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Zdeněk Frolík

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THE TOPOLOGICAL PRODUCT OF COUNTABLY COMPACT SPACES

ZDENĚK FROLÍK, Praha

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In the present paper we investigate the class of all completely regular spaces P such that the topological product $P \times Q$ is countably compact for all countably compact and completely regular spaces Q. Moreover we give a necessary and sufficient condition for the embedding of a completely regular space as a closed subspace into the topological product of two countably compact spaces. Every separable metrizable and every discrete space may be embedded as a closed subspace in the topological product of two countably compact spaces. In this connection we derive a theorem on the Stone-Čech compactification of metrizable separable spaces.

The terminology of J. KELLEY, *General Topology*, is used throughout. The closure of a subset M of a space P will be denoted by $\overline{M}^P$ or merely $\overline{M}$. The Stone — Čech compactification of a completly regular space P will be denoted by βP. Unless otherwise stated, subsets of a space will be considered to be subspaces. The potency of a set M will be denoted by card M.

1. SOME PRELIMINARIES

In this section we recall some definitions and theorems. Let $\mathfrak{m}$ be an infinite cardinal number. A space P is said to be $\mathfrak{m}$-*compact* if it satisfies the following condition: If $\{F\}$ is a family of closed subsets of P such that the potency of $\{F\}$ is $\leqq \mathfrak{m}$ and the family $\{F\}$ has the finite intersection property, then the intersection $\bigcap \{F\}$ is non-void. $\aleph_0$-compact spaces are called countably compact. A map f from a space P into a space Q is said to be *closed* if the image $f[F]$ of every closed subset F of P is closed in Q.

1.1. Proposition. *Let f be a closed map from P into Q. If Q is a $\mathfrak{m}$-compact space and $f^{-1}[y]$ is $\mathfrak{m}$-compact for each y in Q, then P is $\mathfrak{m}$-compact.*

Proof. Suppose that a family $\{F\}$ of closed subsets of P has the finite intersection property and that the potency of $\{F\}$ is $\leqq \mathfrak{m}$. Without loss of generality we may assume that if both F_1 and F_2 belong to $\{F\}$, then their intersection $F_1 \cap F_2$ also belongs to $\{F\}$. Choose a point y in $\bigcap \{f[F]\}$. The space $E = f^{-1}[y]$ is $\mathfrak{m}$-compact, the family $\{F \cap E\}$ consisting of closed subsets of E has the finite intersection property and it has potency $\leqq \mathfrak{m}$. Hence

$$\bigcap \{E \cap F\} \neq \emptyset .$$

In consequence we have $\bigcap \{F\} \neq \emptyset$.

1.2. Proposition. *Let K be a compact space and let P be a space. The projection π of the topological product $P \times K$ onto P is a closed map.*

Proof. Let F be a closed subset of $P \times K$ and $x \in (P - \pi[F])$. For each y in K there exist open sets $U(y) \subset P$ and $V(y) \subset K$ such that

$$x \in U(y) , \quad y \in V(y) , \quad (U(y) \times V(y)) \cap F = \emptyset .$$

Choose a finite subset Y of K so that

$$\bigcup \{V(y) ; \quad y \in Y\} = K .$$

The intersection $U = \bigcap \{U(y);\ y \in Y\}$ is a neighborhood of the point x, and $U \cap \pi[F] = \emptyset$.

As a simple consequence of 1.1 and 1.2 we have

1.3. Theorem. *If P is $\mathfrak{m}$-compact and if K is compact, then the topological product $P \times K$ is $\mathfrak{m}$-compact.*

We shall need the following simple and useful lemma.

1.4. Lemma. *Let A be an index set, and for each a in A let P_a be a subspace of a fixed Hausdorff space R. Consider the topological product*

$$P = X \{P_a;\ a \in A\} .$$

Put $Q = \bigcap \{P_a;\ a \in A\}$. The "diagonal" D of the product P, that is the set of all $y \in P$ such that all coordinates of y are equal and belong to Q, is a closed subset of P.

