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NON-STOCHASTIC UNCERTAINTY QUANTIFICATION OF A MULTI-MODEL RESPONSE

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Abstract: The focus is put on the application of fuzzy sets and Dempster-Shafer theory in assessing the nature and extent of uncertainty in the response of M models that model the same phenomenon and depend on fuzzy input data. Dempster-Shafer theory uses a weighted family of fixed sets called the focal elements to evaluate the relationship between an arbitrarily chosen set and the focal elements. It is proposed to create at least M weighted focal elements on the basis of 1) the responses to fuzzy inputs to the models, and 2) the weights associated with the models. Four variants of this approach are illustrated by academic examples.

Keywords: fuzzy sets, evidence theory, uncertainty

MSC: 03E75, 90C90

1. Introduction

In this contribution, the following situation is addressed: Let one phenomenon be modeled by several models whose input parameters are uncertain. How can the combined responses of the individual models be assessed and their trustworthiness evaluated? In other words, what sort of uncertainty quantification can be applied to the synergy of responses that originates from various models?

An uncertainty analysis applied to one model with uncertain inputs is quite common. Although the above multi-model situation is not frequent, it is not exceptional. Take, for instance, 1D models of elastic beams. One can choose the Euler-Bernoulli beam model, the Timoshenko(-Ehrenfest) model, or the less known nonlinear Gao beam model [6, 8], see also [9]. The 1D models can always be confronted with 3D models or, under special circumstances, with 2D models.

A large variety of models with uncertain input data offers the modeling of a long-term behavior of concrete. They include a number of internationally recognized models, national codes, and models proposed in academia, see [3].

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2. Elements of fuzzy set theory and evidence theory

Let us recall the three key concepts of fuzzy sets and their applications, namely the membership function μ_A of a fuzzy interval A, the α-cut A^α of a fuzzy set A, and Zadeh's extension principle.

2.1. Fuzzy sets, membership functions, α-cuts

Let the membership function μ_A be a continuous and concave function that maps a closed interval $A = [a, b]$ onto the interval $[0, 1]$. For computational purposes, let us limit ourselves to trapezoidal membership functions, i.e., piecewise linear functions identifiable with ordered 4-tuples $(a, c_1, c_2, b) \in \mathbb{R}^4$, where $\mu_A(a) = 0 = \mu_A(b)$, $\mu_A(c_1) = 1 = \mu_A(c_2)$, and $\mathbb{R}$ stands for the set of real numbers. A special, i.e., triangular case is obtained if $c_1 = c_2$.

The subsets of A defined through

$$A^\alpha = \{x \in A|\ \mu_A(x) \geq \alpha\}, \tag{1}$$

where $\alpha \in [0, 1]$, are called the α-cuts of A.

Remark: The abovementioned concept of membership functions is simple and restrictive, but it is tailored to our future computational needs. Another advantage lies in the fact that the existence of extremes is guaranteed, see (5) and (6), and that we can replace suprema and infima by maxima and minima in the theory of fuzzy sets. Nevertheless, a more general concept of fuzzy sets is common, see [5, 13], for example.

Fuzzy intervals can easily be generalized to fuzzy n-dimensional rectangular parallelepiped $A = A_1 \times A_2 \times \cdots \times A_n \subset \mathbb{R}^n$ where each interval A_i is associated with a membership function μ_{A_i} and the fuzzy variables are mutually independent. Then for each $x = (x_1, x_2, \ldots, x_n) \in A$, we can define

$$\mu_A(x) = \min\{\mu_{A_1}(x_1), \mu_{A_2}(x_2), \ldots, \mu_{A_n}(x_n)\}. \tag{2}$$

We also observe that

$$\forall \alpha \in [0, 1] \quad A^\alpha = A_1^\alpha \times A_2^\alpha \times \cdots \times A_n^\alpha. \tag{3}$$

Let g be a continuous function defined on a fuzzy set A (either $A \subset \mathbb{R}$ or a parallelepiped $A \subset \mathbb{R}^n$) and mapping A to a range $R_{g,A}$. Zadeh's extension principle defines the way how to transfer the membership degree from $x \in A$ to $g(x) \in R_{g,A}$. In detail [13],

$$\forall y \in R_{g,A} \quad \mu_{R_{g,A}}(y) = \max_{\{x \in A|\ g(x) = y\}} \mu_A(x). \tag{4}$$

