PANM 22

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In: Jan Chleboun and Jan Papež and Karel Segeth and Jakub Šístek and Tomáš Vejchodský (eds.): Programs and Algorithms of Numerical Mathematics 22. Institute of Mathematics, Czech Academy of Sciences, Prague, 2025. pp. 17–28.

Persistent URL: `http://dml.cz/dmlcz/703228`

Programs and Algorithms of Numerical Mathematics 22
J. Chleboun, J. Papež, K. Segeth, J. Šístek, T. Vejchodský (Eds.)
Institute of Mathematics CAS, Prague 2025

CONTINUOUS ADJOINT APPROACH TO SHAPE OPTIMIZATION WITH RESPECT TO 2D INCOMPRESSIBLE FLUID FLOW

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Abstract: The aim of this article is to briefly introduce the procedure for optimizing water turbine blades, which can lead to an innovative blade design and, consequently, an improvement in the desired properties of the water turbine, such as efficiency or the preferred pressure distribution on the blade. The computational method is based on formulating an objective function under certain constraint conditions, which are governed by the Navier-Stokes equations. This formulation enables the use of the Lagrange multiplier method, which incorporates the constraints into the augmented objective function. We derive the so-called adjoint problem, allowing us to simplify the gradient formulation for the chosen gradient-based optimization method.

Keywords: shape optimization, continuous adjoint

MSC: 49Q10, 49M41

1. Introduction

The problem of shape optimization, i.e., optimization where we try to find the optimal shape of a domain or part of it (e.g., water turbine blades), is a constrained optimization problem. It is necessary to prescribe an objective function (usually in integral form), constraint conditions (in our case, the equations describing the fluid flow), and a set of design parameters describing the optimized shape. Furthermore, for the optimization computational process itself, it is essential to determine the gradient of the objective function (required for any gradient-based optimization method), which includes the so-called shape derivative. For gradient computation, the continuous adjoint method is used, i.e., the adjoint problem is derived at first and then it is discretized. The derivation of the method and of all principles and ideas will be illustrated by a simplified two-dimensional model with laminar flow. The selected solver for solving the state and adjoint problems is described in detail in [3].

DOI: 10.21136/panm.2024.02

Consider the following optimization problem:

$$\min_{\mathbf{u},p,\Omega} F(\mathbf{u}, p, \Omega) \tag{1}$$

subject to incompressible steady-state Navier–Stokes equations (so-called primal or state equations)

$$R_i^u = -\frac{\partial \tau_{ij}}{\partial x_j} + u_j \frac{\partial u_i}{\partial x_j} + \frac{\partial p}{\partial x_i} = 0, \quad i = 1,2 \qquad \mathbf{x} \in \Omega \subset \mathbb{R}^2, \tag{2}$$

$$R^p = \frac{\partial u_j}{\partial x_j} = 0, \qquad \mathbf{x} \in \Omega \subset \mathbb{R}^2, \tag{3}$$

where u_i is a component of the velocity vector, $p := \frac{p}{\rho}$ is static pressure divided by the constant density of the liquid, and constant kinematic viscosity ν is considered in the stress tensor $\tau_{ij} = \nu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$. The Lipschitz domain boundary $\partial\Omega := \Gamma$ consists of several disjoint parts: inflow Γ_{in}, outflow Γ_{out}, periodic parts Γ_1, Γ_2 and optimized (changing) part of the boundary Γ_{opt} with the following boundary conditions:

$$\mathbf{u} = \mathbf{u}_{\text{in}}, \qquad \mathbf{x} \in \Gamma_{\text{in}}, \tag{4}$$

$$\mathbf{u} = \mathbf{0}, \qquad \mathbf{x} \in \Gamma_{opt}, \tag{5}$$

$$\begin{aligned} \mathbf{u}|_{\Gamma_1} &= \mathbf{u}|_{\Gamma_2}, && \mathbf{x} \in \Gamma_1, \Gamma_2 \\ p|_{\Gamma_1} &= p|_{\Gamma_2}, && \mathbf{x} \in \Gamma_1, \Gamma_2 \\ \frac{\partial \mathbf{u}}{\partial \mathbf{n}}|_{\Gamma_1} &= \frac{\partial \mathbf{u}}{\partial \mathbf{n}}|_{\Gamma_2}, && \mathbf{x} \in \Gamma_1, \Gamma_2 \end{aligned} \tag{6}$$

$$\begin{aligned} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) n_j &= 0, \quad i = 1,2 && \mathbf{x} \in \Gamma_{\text{out}}, \\ p &= p_{\text{out}}, && \mathbf{x} \in \Gamma_{\text{out}}, \end{aligned} \tag{7}$$

where n_j is the jth component of the outward unit normal vector to the corresponding part of the boundary. $\mathbf{u}_{\text{in}}$ and p_{out} are given functions and the Einstein convention, where repeated indices imply summation, is used.

The next approach is based on the method of Céa, see [4]. For optimization problems with equality constraints, it is appropriate to formulate the Lagrange function

$$L = F + \int_\Omega \lambda_i R_i^u \, \mathrm{d}\Omega + \int_\Omega \lambda_p R^p \, \mathrm{d}\Omega, \tag{8}$$

where for each flow (or state) variable u_i, $i = 1,2$, and p we define the so-called adjoint variables λ_i, $i = 1,2$, and λ_p. Function F will be described in Section 2.

