Pol equation, random vibration, stochastic differential equation, quasiperiodic response, Fokker-Planck equation, Galerkin method

MSC: 35R60, 37A50, 65C30, 65M60

1. Introduction

Exploring the nonlinear dynamic response on random excitation is an important research subject. There are many analytical, semi-analytical, and numerical methods available to obtain stationary probability density functions (PDF) or statistical moments, particularly focusing on systems influenced by Gaussian white noise. However, the non-stationary case remains the subject of intensive research.

The non-linear van der Pol type single-degree-of-freedom (SDOF) oscillator is often used to represent transverse wind-induced vibrations under additive excitation, including deterministic and random components. This particular type of an oscillator is known and used for the so called *lock-in* or *frequency entrainment* effect, where the response frequency, i.e., vibration frequency of the structure, does not follow the dominant frequency present in the excitation but locks onto the natural frequency

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of the system. This effect appears in a certain neighbourhood of the frequency of the stable limit cycle. Consequently, the oscillator produces very stable frequency output even with noisy harmonic input, provided the driving frequency remains within a certain proximity to the limit cycle frequency. Conversely, the response may attain various types of non-stationary response, including the cyclo-stationary or chaotic type when the driving frequency is far from the natural one.

The literature rarely addresses the van der Pol oscillator subjected to combined harmonic and random excitations. The stationary response case has been explored in [2], where the stochastic averaging method [5] and the equivalent linearization method are used in conjunction. The authors in [7] investigated a similar scenario, providing an explicit solution for the averaged equations in the resonant case. A more general yet stationary case has been outlined by the authors using the Galerkin method [6] to solve the nonlinear Fokker-Planck equation (FPE). The non-stationary case has been presented only recently, [11], where the probabilistic solution of the non-stationary responses is expressed as an exponential function of polynomial with time-variant coefficients and then the FPE is solved approximately.

This contribution reviews several approaches for determining both stationary and non-stationary response characteristics. For the stationary case, a method that refines the analytical solution available under exact resonance conditions is outlined, with a focus on the numerical integration procedure. In the non-stationary case, two approaches based on the Galerkin method are discussed: one utilizes a time-dependent linear combination of Hermite polynomials, while the other is based on exponential polynomials.

2. Mathematical model

Wind-induced vibration due to vortex shedding in slender engineering structures, such as bridge decks, towers, masts, high-rise buildings, or cables, is usually modelled using van der Pol equation. Its self-excitation due to the negative damping closely describes the state when the structure draws energy from the ambient flow. Mathematically,

$$\begin{aligned} \dot{u} &= v\,, \\ \dot{v} &= (\eta - \nu u^2)v - \omega_0^2 u + P\omega^2 \cos\omega t + h\xi(t)\,, \end{aligned} \tag{1}$$

where time differentiation is indicated by a dot above the symbol and the system parameters are:

- u, v – the displacement $[m]$ and velocity $[ms^{-1}]$;
- η, ν – the linear and quadratic damping $[s^{-1}, s^{-1}m^{-2}]$;
- ω_0, ω – the eigen-frequency of the linear SDOF system and frequency of the vortex shedding $[s^{-1}]$;

and the external excitation is described with: $f(t) = P\omega^2 \cos\omega t + h\xi(t)$, where:

- $P\omega^2$ – amplitude of the harmonic excitation $[ms^{-2}]$;
- $\xi(t)$ – the non-dimensional broadband Gaussian random process;
- h – multiplicative constant $[ms^{-2}]$.

In the deterministic case, there are four basic configurations that characterize the solution in terms of frequency content and system solvability:

(i) The resonant case, where the excitation frequency is equal to the natural frequency $\omega_0 = \omega$. In this case the response of the model is periodic and, with random additive excitation, there exists an explicit expression for the stationary probability density of the response amplitude and phase shift [7].

(ii) When the frequency of the harmonic part of the right-hand side is close to the model's natural frequency, a lock-in effect occurs. The amplitudes of the deterministic solution are constant, and the response in the presence of stationary random disturbance remains stationary, [2, 6]. The width of the lock-in interval depends on system parameters.