Proof. Denote by π_a the projection (map) of the product P onto the factor P_a. Let y be an element of $P - D$. Hence there exist indexes a_1 and a_2 such that $\pi_{a_1}[y] = x_1 \neq x_2 = \pi_{a_2}[y]$. The space R is Hausdorff and hence there exist disjoint open (in R) sets U_1 and U_2 containing x_1 and x_2 respectively.

Put

$$W = (\pi_{a_1}^{-1} [U_1 \cap P_{a_1}]) \cap (\pi_{a_2}^{-1}[U_2 \cap P_{a_2}]) .$$

Since $U_1 \cap U_2 = \emptyset$, we have that $W \cap D = \emptyset$. The proof is complete.

We state the following special case of 1,4:

1.5. Lemma. *Let P and Q be subspaces of a Hausdorff space R. Then the "diagonal", that is the set*

$$\{(x, x) ; \quad x \in P \cap Q\} ,$$

is closed in $P \times Q$.

Let us recall that a space R is said to be extension (Hausdorff extension, completely regular extension) of a space P, if P is a dense subspace of R (and R is a Hausdorff space, R is a completely regular space, respectively).

1.6. Lemma. *Let P_1 and P_2 be spaces and let K_1 and K_2 be Hausdorff extensions of P_1 and P_2 respectively. Consider the product spaces*

$$P = P_1 \times P_2 , \quad S_1 = K_1 \times P_2 , \quad S_2 = P_1 \times K_2 , \quad S = S_1 \times S_2 .$$

The set

$$D = \{(x, x) ; \quad x \in P\}$$

is closed in S (and homeomorphic to P).

Proof. Consider $R = (K_1 \times K_2) \times (K_1 \times K_2)$ and apply 1.5.

2. EMBEDDING OF COMPLETELY REGULAR SPACES AS CLOSED SUBSPACES IN THE TOPOLOGICAL PRODUCT OF TWO COUNTABLY COMPACT SPACES

In this section we shall study the class $\mathfrak{F}_{\mathfrak{m}}$ ($\mathfrak{m}$ being an infinite cardinal number) consisting of all spaces such that for some completely regular $\mathfrak{m}$-compact spaces R and S the space P is a closed subspace of the topological product $R \times S$.

In [4] J. Novák constructed countably compact subspaces P and Q of the Stone — Čech compactification βN of the countable infinite discrete space N such that $P \cap Q = N$ (and $P \cup Q = \beta N$). The topological product $P \times Q$ is not countably compact since the diagonal, that is the set

$$\{(n, n) ; \quad n \in N\}$$

is an infinite closed discrete subset of $P \times Q$. This follows from 1.5. First we show that Novák's method of embedding is quite general.

2.1. Theorem. *A completely regular space F belongs to $\mathfrak{F}_{\mathfrak{m}}$ if and only if there exist $\mathfrak{m}$-compact subspaces P and Q of the Stone-Čech compactification βF of F such that $P \cap Q = F$.*

Proof. Sufficiency is a quite elementary consequence of lemma 1.5. Indeed, supposing $P \cap Q = F$ and $P \cup Q \subset \beta F$ we have by 1.5 that the set

$$D = \{(x, x) ; \quad x \in F\}$$

is closed in $P \times Q$ (and homeomorphic to F). Hence if P and Q are $\mathfrak{m}$-compact, then F belongs to $\mathfrak{F}_{\mathfrak{m}}$.

Conversely, suppose that F belongs to $\mathfrak{F}_{\mathfrak{m}}$. There exist completely regular $\mathfrak{m}$-compact spaces P_1 and P_2 such that F is a closed subspace of the topological product $P_1 \times P_2$. Let K_1 and K_2 be compactifications of P_1 and P_2 respectively. Put

$$S_1 = K_1 \times P_2 , \ S_2 = P_1 \times K_2$$

and consider the topological product $S = S_1 \times S_2$. By theorem 1.3 the spaces S_1 and S_2 are $\mathfrak{m}$-compact. Put

$$D = \{(x, x)\ ;\ \ x \in P_1 \times P_2\}\ ,\quad F_1 = \{(x, x)\ ;\ \ x \in F\}\ .$$