Next, it is necessary to choose design variables $\mathbf{q} \in \mathbb{R}^{n_q}$. Complex shapes, such as a turbine blade, are suitably described by B-splines. This description is a linear combination of B-spline basis functions with coefficients known as control points, see [3]. Given the selected solver, we choose the set of the control points (more precisely, coordinates of the control points) as our design parameters. Without loss of generality, we assume in the following text that the vector $\mathbf{q}$ has only one component, i.e., $\mathbf{q} = q$. To determine the shape gradient using the parametric approach, it is necessary to compute the total (or material) derivative of the Lagrange function (8) with respect to the chosen design variables

$$\frac{\mathrm{d}L}{\mathrm{d}q} = \frac{\mathrm{d}F}{\mathrm{d}q} + \frac{\mathrm{d}}{\mathrm{d}q}\int\limits_{\Omega} \lambda_i R_i^u \,\mathrm{d}\Omega + \frac{\mathrm{d}}{\mathrm{d}q}\int\limits_{\Omega} \lambda_p R^p \,\mathrm{d}\Omega. \tag{9}$$

Let us briefly summarize the relations between total (material) and partial derivatives both in the domain Ω and on the boundary Γ, for details see [1]. For an arbitrary quantity $I = I(\mathbf{u}, p)$ defined in Ω it holds (after using the Leibniz theorem)

$$\frac{\mathrm{d}}{\mathrm{d}q}\int\limits_{\Omega} I \,\mathrm{d}\Omega = \int\limits_{\Omega} \frac{\partial I}{\partial q} \,\mathrm{d}\Omega + \int\limits_{\Gamma} I \,\frac{\mathrm{d}x_i}{\mathrm{d}q} n_i \,\mathrm{d}\Gamma, \tag{10}$$

where the partial derivative can be expressed by the chain rule $\frac{\partial I}{\partial q} = \frac{\partial I}{\partial u_i}\frac{\partial u_i}{\partial q} + \frac{\partial I}{\partial p}\frac{\partial p}{\partial q}$ in the first integral. The last boundary integral can be split into a sum over individual segments, but the term $\frac{\mathrm{d}x_i}{\mathrm{d}q}$ is zero everywhere except the optimized moving boundary, leaving only the integral over Γ_{opt}. For an arbitrary quantity $J = J(\mathbf{u}, p)$ defined on the boundary Γ, it holds

$$\frac{\mathrm{d}}{\mathrm{d}q}\int\limits_{\Gamma} J \,\mathrm{d}\Gamma = \int\limits_{\Gamma} \frac{\mathrm{d}J}{\mathrm{d}q} \,\mathrm{d}\Gamma + \int\limits_{\Gamma} J \,\frac{\mathrm{d}\,(\mathrm{d}\Gamma)}{\mathrm{d}q}, \tag{11}$$

$$\frac{\mathrm{d}J}{\mathrm{d}q} = \frac{\partial J}{\partial q} + \frac{\partial J}{\partial x_i} n_i \frac{\mathrm{d}x_j}{\mathrm{d}q} n_j \quad \text{and} \quad \frac{\mathrm{d}\,(\mathrm{d}\Gamma)}{\mathrm{d}q} = -\kappa \frac{\mathrm{d}x_i}{\mathrm{d}q} n_i \,\mathrm{d}\Gamma, \tag{12}$$

where κ denotes the mean curvature of Γ (can be derived using differential geometry, see [2]). We assume that the changes of design variables that produce changes of Γ_{opt} in the tangent direction do not change the shape of the domain Ω, therefore we consider only the normal component of the surface deformation. After the substitution of (12) into (11) we get

$$\frac{\mathrm{d}}{\mathrm{d}q}\int\limits_{\Gamma} J \,\mathrm{d}\Gamma = \int\limits_{\Gamma} \left(\frac{\partial J}{\partial u_i}\frac{\partial u_i}{\partial q} + \frac{\partial J}{\partial p}\frac{\partial p}{\partial q} \right) \mathrm{d}\Gamma + \int\limits_{\Gamma} \frac{\partial J}{\partial x_j} n_j \frac{\mathrm{d}x_i}{\mathrm{d}q} n_i \,\mathrm{d}\Gamma - \int\limits_{\Gamma} \kappa \, J \,\frac{\mathrm{d}x_i}{\mathrm{d}q} n_i \,\mathrm{d}\Gamma \tag{13}$$

and again the last two boundary integrals are nonzero only on Γ_{opt} and again we used the chain rule in the first integral.

Now we continue with (9) and after using (10) with $I = \lambda R$ and since the adjoint variables are independent of the flow variables, we arrive at

$$\frac{\mathrm{d}L}{\mathrm{d}q} = \frac{\mathrm{d}F}{\mathrm{d}q} + \int\limits_{\Omega} \lambda_i \frac{\partial R_i^u}{\partial q} \,\mathrm{d}\Omega + \int\limits_{\Omega} \lambda_p \frac{\partial R^p}{\partial q} \,\mathrm{d}\Omega + \int\limits_{\Gamma_{\mathrm{opt}}} \left(\lambda_j R_j^u + \lambda_p R^p \right) \frac{\mathrm{d}x_i}{\mathrm{d}q} n_i \,\mathrm{d}\Gamma. \tag{14}$$

The next step is to determine the total derivative of the objective function.