(iii) Just beyond the boundary of the lock-in interval, in the deterministic case, a series of frequencies ω_i emerge in the frequency content of the response in addition to the natural frequency ω_0. The new frequencies move away from the natural frequency ω_0, depending on the distance of the excitation frequency from the boundary of the lock-in interval, approximately following the relationship $\omega_i = \omega_0 \pm \beta_i \left(\omega - \gamma^+\right)^{d_i}$ where γ^+ is the upper boundary of the lock-in interval, and β_i, d_i are coefficients characteristic to the new frequencies. The presence of nearby frequencies in the response process results in the emergence of long-period beats at a frequency $|\omega_i - \omega_0|$, which give the response a quasiperiodic character. The analytic examination of this effect using the multiple scales method was recently published, [1].

This phenomenon causes ill conditioning of the behaviour of the van der Pol equation, where small errors in the excitation frequency lead to large changes in the nature of the solution. This effect is amplified in the presence of stochastic noise.

(iv) When the frequency of beats and the excitation frequency are comparable and/or the influence of self-excitation diminishes, the system's response is primarily characterized by the harmonic component of the excitation (forced vibrations). The response is periodic in the deterministic case and stationary in the stochastic case.

3. Stationary case

For weakly nonlinear systems subjected to weak excitations, the *stochastic averaging method* [9] is commonly employed. This method involves replacing fast variables with statistically equivalent stochastic processes to analyse variables evolving on a slower time-scale. The underlying assumption is that the response process can be uniformly approximated over a given time interval.

Using the Itô stochastic calculus, the response PDF of the original differential system Eq. (1) is governed by the *Fokker-Planck Equation*:

$$\frac{\partial p(\mathbf{x},t)}{\partial t} = -\sum_{j=1}^{N} \frac{\partial}{\partial x_j}\left(\kappa_j(\mathbf{x},t)p(\mathbf{x},t)\right) + \frac{1}{2}\sum_{j,k=1}^{N} \frac{\partial^2}{\partial x_j \partial x_k}(\kappa_{jk}(\mathbf{x},t)p(\mathbf{x},t)), \qquad (2)$$

where $\mathbf{x} = (x_1, x_2) = (u, v)$, $N = 2$. The drift coefficients $\kappa_j(\mathbf{x},t)$ correspond to the first moment of the derivative, while the diffusion coefficients $\kappa_{jk}(\mathbf{x},t)$ correspond to the second moment. In the case of a stationary response process, $p(\mathbf{x},t)=p(\mathbf{x})$ and the left-hand side of Eq. (1) vanishes. The resulting equation is referred to as the *reduced Fokker-Planck equation.*

In the stochastic average method, the expressions for the displacement and velocity $u(t), v(t)$ are written in trigonometric form:

$$u(t)=a_c\cos\omega t+a_s\sin\omega t,\qquad v(t)=-a_c\omega\sin\omega t+a_s\omega\cos\omega t, \tag{3a}$$

where partial amplitudes a_c, a_s comply with the additional condition

$$\dot{a}_c\cos\omega t+\dot{a}_s\sin\omega t=0. \tag{3b}$$

In the general case, $a_c(\tau), a_s(\tau)$ are functions of the slow time $\tau=\varepsilon t$, where ε is a small parameter, and may represent non-stationary processes. In the lock-in region (i.e., in cases (i) and (ii) in the previous section), the response process is stationary, and the partial amplitudes a_c and a_s can be assumed stationary.

Based on the approximation Eq. (3), the original stochastic system Eq. (1) can be transformed using the time-averaging operator into the averaged Itô system:

$$\mathsf{d}a_c=\frac{\pi}{\omega}\left[\ \ \eta a_c+2\Delta a_s-\frac{1}{4}\nu\cdot a_c(a_c^2+a_s^2)\right]\mathsf{d}t+\left(\frac{\pi}{\omega}\Phi_{\xi\xi}\right)^{\frac{1}{2}}\mathsf{d}B_c, \tag{4a}$$

$$\mathsf{d}a_s=\frac{\pi}{\omega}\left[-2\Delta a_c+\eta a_s-\frac{1}{4}\nu\cdot a_s(a_c^2+a_s^2)\right]\mathsf{d}t+\frac{\pi}{\omega}P\omega\ \mathsf{d}t+\left(\frac{\pi}{\omega}\Phi_{\xi\xi}\right)^{\frac{1}{2}}\mathsf{d}B_s. \tag{4b}$$

Here $\Phi_{\xi\xi}(\omega)$ is the spectral density of the process $\xi(t)$ at frequency ω, $B_{c,s}(t)$ stands for the Wiener process corresponding to input excitation $\xi(t)$ and $\Delta=(\omega_0^2-\omega^2)/(2\omega)$ is the frequency detuning.