Evidently the spaces D and F_1 are homeomorphic with $P_1 \times P_2$ and F, respectively. Moreover F_1 is closed in D. Let R_1 and R_2 be closures of F in S_1 and S_2, respectively. The spaces R_1 and R_2 are $\mathfrak{m}$-compact as closed subsets of $\mathfrak{m}$-compact spaces S_1 and S_2. Evidently

$$D \cap (R_1 \times R_2) = F_1\ .$$

By lemma 1.5, the set D is closed in S. Hence F_1 is closed in $R_1 \times R_2$, Let φ_i $(i = 1, 2)$ be Stone-Čech mappings from βF onto the Stone-Čech compactification βR_i of R_i (this mapping exists, since F is dense in R_i, and therefore, βR_i is a compactification of F). Put

$$Q_1 = \varphi_1^{-1}\,[R_1]\ ,\quad Q_2 = \varphi_2^{-1}\,[R]_2\ .$$

It is sufficient to prove that Q_1 and Q_2 are both $\mathfrak{m}$-compact and $Q_1 \cap Q_2 = F$. We need the following simple lemma on closed maps.

2.2. Lemma. *Let f be a closed and continuous map from A into B. Let $C \subset B$. Then the restriction g of f onto $f^{-1}\,[C]$ is a closed map.*

The simple proof of this lemma may be left to the reader. From lemma 2.2 and proposition 1.1. it follows that Q_1 and Q_2 are $\mathfrak{m}$-compact. It remains to prove that $Q_1 \cap Q_2 = F$, but this is a consequence of the fact that the set F_1 is closed in $R_1 \times R_2$. The proof is complete.

2.3. Note. Note that theorem 2.1 may be generalized in the following way. Let $\mathfrak{A}$ be a class of completely regular spaces satisfying the following conditions

(a) If $P \in \mathfrak{A}$ and F is closed subspace of P, then $F \in \mathfrak{A}$.

(b) If $P_1 \in \mathfrak{A}$ and $P_2 \in \mathfrak{A}$, then there exists a completely regular extension K of P_1 such that both $K \in \mathfrak{A}$ and $K \times P_2 \in \mathfrak{A}$.

(c) If P is an completely regular extension of Q and $P \in \mathfrak{A}$, then there exists a subspace R of βQ such that $\varphi[R] = P$, where φ denotes the Stone-Čech mapping of βQ onto βP.

Then a space F is a closed subspace of the topological product of two spaces belonging to $\mathfrak{A}$ if and only if there exist subspaces P_1 and P_2 of βF such that

$$P_1 \cap P_2 = F \quad \text{and} \quad P_1 \in \mathfrak{A},\quad P_2 \in \mathfrak{A}\ .$$

In the remainder of this section we shall consider only the class $\mathfrak{F}_{\aleph_0} = \mathfrak{F}$.

2.4. Theorem. *Every discrete space may be embedded as a closed subspace in the topological product of two countably compact completely regular spaces. That is, every discrete space belongs to $\mathfrak{F}$.*

Proof. Let M be an infinite discrete space. By [1], theorem 3.1.6, there exist countably compact subspaces P and Q of the Stone-Čech compafication

βM of M such that $P \cap Q = M$. (The proof of the theorem just quoted is rather involved.)

2.5. Theorem. *Every metrizable separable space may be embedded as a closed subspace in the topological product of two countably compact completely regular spaces. That is, every metrizable separable space belongs to $\mathfrak{F}$.*

By theorem 2.1, the theorem 2.5 is an consequence of the following

2.6. Theorem. *Let P be a metrizable and separable space. There exist countably compact subspaces R and S of the Stone-Čech compactification of P such that $R \cap S = P$.*

We shall prove the following generalisation of 2.6.

2.7. Theorem. *Let P be a separable and metrizable space. Denote by K the Stone-Čech compactification of P. Let A be an index set of potency $2^{2^{\aleph_0}}$. For each a in A there exists a countably compact subspace P_a of K such that*

$$a_1 \in A,\ a_2 \in A,\ a_1 \neq a_2 \Rightarrow P_{a_1} \cap P_{a_2} = P\,.$$

First we prove two lemmas.