2. Objective function

The overall objective function, which is mentioned in (1), can be considered as an appropriate weighted combination of multiple components. In this text, we will introduce four components of the objective function F_1, F_2, F_3, F_4, so that:

$$F = w_1 F_1 + w_2 F_2 + w_3 F_3 + w_4 F_4. \tag{15}$$

1. The function F_1 quantifies the effect of the head. By optimizing this function, we achieve a minimal difference between the target head H_{tar} and the actual head H. The function F_1 is prescribed on the inflow and outflow part of the boundary, Γ_{in} and Γ_{out}. It is defined as follows:

$$F_1 = \frac{1}{2}\left(\frac{H - H_{\mathrm{tar}}}{H_{\mathrm{tar}}}\right)^2, \tag{16}$$

where the head H is defined as follows:

$$H = \frac{1}{\rho g S_{\mathrm{in}}} \int_{\Gamma_{\mathrm{in}}} p_{\mathrm{tot,in}} \,\mathrm{d}\Gamma - \frac{1}{\rho g S_{\mathrm{out}}} \int_{\Gamma_{\mathrm{out}}} p_{\mathrm{tot,out}} \,\mathrm{d}\Gamma, \tag{17}$$

$$\text{for} \quad p_{\mathrm{tot}} = p_{\mathrm{stat}} + \frac{1}{2}\rho v^2,\ p_{\mathrm{stat}} = \rho\, p,\ v = \frac{Q}{S} = \text{const.},\ Q = \int_{\Gamma_{\mathrm{in}}} \mathbf{u}_{\mathrm{in}} \cdot \mathbf{n} \,\mathrm{d}\Gamma, \tag{18}$$

where p_{stat} is static pressure and p kinematic pressure, further ρ denotes the density of the liquid, g gravitational acceleration, Q is the flow rate and S is the length of the respective boundary segment.

After some easy manipulations we can see that the only flow variable on which this term depends is the pressure p. Thus, after a straightforward differentiation of the composite function and using (13), we get

$$\frac{\mathrm{d}F_1}{\mathrm{d}q} = \left(\frac{H}{H_{\mathrm{tar}}} - 1\right) \frac{1}{H_{\mathrm{tar}}} \left[\int_{\Gamma_{\mathrm{in}}} \frac{\partial f_{1,\mathrm{in}}(p)}{\partial p} \frac{\partial p}{\partial q} \,\mathrm{d}\Gamma - \int_{\Gamma_{\mathrm{out}}} \frac{\partial f_{1,\mathrm{out}}(p)}{\partial p} \frac{\partial p}{\partial q} \,\mathrm{d}\Gamma \right], \tag{19}$$

where

$$f_{1,\mathrm{in(out)}}(p) = \frac{1}{\rho g S_{\mathrm{in(out)}}} \left(p + \frac{1}{2}\rho v^2\right), \quad \frac{\partial f_{1,\mathrm{in(out)}}(p)}{\partial p} = \frac{1}{\rho g S_{\mathrm{in(out)}}}. \tag{20}$$

2. The function F_2 is related to the efficiency of the water turbine. The ideal state is 100% efficiency, and therefore, we will minimize the deviation from this ideal state. Thus, we define the function F_2 as follows:

$$F_2 = 1 - \frac{M\omega}{Q\rho g H}, \tag{21}$$

where $\omega = \text{const.}$ denotes the angular velocity, and its value is prescribed by the real situation. The torque M, which acts on the turbine blade, i.e. Γ_{opt}, is defined as follows:

$$M = N \int_{\Gamma_{\text{opt}}} \mathbf{M} \cdot \mathbf{e} \, \mathrm{d}\Gamma, \quad \text{where} \quad \mathbf{M} = \mathbf{r} \times \mathbf{F}, \quad \mathbf{F} = \mathbf{n} \, p_{\text{stat}} \tag{22}$$

and $\mathbf{e}$ is the direction of the axis of rotation, N is the number of blades, $\mathbf{F}$ denotes the force acting on the blade, $\mathbf{r}$ is the position vector perpendicular to the axis of rotation, and $\mathbf{n}$ is the normal vector pointing outward from the suction side. The above formulas are valid for 3D calculations. For our simplified 2D model, we choose $\mathbf{e} = (1, 0, 0)$, $\mathbf{r} = (0, 0, 1)$, $\omega = 1$ and $N = 5$.