The stationary PDF of a_c, a_s follows from the reduced FPE:

$$\begin{aligned}&\frac{\partial}{\partial a_c}\left(\left[\eta a_c+2\Delta a_s-\frac{1}{4}\nu\cdot a_c(a_c^2+a_s^2)\right]p\right)-\frac{1}{2\omega}\Phi_{\xi\xi}(\omega)\frac{\partial^2 p}{\partial a_c^2}\\&+\frac{\partial}{\partial a_s}\left(\left[\eta a_s-2\Delta a_c-\frac{1}{4}\nu\cdot a_s(a_c^2+a_s^2)+P\omega\right]p\right)-\frac{1}{2\omega}\Phi_{\xi\xi}(\omega)\frac{\partial^2 p}{\partial a_s^2}=0,\end{aligned} \tag{5}$$

with boundary conditions assuring vanishing $p(a_c,a_s)$ for $|a_c|+|a_s|\to\infty$. The differential system Eq. (5) admits a closed-form solution under zero detuning (see [6] and Eq. (7)). The existence of such a solution depends on the existence of a probability density potential, which occurs only when $\Delta=0$.

3.1. Galerkin method

For non-zero detuning, but with a stationary response within the lock-in frequency range, a solution to the reduced, stationary Fokker-Planck equation for partial amplitudes can be sought in the form of a Galerkin approximation:

$$p(a_c,a_s)=p_0(a_c,a_s)\sum_{k,l=0}^{M,k}q_{kl}a_c^{k-l}a_s^l, \tag{6}$$

where M is the upper limit of stochastic moments included into the analysis. In Eq. (6), $p_0(a_c, a_s)$ represents the weight function and is selected in the form of the solution to the stationary FPE when $\Delta = 0$:

$$p_0(a_c, a_s) \;=\; C \cdot \exp\left(\frac{\eta}{2S}\left[\left(a_s + \frac{P\omega}{\eta}\right)^2 + a_c^2 - \frac{\nu}{8\eta}\left(a_c^2 + a_s^2\right)^2\right]\right), \tag{7}$$

where $S = \Phi_{\xi\xi}(\omega)/(2\omega)$ and the normalizing factor C is to be determined numerically.

When a harmonic component is present in the excitation, $P \neq 0$, the unsymmetric weight function fails to ensure the orthogonality of Hermite polynomials. Thus, for simplicity, standard polynomial basis and test functions are used, which, by virtue of the weight function p_0, satisfy the zero boundary conditions at infinity. For individual values of k, the terms in the sum in Eq. (6) represent the k-th stochastic moment and act as correction terms to the analytic solution for $\Delta = 0$.

3.2. Numerical integration

Integration in the Galerkin method takes place over the entire space $\mathbb{R}^2$, and the coefficients $q_{k,l}$ for $k, l = 0, \ldots, M; k + l \leq M$ are determined from the linear system obtained by substituting Eq. (6) into the FPE (5), followed by several steps of integration by parts and the application of homogeneous boundary conditions, where the specific forms of the partial derivatives of $p_0(a_c, a_s)$ were also taken into account:

$$\begin{aligned} 0 = \iint_{\mathbb{R}\times\mathbb{R}} \Bigg\{ & \left[a_c^{\sigma-2} a_s^{s-2}\left(\sigma(\sigma-1)a_s^2 - s(s-1)a_c^2\right) S + \Delta a_c a_s \left(\sigma a_s^2 - s a_c^2\right)\right] \sum_{k,l=0}^{M,k} q_{kl} a_c^{k-l} a_s^l \\ & - S\left[s \frac{\mathrm{d}}{\mathrm{d}a_s}\left(a_c^{\sigma} a_s^{s-1} \sum_{k,l=0}^{M,k} q_{kl} a_c^{k-l} a_s^l\right) - \sigma \frac{\mathrm{d}}{\mathrm{d}a_c}\left(a_c^{\sigma-1} a_s^{s} \sum_{k,l=0}^{M,k} q_{kl} a_c^{k-l} a_s^l\right)\right]\Bigg\} p_0 \,\mathrm{d}a_c \mathrm{d}a_s. \end{aligned} \tag{8}$$

where $\sigma = (r - s)$, $p_0 = p_0(a_c, a_s)$.