2.8. Lemma. *Let P be* a *metrizable and separable space. Every infinite subset M of βP has $2^{2^{\aleph_0}}$ accumulation points (in βP) provided that it has no accumulation point in P. Thus, if K is a compact subspace of $\beta P - P$, then K is either finite or has potency $2^{2^{\aleph_0}}$.*

Proof.[1]) The space βP is a Hausdorff space and it contains countable dense sets (since P contains countable dense sets and P is dense in βP); consequently, the potency of βP is $\leqq 2^{2^{\aleph_0}}$. It follows that the assertion about compact subspaces is an immediate corollary of the first assertion; and that to prove the first assertion it is sufficient to show that the potency of $\overline{M}^{\beta P}$ is $\geqq 2^{2^{\aleph_0}}$.

Suppose that M is an infinite subset of βP possessing no accumulation point in P. Choose a countable infinite discrete subset X of M. Consider the subspace $R = P \cup X$ of βP. First we note that R is a Lindelöf space, that is, every open covering of R contains a countable subcovering. Indeed, $R = P \cup X$ and P is Lindelöf space since it is separable and metrizable and X, being countable, is a Lindelöf space. R is a regular (moreover completely regular) Lindelöf space, and consequently, by a well-known of theorem Tychonoff, R is a normal space. By assumption X is a closed subset of R. It follows that for every bounded continuous real function f on X there exists a bounded continuous function f^* on R such that f is a restriction of f^*. Therefore every bounded continuous function on X has a continuous extension to βP. Thus the closure $\overline{X}^{\beta P}$ of X in βP is the Stone-Čech compactification of X. By a well-known

[1]) This simple proof was communicated to me by prof. M. KATĚTOV.

theorem of Pospíšil, the potency of the Stone-Čech compactification of the countable discrete space is $2^{2^{\aleph_0}}$, and hence the potency of $\overline{X}^{\beta P}$ is $2^{2^{\aleph_0}}$. In consequence the potency of $\overline{M}^{\beta P}$ is $\geqq 2^{2^{\aleph_0}}$. The proof is complete.

2.9. Lemma. *Let P be a metrizable and separable space. Let $M \subset \beta P - P$ be a set of the potency $< 2^{2^{\aleph_0}}$. There exists a countably compact space R such that $P \subset R \subset \beta P - M$ and that the potency of R is $\leqq 2^{\aleph_0}$.*

Proof. For every infinite subset X of βP choose in $\beta P - M$ an accumulation point $x(S)$ of X. By 2.8 such a function x exists. For convenience denote by $\mathfrak{N}(N)$ the family of all infinite subsets of the set N. Put $P_0 = P$ and for every ordinal number α put

$$P_\alpha = U \{x [\mathfrak{N}(R_\beta)];\ \beta < \alpha\}\ .$$

We assert that the subspace $R = P_{\omega_1}$ (ω_1 denotes the first uncountable ordinal number) of βP satisfies all the requirements of 2.9. Evidently the potency of P_0 is $\leqq 2^{\aleph_0}$ and by induction ($\alpha \leqq \omega_1$)

$$\text{card } R_\alpha \leqq \sum_{\beta < \alpha} \text{card } \mathfrak{N}(R_\beta) \leqq (2^{\aleph_0})^{\aleph_0} \cdot \aleph_1 = 2^{\aleph_0}\ .$$

Clearly $R \cap M = \emptyset$. If S is an countably infinite subset of R, then for some $\alpha < \omega_1$ we have that $S \subset P_\alpha$ and hence, the point $x(S) \in P_{\alpha+1}$. is an accumulation point of S. Thus R is a countably compact space. The proof is complete.