The original definition (4) is not computation-friendly. This is why we will use an equivalent approach based on the fact that if the α-cuts $R_{g,A}^\alpha$ are known for all $\alpha \in [0, 1]$ and if $y \in R_{g,A}$, then

$$\mu_{R_{g,A}}(y) = \max\{\alpha \in [0, 1]|\ y \in R_{g,A}^\alpha\}. \tag{5}$$

It is not difficult to infer, see [10] or elsewhere, that

$$\forall \alpha \in [0,1] \quad R^{\alpha}_{g,A} = \left[\min_{x\in A^{\alpha}} g(x), \max_{x\in A^{\alpha}} g(x) \right]. \tag{6}$$

In other words, to obtain $R^{\alpha}_{g,A}$, we have to solve worst-case and best-case scenario problems (6).

2.2. Evidence theory, focal elements, Belief, Plausibility

The origin of the Dempster-Shafer theory of evidence [4, 11, 1, 12] can be traced back to considerations about lower and upper bounds of probabilities. In our approach, we interpret the weights forming the basic probability assignment as the amounts of trustworthiness assigned to fixed significant sets called focal elements, see the next paragraphs.

To this end, we assume that a set $\mathcal{S}$ of chosen intervals $I_1, I_2, \ldots, I_s$ is given together with the weight map $w\colon I \mapsto (0,1]$ where $I \in \mathcal{S}$ and $\sum_{i=1}^{s} w(I_i) = 1$. In the evidence theory, the intervals and the map are called the focal elements and the basic probability assignment, respectively.

Two values can be associated with an arbitrary subset $B \subset \mathbb{R}$, namely *Belief* and *Plausibility*

$$Bel(B) = \sum_{\{I\in\mathcal{S}\colon I\subseteq B\}} w(I) \quad \text{and} \quad Pla(B) = \sum_{\{I\in\mathcal{S}\colon I\cap B\neq\emptyset\}} w(I). \tag{7}$$

We observe that $Bel(B)$ collects the weights of those focal elements that are fully covered by B. That is, if these focal elements are outputs of some weighted models, then B fully represents all of these outputs. In contrast, $Pla(B)$ is less strict as it allows for both full (subset) and partial (nonempty intersection) representation.

3. Uncertainty quantification in multi-modeling

The background idea is not new. It associates α-cuts of a fuzzy set with focal elements [2]. A rather straightforward modification leads to an application to responses of several models. The method will be explained and illustrated on a particular example.

Let us consider $M = 3$ models represented by the following respective functions

$$m_1(p) = 7.3 + 0.02 p_3 (p_1 p_2)^{(p_3+p_4)}, \qquad m_2(p) = 7.3 + 0.02 p_2 (p_1 + p_3),$$
$$m_3(p) = 6 + 0.4 \frac{p_2 p_3 p_4}{p_1},$$

where $N_p = 4$ parameters form the vector $p = (p_1, p_2, p_3, p_4)$. If $\widehat{p} = (1.2, 2.1, 1.5, 1.2)$, then the response of all three models is roughly equal to 7.5 as is also indicated in Figure 1, the details of which will be given later.

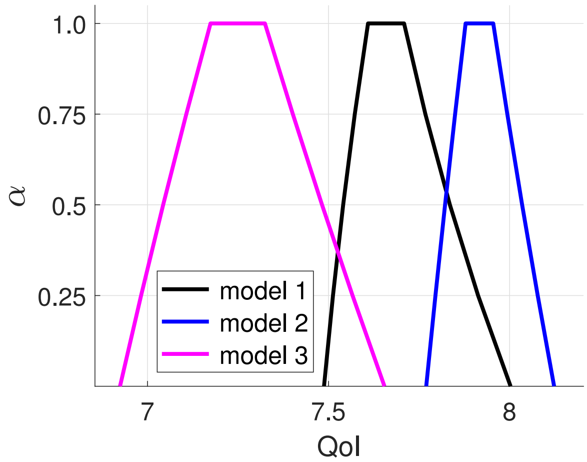

Figure 1: Membership functions of fuzzy responses.

Let the input parameters p_i, $i = 1, \ldots, N_p$, belong to intervals $A_i \equiv A_i^0$ provided with membership functions μ_i. The product of the intervals forms the set $A = A_1 \times \cdots \times A_{N_p}$.