Basis functions in the form of polynomials have poor numerical properties because the corresponding Gram matrix is usually ill-conditioned. However, for low values of M and with careful handling of the numerical integration, constructing the system matrix is feasible, especially when the following considerations are taken into account: Due to symmetry properties, terms involving odd powers of a_c do not contribute to the total value of the integral and should be skipped during integration to avoid numerical cancellation. Additionally, the integral should be computed over the half-plane $a_c > 0$, with the result doubled. It is also convenient to transform the variables into polar coordinates centred at the maximum value of the weight function. In this way, the decrease of the integrand in the radial direction becomes roughly uniform.

The numerical integration in Eq. (6) involves a large number of terms of the form $z_{kl} = p_0(a_c, a_s)a_c^k a_s^l$; each of them approximately bounded from above on a logarith-

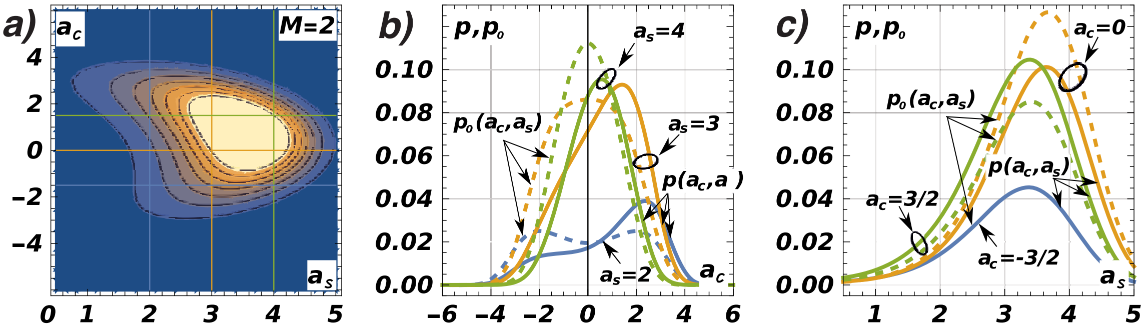

Figure 1: The Galerkin approximation of the stationary PDF for $M = 2$ and detuning value $\Delta = 0.10$. a) Contour plot of the PDF. b) "Vertical" sections of the PDF; $a_s = \{2,\ 3,\ 4\}$. c) "Horizontal" sections of the PDF; $a_c = \{-3/2, 0, 3/2\}$. In plots b,c: dashed is analytical solution $p_0(a_c, a_s)$, solid is Galerkin solution $p(a_c, a_s)$.

mic scale by the following estimate

$$\begin{aligned} \log|z_{kl}| \leq& \frac{1}{2S}\left(\eta\varrho^2 - \frac{\nu}{8}\left(\frac{P\omega}{\eta} - \varrho\right)^4\right) + l\log\left(\varrho - \frac{P\omega}{\eta}\right) + k\log(\varrho); \\ a_c =& \varrho\cos\varphi, \quad a_c = \varrho\sin\varphi - \frac{P\omega}{\eta}. \end{aligned} \tag{9}$$

The estimate Eq. (9) is useful for determining the required integration radius ϱ and identifying the terms that contribute to the total value of the integral.

3.3. Numerical example

The PDF of the stochastic van der Pol oscillator response with respect to partial amplitudes a_c, a_s is shown for $M = 2$ in Figure 1. The value of detuning $\delta = 0.10$ still represents the lock-in response. The contour plot of the estimated cross-PDF $p(a_c, a_s)$ is shown on the left. Plot b) depicts the sections of the PDF for fixed values $a_s = \{2,\ 3,\ 4\}$ and plot c) show sections for $a_c = \{-3/2,\ 0,\ 3/2\}$. The sections and the corresponding colors are indicated as horizontal/vertical lines in the left-hand plots. The dashed curves show the basic analytical solution which is valid for the case $\delta = 0$, i.e., no detuning is assumed. The estimates including the $M = 2$ Galerkin approximations are shown in solid.