If we substitute the formulas (22) and (17) into (21), then again the resulting expression depends only on the pressure p, but this time there are integrals over Γ_{in}, Γ_{out} and Γ_{opt}. Thus, after differentiation of the quotient and using (13) we obtain

$$\begin{aligned} \frac{\mathrm{d}F_2}{\mathrm{d}q} = &- \frac{N\omega}{Q\rho g H} \left[\int_{\Gamma_{\text{opt}}} \frac{\partial f_{2,\text{opt}}(p)}{\partial p} \frac{\partial p}{\partial q} \, \mathrm{d}\Gamma + \int_{\Gamma_{\text{opt}}} \left(\frac{\partial f_{2,\text{opt}}}{\partial x_j} n_j - \kappa_{\text{opt}} f_{2,\text{opt}} \right) \frac{\mathrm{d}x_i}{\mathrm{d}q} n_i \, \mathrm{d}\Gamma \right] \\ &+ \frac{N\omega}{Q\rho g H^2} \int_{\Gamma_{\text{opt}}} f_{2,\text{opt}}(p) \, \mathrm{d}\Gamma \left[\int_{\Gamma_{\text{in}}} \frac{\partial f_{1,\text{in}}(p)}{\partial p} \frac{\partial p}{\partial q} \, \mathrm{d}\Gamma - \int_{\Gamma_{\text{out}}} \frac{\partial f_{1,\text{out}}(p)}{\partial q} \frac{\partial p}{\partial q} \, \mathrm{d}\Gamma \right], \end{aligned} \tag{23}$$

where

$$f_{2,\text{opt}}(p) = (\mathbf{r} \times \mathbf{n}) \cdot \mathbf{e} \, p = n_2 p, \qquad \frac{\partial f_{2,\text{opt}}(p)}{\partial p} = (\mathbf{r} \times \mathbf{n}) \cdot \mathbf{e} = n_2. \tag{24}$$

3. The function F_3 represents the pressure distribution on the blade. The optimization aims to match this distribution as closely as possible to the target pressure, p_{tar}. Hence, F_3 is defined as

$$F_3 = \frac{1}{2} \int_{\Gamma_{\text{opt}}} \frac{(p - p_{\text{tar}})^2}{p_{\text{tar}}^2} \, \mathrm{d}\Gamma. \tag{25}$$

In this function, the dependence on pressure is clear, so the total derivative with respect to q is determined using the same procedure as before and we get

$$\frac{\mathrm{d}F_3}{\mathrm{d}q} = \int_{\Gamma_{\text{opt}}} \frac{\partial f_{3,\text{opt}}(p)}{\partial p} \frac{\partial p}{\partial q} \, \mathrm{d}\Gamma + \int_{\Gamma_{\text{opt}}} \left(\frac{\partial f_{3,\text{opt}}(p)}{\partial x_j} n_j - \kappa_{\text{opt}} f_{3,\text{opt}}(p) \right) \frac{\mathrm{d}x_i}{\mathrm{d}q} n_i \, \mathrm{d}\Gamma, \tag{26}$$

where

$$f_{3,\text{opt}}(p) = \frac{1}{2} \frac{(p - p_{\text{tar}})^2}{p_{\text{tar}}^2}, \qquad \frac{\partial f_{3,\text{opt}}(p)}{\partial p} = \frac{(p - p_{\text{tar}})}{p_{\text{tar}}^2}. \tag{27}$$

4. The final part of the objective function, F_4, minimizes the difference between the outflow boundary velocity and a given target outflow velocity $\mathbf{u}_{\text{tar}}$. This prevents undesirable turbulence behind the runner, thereby improving overall efficiency. Thus, F_4 is defined as:

$$F_4 = \frac{1}{2} \int\limits_{\Gamma_{\text{out}}} \frac{\|\mathbf{u} - \mathbf{u}_{\text{tar}}\|^2}{\|\mathbf{u}_{\text{tar}}\|} \, \mathrm{d}\Gamma. \tag{28}$$

This function is the only one dependent on the flow variables u_i. Using the same procedure as before we get

$$\frac{\mathrm{d}F_4}{\mathrm{d}q} = \int\limits_{\Gamma_{\text{out}}} \frac{\partial f_{4,\text{out}}(\mathbf{u})}{\partial u_i} \frac{\partial u_i}{\partial q} \, \mathrm{d}\Gamma, \tag{29}$$

where

$$f_{4,\text{out}}(\mathbf{u}) = \frac{1}{2} \frac{\|\mathbf{u} - \mathbf{u}_{\text{tar}}\|^2}{\|\mathbf{u}_{\text{tar}}\|}, \qquad \frac{\partial f_{4,\text{out}}(\mathbf{u})}{\partial u_i} = \frac{u_i - u_{i,\text{tar}}}{\|\mathbf{u}_{\text{tar}}\|}. \tag{30}$$

3. Adjoint problem derivation

For the derivation of the adjoint problem, the expressions (19), (23), (26), (29), (2) and (3) are substituted into (14), the interchangeability of derivatives is used, i.e. $\frac{\partial}{\partial q}\frac{\partial J}{\partial x_i} = \frac{\partial}{\partial x_i}\frac{\partial J}{\partial q}$ for any function J, and the Green-Gauss theorem is applied. After appropriate term rearranging and relabeling to simplify the formulas, and noting that $\tau^a_{ij} = \nu\left(\frac{\partial \lambda_i}{\partial x_j} + \frac{\partial \lambda_j}{\partial x_i}\right)$ is representing the adjoint stress tensor and again the Einstein convention is used, we arrive at