Next, let each model be associated with a positive weight $w_i \in \mathbb{R}$ such that $\sum_{j=1}^{M} w_j = 1$.

In the following numerical examples, we use these membership functions

$$\mu_1 = \widehat{p_1}(0.95, 0.99, 1.01, 1.05), \qquad \mu_2 = \widehat{p_2}(0.9, 0.98, 1.02, 1.1),$$
$$\mu_3 = \widehat{p_3}(0.92, 0.97, 1.01, 1.08), \qquad \mu_4 = \widehat{p_4}(0.93, 0.99, 1.01, 1.05);$$

and the basic probability assignment defined as $w_1 = 0.25$, $w_2 = 0.4$, and $w_3 = 0.35$.

The partial derivatives of m_i allow us to conclude that the functions m_i are monotone in each p_j on the supports A_i of the membership functions. As a consequence, the extremes of m_i are attained at the ends of the interval A_i^α, thus solving (6) for various values of α is easy. Based on (6) with $\alpha = \alpha_\ell = \ell/N_\alpha$, $\ell = 0, 1, \ldots, N_\alpha$, $N_\alpha = 4$, the approximate piecewise linear membership functions of the ranges of the models' responses are depicted in Figure 1. The value of the quantity of interest (QoI) is simply the scalar response of the models to the input data $p \in A$.

We are ready to introduce **Algorithm 1**:

Step 1: *Fix $\alpha \in [0, 1]$ and infer the α-cut A^α by using (3) and the α-cuts A_i^α, $i = 1, \ldots, N_p$.*
Step 2: *By setting $g = m_j$ and using (6), calculate $R^\alpha_{m_j,A}$, $j = 1, \ldots, M$.*
Step 3: *Interpret the intervals $R^\alpha_{m_j,A}$, $j = 1, \ldots, M$ as focal elements with the respective weights w_j.*
Step 4: *Choose an interval $B \subset \mathbb{R}$ and calculate $Bel(B)$ and $Pla(B)$ by using (7) where $\mathcal{S} = \{R^\alpha_{m_j,A}\}_{j=1}^{M}$.*
Step 5: *Repeat Step 4 several times with the aim to increase Bel and Pla and to identify a set B that satisfactorily represents the joined responses of the models on the level α.*

The algorithm needs some comments. First, the goal of Step 2 can be quite challenging if the models (unlike our case) are not trivial. It may happen, for instance, that p_i are parameters of a problem driven by differential equations whose solution is then post-processed to obtain a value of $m_i(p)$, a quantity of interest. As a consequence, the minimization and maximization in (6) can be a difficult task.

Second, obtaining the weights w_i is a delicate matter. Although measurement-based approaches can be available, see [7] aiming at stochastic uncertainty, expert opinion can often be a substantial, if not sole, source of information.

Third, the goal of Step 4 and Step 5 is to find an interval B that best characterizes the ensemble of output intervals $R^{\alpha}_{m_j,A}$. It commonly happens that there is no such "best" interval available. By taking a sufficiently large and appropriately positioned interval B, we can obtain $Bel(B) = 1 = Pla(B)$. The interval, however, might be so large that its practical value as a representative of key models' responses is questionable. Although it shows the total extent of uncertain responses, it does not indicate the subsets where the responses overlap, that is, the responses of at least some models are not too distinct from each other. To identify such intervals, shorter intervals B must also be tested by the focal elements. Again, the results can prevent an unequivocal conclusion. Take, for instance, $Bel(B_1) < Bel(B_2)$ and $Pla(B_1) > Pla(B_2)$ for some two intervals B_1 and B_2 of the same length.

If the number of the output intervals $R^{\alpha}_{m_j,A}$ (i.e., output focal elements) is small, then the analysis of their intersections and unions can lead to the sets maximizing *Belief* and *Plausibility*. Such analysis is more and more challenging if the number of output focal elements increases. *Bel* and *Pla* values calculated for a family of intervals is then an option that offers both a general view and sufficiently accurate information on the synergy of joint responses. This approach will be in the focus of the next paragraphs.

We define intervals $B^d_{s,k} = (a + ks, a + ks + d)$ of the length $d > 0$. The position of $B^d_{s,k}$ is controlled by the fixed parameters $a \in \mathbb{R}$ and $s \in \mathbb{R}$ as well as by the parameter $k = 0, 1, \ldots, K$. The intervals $B^d_{s,k}$ play the role of B in Algorithm 1. Some results are depicted in Figure 2 where the points $[a + ks, Y]$ represent the values $Y = Bel(B^d_{s,k})$ and $Y = Pla(B^d_{s,k})$. The parameters α and a are fixed to 0.5 and 6.8, respectively.