4. Non-stationary response case

When studying the non-stationary case, the dependence on the original time coordinate must be retained. The FPE reflecting the original stochastic problem Eq. (1) in the original coordinates reads:

$$\frac{\partial p(\mathbf{x}, t)}{\partial t} = -\sum_{j=1}^{2}\frac{\partial}{\partial x_j}\left(\kappa_j(\mathbf{x}, t)p(\mathbf{x}, t)\right) + \frac{1}{2}\sum_{j,k=1}^{2}\frac{\partial^2}{\partial x_j \partial x_k}(\kappa_{jk}(\mathbf{x}, t)p(\mathbf{x}, t)), \tag{10}$$

where $\mathbf{x} = (u, v)$; $x_1 = u$, $x_2 = v$. The input random process $\xi(t)$ is considered stationary and ergodic and the drift and diffusion coefficients can be written in a form:

$$\kappa_j(\mathbf{x}_t, t) = f_j(\mathbf{x}_t, t)\,, \quad \kappa_{jk}(\mathbf{x}_t, t) = \sum_{r=1}^{2} g_{jr}(\mathbf{x}_t, t) \int_{-\infty}^{\infty} g_{kr}(\mathbf{x}_{t+\tau}, t+\tau) R(\tau) \mathsf{d}\tau, \quad j, k = 1, 2, \tag{11}$$

where $R(\tau)$ is the auto-correlation function of $\xi(t)$.

Assuming that the detuning $\Delta \sim \varepsilon$ and the terms $(\eta - \nu u^2)\dot{u}$ and $P\omega^2$ are of a small order ε, and $h\xi(t)$ is of order $\varepsilon^{1/2}$. In such a case the FPE can be constructed for the SDE Eq. (1). It holds obviously:

$$\begin{aligned}
&\kappa_1 = v\,, && \kappa_2 = (\eta - \nu u^2)v - \omega_0^2 u - P\omega^2 \cos\omega t,\\
&g_{11} = g_{12} = g_{21} = 0\,, && g_{22} = h,\\
&\kappa_{11} = \kappa_{12} = \kappa_{21} = 0\,, && \kappa_{22} = g_{22} \int_{-\infty}^{\infty} g_{22} R_{vv}(\tau) \mathsf{d}\tau = h^2 \sigma_{\xi\xi}^2 = h^2\, S,
\end{aligned} \tag{12}$$

where S is the variance of the process $\xi(t)$. Take a note that $\kappa_{22} = h^2 S$ is valid independently from a particular shape of the input process spectral density and formally it corresponds to the special case of ξ, which is the white noise (δ correlated), provided the excitation is a non-modulated additive stationary ergodic process. Anyway, the FPE can be readily written out as follows:

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial u}(v\,p) - \frac{\partial}{\partial v}\Big(\big((\eta - \nu u^2)v - \omega_0^2 u - P\omega^2 \cos\omega t\big)\, p\Big) + \frac{1}{2} h^2 S \frac{\partial^2 p}{\partial v^2}, \tag{13}$$

together with initial and boundary conditions:

$$\lim_{u,v \to \pm\infty} p(u, v, t) = 0, \qquad p(u, v, 0) = \delta(u, v). \tag{14}$$

Near the boundary of the lock-in interval, the solution exhibits a quasi-periodic nature, which can be identified using a Galerkin-series-based solution in a form:

$$p(u, v, t) = p_0(u, v) \sum_{k=0}^{M} \sum_{l=0}^{k} q_{kl}(u, v, t). \tag{15}$$

The series Eq. (15) represents a weak solution to the FPE in the probabilistic sense. Choices of the weight function $p_0(u, v)$ and an approximation scheme used for terms q_{kl} classify the available methods.

4.1. Galerkin solution based on Hermite polynomials

The challenges associated with numerical integration, discussed in the preceding section, have motivated the use of Hermite polynomials as basis functions which approximate the residuum between the weight function in the Galerkin method and the solution of the FPE. However, the weight function must be adjusted to maintain the orthogonality property of the Hermite polynomials.

The elements q_{kl} are formulated as follows:

$$q_{kl}(u,v,t) = q_{kl}(t) L_{k-l}(\alpha u) L_l(\beta v), \qquad \alpha^2 = \frac{\eta\omega_0^2}{h^2 S}, \quad \beta^2 = \frac{\eta}{h^2 S}, \tag{16}$$

where $L_k(x)$ are Hermite polynomials.

The weight function $p_0(u,v)$ is adopted in a form of the Boltzmann's solution to a related problem without damping and external excitation, [3]. In particular:

$$p_0(u,v) = C \exp\left(-\frac{2\eta}{h^2 S} H(u,v)\right), \tag{17}$$

where C is the dimensionless normalizing constant, which can be put for now $C = 1$. $H(u,v)$ represents the Hamiltonian function of the basic system:

$$H(u,v) = \frac{1}{2}\omega_0^2 u^2 + \frac{1}{2} v^2, \tag{18}$$

which implicates $p_0(u,v) = p_u(u) p_v(v)$, so that u, v are stochastically independent Gaussian processes on a level of the zero-th approximation.