Proof of theorem 2,7. If P is a compact space, then theorem 2.7 is trivial. Suppose that the space P is not compact. Denote by $\mathfrak{A}$ the family of all countably compact spaces R such that $P \subset R \subset \beta P$ and that the potency of R is $\leqq 2^{\aleph_0}$. By 2.9 the family $\mathfrak{A}$ is non-void. Let $\mathfrak{L}$ be a maximal subfamily of $\mathfrak{A}$ with the property

$$(*) \qquad R_1 \in \mathfrak{B},\quad R_2 \in \mathfrak{B},\quad R_1 \neq R_2 \Rightarrow R_1 \cap R_2 = P$$

Since the condition (*) is of finite character we conclude from Tukey's lemma that such a family $\mathfrak{B}$ exists. We shall now prove that the potency of $\mathfrak{B}$ is $2^{2^{\aleph_0}}$. Evidently this potency is $\leqq 2^{2^{\aleph_0}}$. Suppose that, on the contrary, card $\mathfrak{B} = \mathfrak{m} < 2^{2^{\aleph_0}}$. Then the set $C = U \{R;\ R \in \mathfrak{B}\}$ has potency at most

$$\max\ [2^{\aleph_0}, \mathfrak{m}] < 2^{2^{\aleph_0}}\ .$$

By lemma 2.9 there exists a space $R \in \mathfrak{A}$ such that $R \cap (C - P) = \emptyset$. This contradicts the maximality of $\mathfrak{B}$.

2.10. Note. According to F. Hausdorff [2], a space P is said to be F_{II} if every closed subset of P is nonmeager in itself. It is easy to prove that every countably compact regular space is F_{II}. By 2.5 the space of all rational numbers may be imbedded as a closed subset in the topological product of two countably compact spaces. It follows that the topological product of two spaces F_{II} may fail to be F_{II}.

3.1. Definition. *Denote by $\mathfrak{C}$ the class of all completely regular spaces P such that, for every countably compact completely regular space Q, the topological product $P \times Q$ is countably compact.*

First we state the following simple proposition.

3.2. Proposition. *If P belongs to $\mathfrak{C}$ and F is a closed subspace of P, then F belongs to $\mathfrak{C}$. If both P_1 and P_2 belong to $\mathfrak{C}$, then the topological product $P_1 \times P_2$ belongs to $\mathfrak{C}$.*

The following theorem is the main result of this section.

3.3. Theorem. *A completely regular space P does not belong to $\mathfrak{C}$ if and only if it satisfies the following condition:*

There exists an infinite discrete subset N of P such that for every compactification K of P there exists a subset S of $K - P$ such that the subspace $N \cup S$ of K is countably compact.

Proof. Suppose that the condition is satisfied. By 1.4 the diagonal

$$\{(n, n) ;\quad n \in N\}$$

is an infinite discrete closed subset of $P \times (N \cup S)$ and consequently, the space P does not belong to $\mathfrak{C}$.

To prove the necessity of the condition, suppose that a completely regular space P does not belong to $\mathfrak{C}$. Hence there exists a countably compact completly regular space Q such that the topological product $P \times Q$ is not countably compact. If P is not countably compact, then it contains an infinite closed discrete subset N and we may put $S = \overline{N}^K - N$. The space $N \cup S$ being compact the condition is satisfied. Now suppose that P is countably compact. $P \times Q$ is not countably compact and therefore there exists an infinite closed discrete subset N' of $P \times Q$. Denote by π and ν the projections of $P \times Q$ onto P and Q, respectively. The mappings π and ν are open and continuous. The spaces P and Q being countably compact the sets $\pi^{-1}[x] \cap N'$ and $\nu^{-1}[y] \cap \cap N'$ are finite for each x in P and y in Q. It follows that for some infinite subset N'' of N' the sets $\pi^{-1}[x] \cap N''$ and $\nu^{-1}[y] \cap N''$ contain at most one point. That is,

$$(x, y) \in N'',\ (x', y') \in N'',\ (x, y) \neq (x', y') \Rightarrow x \neq x', y \neq y' .$$

Since every infinite Hausdorff space contains an infinite discrete subset, we can choose an infinite subset N of N'' such that the sets $\pi[N]$ and $\nu[N]$ are discrete. Put

$$N_1 = \pi[N], \quad N_2 = \nu[N]$$

Thus N is an one-to-one mapping from N_2 onte N_1. For each y in $\overline{N}_2 - N_2$ denote be $\mathfrak{A}(y)$ the family of all neighborhoods of the point y in Q.