In the left graph, we observe that $k = 15$ and $k = 19, 20, 21$ indicate the intervals that are worth attention. Although $Pla([7.45, 7.825]) = 1$, $Bel([7.45, 7.825]) = 0$ might suggests that the intervals with nonzero *Belief* could be a better representation of the combined responses since their *Bel* and *Pla* values are more balanced. Similar ambiguity shows the right graph. The analyst can choose either the maximum of *Pla* with a rather low *Bel* value or the maximum of *Bel* accompanied by a decreased *Pla* value. The interval $B^d_{s,15} = [7.5, 8.05]$ shows a balanced assessment in both respects. Naturally, the use of longer intervals ($d = 0.55$) increases the *Bel* value and increases the number of positions where *Pla* is equal to one.

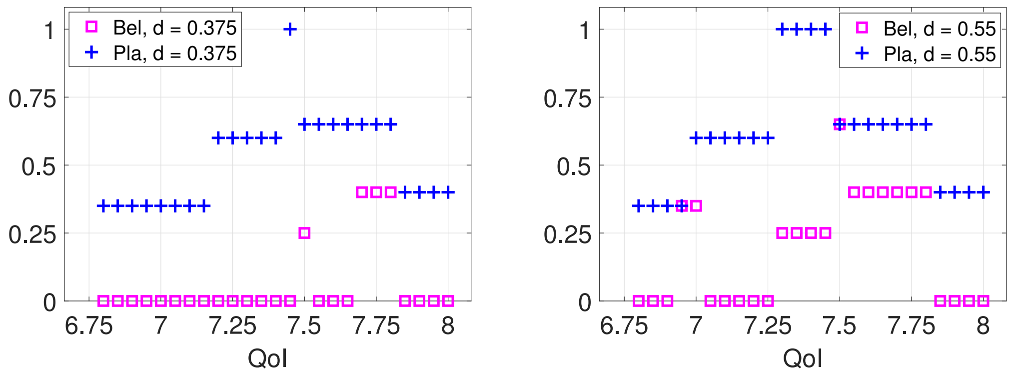

Figure 2: Algorithm 1. $Bel(B^d_{s,k})$ and $Pla(B^d_{s,k})$ for $s = 0.05$, $d = 0.375$ (left) and $s = 0.05$, $d = 0.55$ (right).

3.1. Modifications of Algorithm 1

The standard definition (7) shows a shortcoming that becomes more visible especially in our application where we wish to assess the extent of joint responses of the models. In (7), there is no difference between a very short intersection $I \cap B$ and a full set intersection; both cases are evaluated by the full weight $w(I)$.

To take into account the relative extent of intersection, let us redefine Pla in (7) as Pla^{new}

$$Bel(B) = \sum_{\{I \in \mathcal{S}:\ I \subseteq B\}} w(I) \quad \text{and} \quad Pla^{\mathrm{new}}(B) = \sum_{\{I \in \mathcal{S}:\ I \cap B \neq \emptyset\}} w(I) \frac{\mathrm{meas}_1(I \cap B)}{\mathrm{meas}_1 I} \qquad (8)$$

where meas_1 stands for the one-dimensional Lebesgue measure, which turns into the length of intervals in our calculations.

Algorithm 2 coincides with Algorithm 1 except for
Step 4: *Choose an interval* $B \subset \mathbb{R}$ *and calculate* $Bel(B)$ *and* $Pla^{\mathrm{new}}(B)$ *by using* (8) *where* $\mathcal{S} = \{R^{\alpha}_{m_j,A}\}_{j=1}^{M}$.

We observe in Figure 3 that if $d = 0.55$, then the interval $B^s_{d,15} = [7.5, 8.05]$ is the best representation of the joint model response on the uncertainty level $\alpha = 0.5$. For $d = 0.375$, the analyst would see the interval $[7.7, 8.075]$ as the best representative though its Pla^{new} does not reach the maximum. However, any increase in Pla^{new} is paid for by the zero Bel value.