$$\begin{aligned}
\frac{\mathrm{d}L}{\mathrm{d}q} = \; & \underbrace{\left(\frac{H}{H_{\text{tar}}} - 1\right)\frac{w_1}{H_{\text{tar}}}}_{\text{C}_1} \left[\int\limits_{\Gamma_{\text{in}}} \frac{\partial f_{1,\text{in}}(p)}{\partial p}\frac{\partial p}{\partial q}\,\mathrm{d}\Gamma - \int\limits_{\Gamma_{\text{out}}} \frac{\partial f_{1,\text{out}}(p)}{\partial p}\frac{\partial p}{\partial q}\,\mathrm{d}\Gamma\right] \\
& - \underbrace{\frac{w_2 N\omega}{Q\rho g H}}_{\text{C}_2} \left[\int\limits_{\Gamma_{\text{opt}}} \frac{\partial f_{2,\text{opt}}(p)}{\partial p}\frac{\partial p}{\partial q}\,\mathrm{d}\Gamma + \int\limits_{\Gamma_{\text{opt}}} \left(\frac{\partial f_{2,\text{opt}}}{\partial x_j} n_j - \kappa_{\text{opt}} f_{2,\text{opt}}\right)\frac{\mathrm{d}x_i}{\mathrm{d}q} n_i\,\mathrm{d}\Gamma\right] \\
& + \underbrace{\frac{w_2 N\omega}{Q\rho g H^2} \int\limits_{\Gamma_{\text{opt}}} f_{2,\text{opt}}(p)\,\mathrm{d}\Gamma}_{\text{C}_3} \left[\int\limits_{\Gamma_{\text{in}}} \frac{\partial f_{1,\text{in}}(p)}{\partial p}\frac{\partial p}{\partial q}\,\mathrm{d}\Gamma - \int\limits_{\Gamma_{\text{out}}} \frac{\partial f_{1,\text{out}}(p)}{\partial q}\frac{\partial p'}{\partial q}\,,\mathrm{d}\Gamma\right] \\
& + w_3 \int\limits_{\Gamma_{\text{opt}}} \frac{\partial f_{3,\text{opt}}(p)}{\partial p}\frac{\partial p}{\partial q}\,\mathrm{d}\Gamma + w_3 \int\limits_{\Gamma_{\text{opt}}} \left(\frac{\partial f_{3,\text{opt}}(p)}{\partial x_j} n_j - \kappa_{\text{opt}} f_{3,\text{opt}}(p)\right)\frac{\mathrm{d}x_i}{\mathrm{d}q} n_i\,\mathrm{d}\Gamma \\
& + w_4 \int\limits_{\Gamma_{\text{out}}} \frac{\partial f_{4,\text{out}}(\mathbf{u})}{\partial u_i}\frac{\partial u_i}{\partial q}\,\mathrm{d}\Gamma + \int\limits_{\Omega} \frac{\partial \tau^a_{ij}}{\partial x_j}\frac{\partial u_i}{\partial q}\,\mathrm{d}\Omega + \int\limits_{\Omega} \lambda_j \frac{\partial u_j}{\partial x_i}\frac{\partial u_i}{\partial q}\,\mathrm{d}\Omega
\end{aligned}$$

$$
\begin{aligned}
&- \int\limits_{\Omega} u_j \frac{\partial \lambda_i}{\partial x_j} \frac{\partial u_i}{\partial q}\, \mathrm{d}\Omega - \int\limits_{\Omega} \frac{\partial \lambda_p}{\partial x_i} \frac{\partial u_i}{\partial q}\, \mathrm{d}\Omega - \int\limits_{\Omega} \frac{\partial \lambda_j}{\partial x_j} \frac{\partial p}{\partial q}\, \mathrm{d}\Omega + \int\limits_{\Gamma} \tau^a_{ij} n_j \frac{\partial u_i}{\partial q}\, \mathrm{d}\Gamma \\
&+ \int\limits_{\Gamma} u_j n_j \lambda_i \frac{\partial u_i}{\partial q}\, \mathrm{d}\Gamma + \int\limits_{\Gamma} \lambda_p n_i \frac{\partial u_i}{\partial q}\, \mathrm{d}\Gamma - \int\limits_{\Gamma} \frac{\partial \tau_{ij}}{\partial q} n_j \lambda_i\, \mathrm{d}\Gamma + \int\limits_{\Gamma} \lambda_i n_i \frac{\partial p}{\partial q}\, \mathrm{d}\Gamma \\
&+ \int\limits_{\Gamma_{\mathrm{opt}}} \left(\lambda_j R^u_j + \lambda_p R^p\right) \frac{\mathrm{d}x_i}{\mathrm{d}q} n_i\, \mathrm{d}\Gamma. \qquad (31)
\end{aligned}
$$

First of all, we will focus on the volume integrals in (31). It is useful to avoid calculations the derivatives of the flow variables with respect to the design parameters, i.e., $\frac{\partial u_i}{\partial q}$ and $\frac{\partial p}{\partial q}$. This can be achieved by setting all the terms that involve these derivatives to zero. This leads to the adjoint set of equations

$$
R^\lambda_i = -\frac{\partial \tau^a_{ij}}{\partial x_j} + \lambda_j \frac{\partial u_j}{\partial x_i} - u_j \frac{\partial \lambda_i}{\partial x_j} - \frac{\partial \lambda_p}{\partial x_i} = 0, \quad i = 1, 2, \qquad \mathbf{x} \in \Omega \subset \mathbb{R}^2, \qquad (32)
$$

$$
R^{\lambda_p} = \frac{\partial \lambda_j}{\partial x_j} = 0, \qquad \mathbf{x} \in \Omega \subset \mathbb{R}^2. \qquad (33)
$$

Thus, only the boundary integrals remain and it is necessary to set the correct boundary conditions in order to reduce the number of integrals as much as possible.