Let K be a compactification of P. For each y in $\overline{N}_2 - N_2$ put

$$\alpha(y) = \bigcap \{\overline{N[U \cap N_2]}^K;\ U \in \mathfrak{A}(y)\} .$$

The space K being compact the sets $\alpha(y)$ are compact and non-void. Moreover, the sets $\alpha(y)$ are disjoint with P. Suppose, on the contrary, that there exists a point x in $\alpha(y) \cap P$. We assert that the point (x, y) is an accumulation point of the set N, which is impossible. Indeed let U be a neigborhood of the point x in P and let V be a neighborhood of the point y in Q. According to the definition of $\alpha(y)$ the set $U \cap N[V \cap N_2]$ is infinite, and clearly

$$N \cap (U \times V) \supset \{(x, N^{-1}[x]);\quad x \in U \cap N[V \cap N_2]\} .$$

Put

$$S = \bigcup \{\alpha(y);\quad y \in \overline{N}_2 - N_2\} .$$

It remains to prove that the space $N_1 \cup S = R$ is countably compact.

First let N' be an infinite subset of N_1. The set $N^{-1}[N']$ has an accumulation point y in $\overline{N}_2 - N_2$. It easy to see that $\alpha(y) \cap \overline{N}'^K \neq \emptyset$. Indeed, the family

$$\{N' \cap N\,[U \cap N_2]\,;\quad U \in \mathfrak{A}(y)\}$$

has the finite intersection property.

Now let N' be an infinite discrete subset of S. If for some y in $\overline{N}_2 - N_2$ the set $N' \cap \alpha(y)$ is infinite, then N' has an accumulation point in $\alpha(y)$ since $\alpha(y)$ is a compact space. In the other case the sets $\alpha(y) \cap N'$ are finite and without loss of generality they may be supposed to contain at most one point. For each x in N' choose a point $\beta(x)$ in $\overline{N}_2 - N_2$ such that $x \in \alpha(\beta(x))$. By our assumption the function β is one-to-one. In consequence the set $\beta[N']$ is infinite. Let y by an accumulation point of $\beta[N']$ in $\overline{N}_2 - N_2$. We shall prove that

(**) $$\alpha(y) \cap \overline{N}'^K \neq \emptyset$$

Denote by $\mathfrak{B}$ the family

$$\{\overline{N[U \cap N_2]}^K;\quad U \in \mathfrak{A}(y)\} .$$

By construction we have $\alpha(y) = \bigcap \{B; B \in \mathfrak{B}\}$. To prove (**) it is sufficient to show that $B \in \mathfrak{B}$ implies $B \cap \overline{N}'^K \neq \emptyset$. Suppose that $B = \overline{N[U \cap N_2]}^K$, where $U \in \mathfrak{A}(y)$. The point y is an accumulation point of the set $\beta[N']$ and therefore we may chosse an interior point y' of U belonging to $\beta[N']$. Hence $U \in \mathfrak{A}(y')$ and $\alpha(y') \subset B$. In consequence $B \cap N' \neq \emptyset$. We have thus proved that if y is an accumulation point of $\beta[N']$ in $\overline{N}_2 - N_2$, then (**) holds. An analoguous assertion holds for every infinite subset N'. Choose an accumulation point y of $\beta[N']$. It is easy to conclude that $\alpha(y) \cap N' = \emptyset$. Indeed,

$$\alpha(y) \subset \overline{N' - (x)}^K$$

for each x in N'. It follows that every point of $\alpha(y)$ is an accumulation point of N'. The proof is complete.

3.4. Lemma. *If a completly regular countably compact space P does not belong to $\mathfrak{E}$, then for some infinite discrete subset N of P every infinite subset of N has an infinite number of accumulation points.*

Proof. This is a consequence of the proof of necessity of 3.3.