Both algorithms focus on uncertainty quantification in model responses restricted to a fixed input uncertainty level, that is, $\alpha = 0.5$ in our examples. By taking into account all the α-cuts of the fuzzy inputs and by modifying the standard transformation [2] of one membership function to a set of focal elements, we arrive at an extended set of focal elements with an associated basic probability assignment.

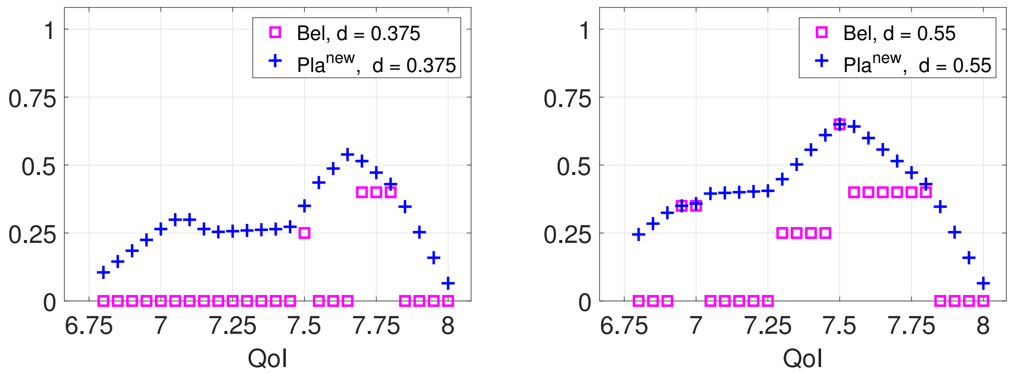

Figure 3: Algorithm 2. $Bel(B^d_{s,k})$ and $Pla^{\text{new}}(B^d_{s,k})$ for $s = 0.05$, $d = 0.375$ (left) and $s = 0.05$, $d = 0.55$ (right).

Algorithm 3:
Step 1: *For α_ℓ, $\ell = 0, 1, \dots, N_\alpha$, infer the α_ℓ-cut A^{α_ℓ} by using (3) and the α_ℓ-cuts $A_i^{\alpha_\ell}$, $\ell = 0, 1, \dots, N_\alpha$.*
Step 2: *By setting $g = m_j$ and using (6), calculate $R^{\alpha_\ell}_{m_j,A}$ for $j = 1, \dots, M$ and $\ell = 0, 1, \dots, N_\alpha$.*
Step 3: *Interpret the intervals $R^{\alpha_\ell}_{m_j,A}$ as focal elements with the respective weights w_j/N_α.*
Step 4: *Choose an interval $B \subset \mathbb{R}$ and calculate $Bel(B)$ and $Pla(B)$ by using (7) where $\mathcal{S} = \{R^{\alpha_\ell}_{m_j,A}\}_{j=1,\dots,M;\,\ell=0,\dots,N_\alpha}$.*
Step 5: *Repeat Step 4 several times with the aim to increase Bel and Pla and to identify an interval B that satisfactorily represents the joined responses of the models.*

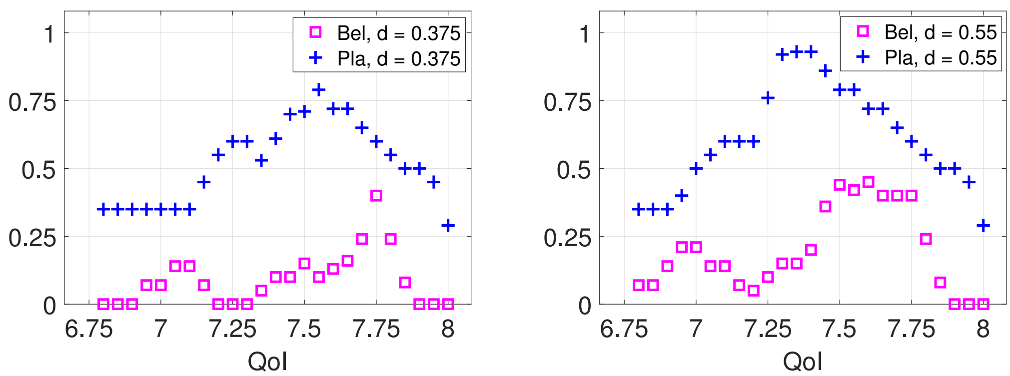

Figure 4: Algorithm 3. $Bel(B^d_{s,k})$ and $Pla(B^d_{s,k})$ for $s = 0.05$, $d = 0.375$ (left) and $s = 0.05$, $d = 0.55$ (right).