3.1. Boundary conditions for the adjoint problem

Recall that $\Gamma = \Gamma_{\mathrm{in}} \cup \Gamma_1 \cup \Gamma_2 \cup \Gamma_{\mathrm{opt}} \cup \Gamma_{\mathrm{out}}$, i.e. the boundary integral over the entire boundary is a sum of integrals over the individual parts of the boundary:

1. Γ_{in}: For the inlet boundary we set (4) and, therefore, it is easy to see that $\frac{\partial u_i}{\partial q} = 0$ and $\frac{\partial \tau_{ij}}{\partial q} = 0$ holds. Thus, only the following nonzero integrals over inlet boundary remain in (31) (again after appropriate term rearranging and factoring out)

$$
\int\limits_{\Gamma_{\mathrm{in}}} \left[(C_1 + C_3) \frac{\partial f_{1,\mathrm{in}}(p)}{\partial p} + \lambda_i n_i\right] \frac{\partial p}{\partial q}\, \mathrm{d}\Gamma. \qquad (34)
$$

So we set

$$
\lambda_i n_i = -(C_1 + C_3) \frac{\partial f_{1,\mathrm{in}}(p)}{\partial p}, \quad \lambda_i t_i = 0, \quad \mathbf{x} \in \Gamma_{\mathrm{in}}, \qquad (35)
$$

to set the integral (34) to zero.

2. Γ_{out}: We set the conditions (7) for the flow variables, thus for differentiation w.r.t. design parameters at the outflow boundary it holds

$\left(\frac{\partial}{\partial x_j}\frac{\partial u_i}{\partial q}+\frac{\partial}{\partial x_i}\frac{\partial u_j}{\partial q}\right)n_j=0$, $i=1,2$ and $\frac{\partial p}{\partial q}=0$. Thus, it is necessary to handle only the following nonzero integrals in (31) (again after appropriate term rearranging and factoring out)

$$\int\limits_{\Gamma_{\text{out}}}\left(w_4\frac{\partial f_{4,\text{out}}(\mathbf{u})}{\partial u_i}+\tau^a_{ij}n_j+u_jn_j\lambda_i+\lambda_pn_i\right)\frac{\partial u_i}{\partial q}\,\mathrm{d}\Gamma. \tag{36}$$

So if we set

$$\tau^a_{ij}n_j+u_jn_j\lambda_i+\lambda_pn_i=-w_4\frac{\partial f_{4,\text{out}}(\mathbf{u})}{\partial u_i},\quad i=1,2,\quad \mathbf{x}\in\Gamma_{\text{out}}, \tag{37}$$

all the integrals over the output boundary vanish from (31).

3. Γ_1, Γ_2: For the periodic boundaries, none of the boundary integrals is equal to zero, so we obtain

$$\int\limits_{\Gamma_1}\left(\tau^a_{ij}n_j\frac{\partial u_i}{\partial q}+u_jn_j\lambda_i\frac{\partial u_i}{\partial q}+\lambda_pn_i\frac{\partial u_i}{\partial q}-\frac{\partial\tau_{ij}}{\partial q}n_j\lambda_i+\lambda_in_i\frac{\partial p}{\partial q}\right)\mathrm{d}\Gamma+$$
$$\int\limits_{\Gamma_2}\left(\tau^a_{ij}n_j\frac{\partial u_i}{\partial q}+u_jn_j\lambda_i\frac{\partial u_i}{\partial q}+\lambda_pn_i\frac{\partial u_i}{\partial q}-\frac{\partial\tau_{ij}}{\partial q}n_j\lambda_i+\lambda_in_i\frac{\partial p}{\partial q}\right)\mathrm{d}\Gamma. \tag{38}$$

For periodic boundary it holds that $\Gamma_2=T(\Gamma_1)$ is a translational copy of Γ_1 under a map T with the opposite normal vector with respect to Γ_1 at the corresponding points of both boundaries. Each pair of integrals for Γ_1 and the same one for Γ_2 vanishes if we set

$$\begin{aligned}\tau^a_{ij}(\mathbf{x})&=\tau^a_{ij}(T(\mathbf{x})), && \mathbf{x}\in\Gamma_1,\\ \lambda_i(\mathbf{x})&=\lambda_i(T(\mathbf{x})),\quad i=1,2, && \mathbf{x}\in\Gamma_1,\\ \lambda_p(\mathbf{x})&=\lambda_p(T(\mathbf{x})), && \mathbf{x}\in\Gamma_1.\end{aligned} \tag{39}$$

4. Γ_{opt}: Optimized boundary is the only one which is assumed to be moving. For velocity, we set homogeneous boundary condition (5), so total derivation w.r.t. q is equal to zero. Thus, using (12) we obtain the following expression for the partial derivative w.r.t. q

$$\frac{\partial u_i}{\partial q}=-\frac{\partial u_i}{\partial x_k}n_k\frac{\mathrm{d}x_l}{\mathrm{d}q}n_l,\quad i=1,2. \tag{40}$$