3.5. Lemma. *Let P be an infinite regular countably compact space which contains a dense subset N such that every infinite subset of N has an infinite number of accumulation points. Then the potency of P is $\geqq 2^{\aleph_0}$.*

Proof. It is easy to show that every neighborhood of an accumulation point of P contains an infinite number of accumulation points. Choose open sets U_1 an U_2 such that $\overline{U}_1 \cap \overline{U}_2 = \emptyset$ and both U_1 and U_2 contain accumulation points of P. Then we choose U_{11}, U_{12}, U_{21} and U_{22} such that $U_{12} \subset U_1$, $U_{22} \subset U_2$ and $U_{j1} \cap U_{j2} = \emptyset$ $(i, j = 1, 2)$. Proceeding by induction we obtain open sets $U_{i_1 i_2 \ldots i_n}$ $(n = 1, 2, \ldots)$ such that $i_j = 1, 2$ and

$$U_{i_1 \ldots i_{n+1}} \subset U_{i_1 \ldots i_n}, \quad \overline{U}_{i_1 \ldots i_n 1} \cap \overline{U}_{i_1 \ldots i_n 2} = \emptyset .$$

For every sequence $\mathfrak{z} = \{i_n\}$ of the integers 1 and 2, the intersection

$$F\mathfrak{z} = \bigcap_{n=1}^{\infty} U_{i_1 \ldots i_n}$$

is non-void. If $\mathfrak{z}_1 \neq \mathfrak{z}_2$, then $F\mathfrak{z}_1 \cap F\mathfrak{z}_2 = \emptyset$. The potency of the set of all such sequences $\mathfrak{z}$ is $2^{\aleph_0}$. It follows that the potency of P is $\geqq 2^{\aleph_0}$.

In consequence of 3.4 and 3.5 we have:

3.6. Theorem. *If a completly regular countably compact space P does not belong to $\mathfrak{E}$, then for some infinite discrete subset N of P the potency of $\overline{N}^P$ is $2^{\aleph_0}$.*

From the proof of 3.3 and from 3.5 we may conclude the following theorem.

3.7. Theorem. *If there exists a compactification K of a countably compact space P such that the potency of $K - P$ is $< 2^{\aleph_0}$, then P belongs to $\mathfrak{E}$.*

3.8. Examples. Let N be the countable infinite discrete space. Let K be the Stone-Čech compactification of N. Denote by K_x the subspace $K - (x)$ of K. The spaces K_x belong to $\mathfrak{E}$. Put

$$P_1 = X\,\{K_x;\, x \in K - N\} .$$

The diagonal of this topological product, that is the set

$$\{y;\, \pi_x[y] = n,\ \ n \in N\}$$

(π_x denotes the projection of P_1 onto K_x) is an infinite closed discrete subset of P_1. Hence *the topological product of $2^{2^{\aleph_0}}$ spaces belonging to $\mathfrak{E}$ may fail to be countably compact.* Now let $Q \supset N$ be a countable compact subspace of K. Put

$$P_2 = X\,\{K_x;\ x \in Q - N\} .$$

The diagonal of this product, that is the set

$$\{y;\ \pi_x[y] = z \in N \cup (K - Q)\}$$

is closed in P_2 and homeomorphic with $N \cup (K - Q)$. It follows that P does not belong to $\mathfrak{C}$. By 2.8 there exists such a space Q with potency $2^{\aleph_0}$. Hence, the topological product of $2^{\aleph_0}$ spaces belonging to $\mathfrak{C}$ may fail to belong to $\mathfrak{C}$.

3.9. It may be proved that the topological product of a countable subfamily of $\mathfrak{C}$ belongs to $\mathfrak{C}$.

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Резюме

ТОПОЛОГИЧЕСКОЕ ПРОИЗВЕДЕНИЕ СЧЕТНО КОМПАКТНЫХ ПРОСТРНСТВ

ЗДЕНЕК ФРОЛИК (Zdeněk Frolík), Прага

Как известно, топологическое произведение двух счетно компактных пространств может не быть счетно компактным пространством. В работе рассматривается класс C всех вполне регулярных пространств P, для которых пространство $P \times Q$ счетно компактно для всякого счетно компактного пространства Q. Рассматриваются также пространства, гомеоморфные замкнутому подмножеству топологического произведения двух счетно компактных вполне регулярных пространств. Показывается, что таковы, например, все сепарабельные метрические пространства.