The output of Algorithm 3 is depicted in Figure 4. Although more information on fuzzy inputs was taken into account, i.e., more focal elements entered the calculations, the graphs do not offer a definite identification of the intervals that best characterize the join models' outputs. Owing to a rather strong gain in *Bel* and a not bad *Pla* level, one would probably prefer $[7.75, 8.125]$ over the other intervals in the $d = 0.375$ family. If $d = 0.55$, then $[7.45, 8]$ and $[7.5, 8.05]$ seem to be equal candidates because the loss in *Bel* is compensated by the gain in *Pla* and vice versa.

Finally, we can modify Algorithm 3 to get **Algorithm 4**. To this end, we refer to (8) instead to (7) in Step 4. The output of Algorithm 4 is depicted in Figure 5.

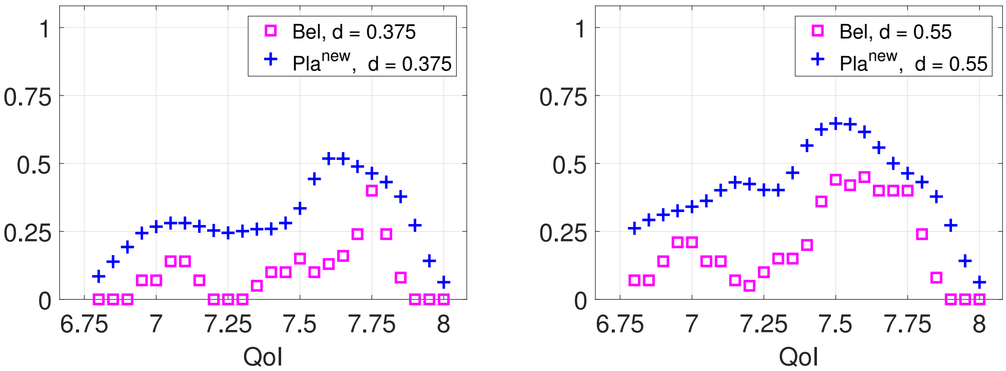

Figure 5: Algorithm 4. $Bel(B^d_{s,k})$ and $Pla^{\text{new}}(B^d_{s,k})$ for $s = 0.05$, $d = 0.375$ (left) and $s = 0.05$, $d = 0.55$ (right).

Now, clearer conclusions can be made than in the case of Figure 4. The intervals $[7.75, 8.125]$ and $[7.5, 8.05]$ seem to guarantee the strongest combination of the *Bel* and *Pla* assessments within the two sequences of intervals.

4. Comments and conclusions

The advantage of Algorithm 1 and Algorithm 3 is not only computational (they use the lowest number of focal elements) but also analytical because the uncertainty analysis is limited to a particular α-cut of input data. Although Algorithm 2 and Algorithm 4 make use of a richer family of focal elements, the picture of a multi-model synergy might not be clearer. Take, for instance, a high value of $Bel(B)$ for some interval B. Then, the questions arise: What is the cause? Does B cover a significant number of focal elements originating in several models, or does B cover a high number of focal elements belonging to only one model? Remember, that the focal elements associated with one model m_j, i.e., j fixed, form a chain of intervals for which $R^{\alpha_1}_{m_j,A} \subset R^{\alpha_2}_{m_j,A}$ if $\alpha_2 < \alpha_1$.

The probabilistic background of the evidence theory has been neglected in our exposition. Nevertheless, Pla^{new} in (8) could be interpreted as the probability that the crisp model response uniformly distributed in the interval I also falls into the interval B.

The reader might propose a modification of Algorithm 2 and Algorithm 4: to infer the focal elements of the models' responses, reduce the range of alphas and use, for instance, $\alpha = 0.5$, $\alpha = 0.75$, and $\alpha = 1$. This would certainly be possible, but we can get the same effect by reshaping the membership functions and considering $\alpha = 0$, $\alpha = 0.5$, and $\alpha = 1$. In this way, we obtain the standard Algorithm 2 and Algorithm 4.

What final conclusions can be made? To identify the intervals that most agree with multi-model responses, it is advisable to apply Algorithm 2 for various but individual values of α, and then Algorithm 4. Sufficiently rich and fine sequences of intervals determined by various values of s and d should be used in the analysis.

Acknowledgments

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