Since no terms vanish on this boundary, thus in (31) we can take care only for the following integral

$$\int\limits_{\Gamma_{\text{opt}}}\left(-C_2\frac{\partial f_{2,\text{opt}}(p)}{\partial p}+w_3\frac{\partial f_{3,\text{opt}}(p)}{\partial p}+\lambda_in_i\right)\frac{\partial p}{\partial q}\,\mathrm{d}\Gamma. \tag{41}$$

For vanishing of (41), we set

$$\lambda_in_i=C_2\frac{\partial f_{2,\text{opt}}(p)}{\partial p}-w_3\frac{\partial f_{3,\text{opt}}(p)}{\partial p},\quad \lambda_it_i=0,\quad \mathbf{x}\in\Gamma_{\text{opt}}. \tag{42}$$

4. Gradient

After setting the boundary conditions as described above and substituting (40) into (31), we obtain the expression for the gradient in the form

$$\begin{aligned}\frac{\mathrm{d}L}{\mathrm{d}q} &= C_2 \int\limits_{\Gamma_{\text{opt}}} \left(\frac{\partial f_{2,\text{opt}}}{\partial x_j} n_j - \kappa_{\text{opt}} f_{2,\text{opt}} \right) \frac{\mathrm{d}x_i}{\mathrm{d}q} n_i \,\mathrm{d}\Gamma + w_3 \int\limits_{\Gamma_{\text{opt}}} \left(\frac{\partial f_{3,\text{opt}}(p)}{\partial x_j} n_j - \right. \\ &- \left. \kappa_{\text{opt}} f_{3,\text{opt}}(p) \right) \frac{\mathrm{d}x_i}{\mathrm{d}q} n_i \,\mathrm{d}\Gamma - \int\limits_{\Gamma_{\text{opt}}} \left(\tau^a_{ij} n_j + u_j n_j \lambda_i + \lambda_p n_i \right) \frac{\partial u_i}{\partial x_k} n_k \frac{\mathrm{d}x_l}{\mathrm{d}q} n_l \,\mathrm{d}\Gamma \\ &- \int\limits_{\Gamma_{\text{opt}}} \frac{\partial \tau_{ij}}{\partial q} n_j \lambda_i \,\mathrm{d}\Gamma + \int\limits_{\Gamma_{\text{opt}}} \left(\lambda_j R^u_j + \lambda_p R^p \right) \frac{\mathrm{d}x_i}{\mathrm{d}q} n_i \,\mathrm{d}\Gamma, \end{aligned} \tag{43}$$

where the term $\frac{\partial \tau_{ij}}{\partial q} n_j \lambda_i$ can be rewritten as follows (after substituting (40) into the stress tensor and considering that the boundary conditions (42) were set for λ_i on Γ_{opt} boundary and tedious computation)

$$\frac{\partial \tau_{ij}}{\partial q} n_j \lambda_i = \lambda_i n_i \left(\tau_{ij} \frac{\mathrm{d}(n_i n_j)}{\mathrm{d}q} + \frac{\partial \tau_{ij}}{\partial x_m} n_m \frac{\mathrm{d}x_k}{\mathrm{d}q} n_k n_i n_j \right). \tag{44}$$

The numerical computation proceeds as follows: we set the initial shape of the blade, i.e., the boundary Γ_{opt}, and solve the primal problem (2) and (3) with the boundary conditions (4), (5), (6), (7). This provides the state variables u_1, u_2 and p. The adjoint quantities λ_1, λ_2 and λ_p are obtained by solving the adjoint problem (32) and (33) with the boundary conditions (35), (37), (39) and (42). Then, the gradient is computed by using the equation (43) and (44) and the shape of the blade is adjusted by using any gradient-based method (here, for simplicity, the steepest descent method is used).

5. Numerical experiment

We test our optimization approach on the simplified problem of flow in a domain which is a part of a 2D blade cascade. This cascade is obtained by unfolding a cylindrical cross-section of the turbine and it is illustrated in Figure 1 (left). The computational domain is then a passage between two blade profiles, see Figure 1 (right). The domain consists of three B-spline patches of degree 3. Γ_{opt} corresponds to the upper (suction side) and lower (pressure side) boundaries of the middle patch which form the blade profile. Left and right patches are bounded by periodic boundaries Γ_1 and Γ_2 and inlet boundary (the left-most) Γ_{in} and outlet boundary (the right-most) Γ_{out}. We use $\mathbf{u}_{in} = [7.76, -0.28]$ on the inlet and kinematic viscosity $\nu = 0.01$ in this example.