The unknown functions $q_{kl}(u,v,t)$ in Eq. (16) are determined using the generalized method of stochastic moments [8]. The expression from Eq. (15) is substituted into Eq. (13), and both sides are multiplied by the test functions $\Phi_{rs}(u,v)$, which has the same formal expression as Eq. (16):

$$\Phi_{rs}(u,v) = L_{r-s}(\alpha u) L_s(\beta v), \quad r = 0,\dots,M;\ s = 0,\dots,r. \tag{19}$$

Subsequently, applying the expectation operator (which, in fact, involves integration over $\mathbb{R}^2$) to all permutations of the subscripts r and s establishes a sufficient number of ordinary differential equations for the unknown functions $q_{kl}(u,v,t)$.

Employing Hermite polynomials reduces computational cost and associated numerical errors. However, empirical evidence suggests that the convergence is relatively slow and, moreover, these basis functions do not guarantee the non-negativity of the computed PDF estimates, which can pose a substantial problem.

4.2. Exponential-polynomial-closure method

The issue of negative PDF estimates does not arise when using the exponential-polynomial-closure method (EPC), [4]. In the original stationary setting, it assumes the sought PDF of an approximate solution in the form of an exponential polynomial:

$$\tilde{p}(u,v;\mathbf{c}) = C \exp\left(Q_n(u,v;\mathbf{c})\right). \tag{20}$$

Here, $\mathbf{c}$ is the unknown parameter vector, and $Q_n(u,v;\mathbf{c})$ is a polynomial function. The algebraic system for unknown parameters $\mathbf{c}$ results from the Galerkin approximation with respect to basis functions $h_k(u,v) = u^r v^s f_N(u,v)$, where $k = r + s$ and f_N is the PDF solution of the linearised Eq. (1) assuming the Gaussian response.

Multiple variants of the EPC method have been proposed for different settings of the stationary PDF solutions of nonlinear stochastic oscillators. Modifications for the non-linear, non-stationary case have only recently emerged, implicitly allowing for non-Gaussian excitation, [10]. The solution is assumed in an evolutionary form:

$$\tilde{p}(u, v, t; \mathbf{c}) = C \exp\left(Q_n(u, v, t; \mathbf{c})\right), \qquad Q_n(u, v, t; \mathbf{c}) = \sum_{i=1}^{n} \sum_{j=1}^{i} c_{ij}(t) u^{i-j} v^j. \tag{21}$$

Denoting by $\Delta(u, v, t; \mathbf{c})$ the residuum obtained by substitution Eq. (21) into the FPE (13), a set of ODEs for unknown parameters $\mathbf{c}(t) = \{c_{ij}(t)\}$ result from

$$\iint_{\mathbb{R}\times\mathbb{R}} \Delta(u, v, t; \mathbf{c}) h_k(u, v) \mathsf{d}u \mathsf{d}v = 0, \quad k = 1 \dots M, \tag{22}$$

where M indicates number of stochastic moments included into the solution.

5. Conclusions

The solution to the stochastic van der Pol equation is generally non-stationary and non-Gaussian, making its characterization a significant challenge. This paper reviews several approaches for determining both stationary and non-stationary response characteristics.

For the stationary case, the presented method is based the stochastic averaging method. The PDF for non-resonant configurations is approximated using the Galerkin method, where improper integrals are evaluated numerically. For this case, some new remarks regarding numerical integration were presented. However, due to the limitations of numerical integration for higher-degree polynomials, alternative basis functions are essential for exploiting higher stochastic moments.

Determining the non-stationary response relies on the Galerkin method, which must account for the time-dependence of the probability density. The paper explores two implementations. One approach utilizes a Boltzmann-type solution as the weight function and Hermite polynomials as basis and test functions in the Galerkin approximations. However, Hermite polynomials do not guarantee the non-negativity of the estimated PDF. As an alternative, the exponential-polynomial closure method is reviewed. It employs a Gaussian-closure solution of the linearised system as the weight function and exponential polynomials as basis and test functions. Based on existing literature, the EPC method is expected to outperform the previous approach. A comparative analysis of these implementations will be addressed in future work.

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