The objective function is defined by its components and corresponding weights in (15). In this example, we use the following weights

$$w_1 = 1, \quad w_2 = 1.8, \quad w_3 = 1, \quad w_4 = 0.2,$$

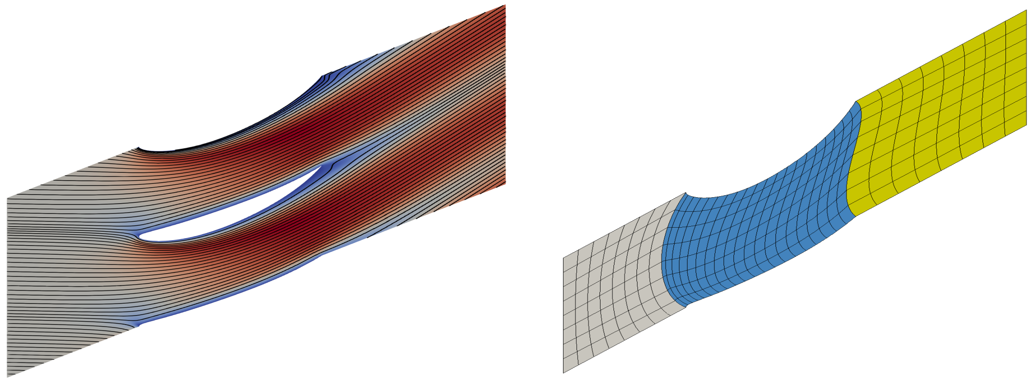

Figure 1: Illustration of the flow in the blade raw (left). Computational domain (right).

which prefers the efficiency component. The target values of pressure (p_{tar}) on pressure and suction sides of the blade profile and velocity ($\mathbf{u}_{\text{tar}}$) at the output are equal to the integral mean values of particular quantities over the corresponding boundary for the initial geometry. For simplicity we use steepest descent method with constant step $\gamma = 5 \cdot 10^{-4}$. Therefore the control points of B-spline curves describing both parts of Γ_{opt} are updated during the iteration process by the formula

$$\mathbf{q}^{\text{new}} = \mathbf{q} - \gamma \nabla_{\mathbf{q}} L, \tag{45}$$

where the length of the vector $\mathbf{q}$ is 42.The optimization is stopped after 40 iterations.

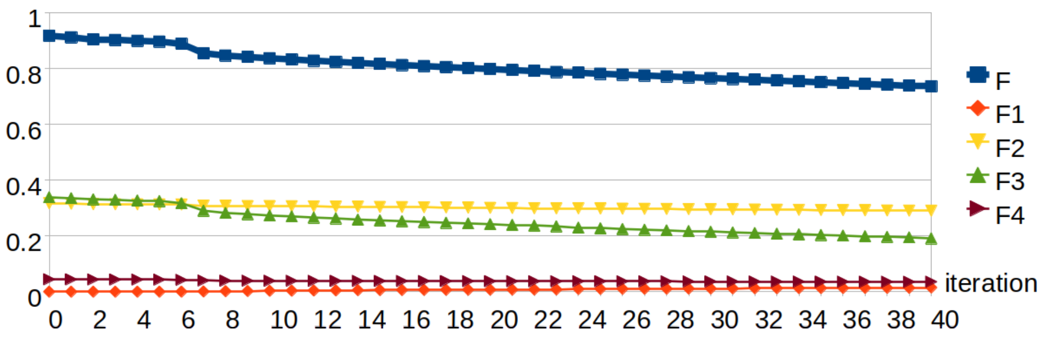

Figure 2: Evolution of the objective function and its components.

The values of objective function and its components are shown in Figure 2. We can see that the objective function as well as its individual components are decreasing, except for F_1. That is obvious, because H_{tar} is defined as the head of the problem with the initial geometry and therefore $F_1 = 0$ for the initial geometry.

The comparison of the initial blade profile and the optimized one is shown in

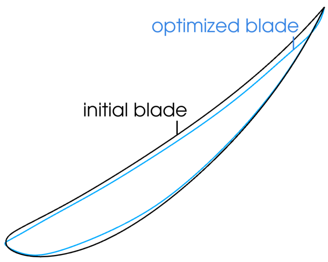

Figure 3: Initial and optimized blade.

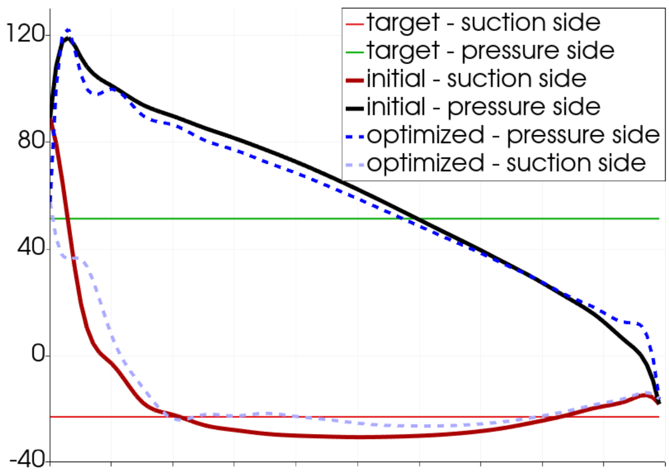

Figure 4: Pressure distribution over the blade profile.

Figure 3. The initial and optimized pressure distribution over the profile is shown in Figure 4 together with the values of the pressure targets.

6. Conclusion

In conclusion, this method shows great promise for further development (especially into 3D and turbulent flow models), as even in the presented simple 2D model with laminar flow and a basic optimization method, it was able to reduce the objective function and adjust the blade in a meaningful way.

Acknowledgements

This work was supported by Technology Agency of the Czech Republic (TA CR) grant No. TK04020250.

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