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BAIRE SPACES AND SOME GENERALIZATIONS OF COMPLETE METRIC SPACES

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A space P is said to be a Baire space if every non-void open subset of P is of the 2nd category in itself. In the present note Baire spaces are investigated. Every complete metric space is a Baire space; with this in view, we introduce in sections 2 and 3 some generalizations of complete metric spaces.

The terminology and notation of J. KELLEY "General Topology", are used troughout. For convenience, the following convention will be used: If $\mathfrak{A}$ is a family of sets and if B is a set, then $\mathfrak{A} \cap B = B \cap \mathfrak{A}$ is the family of all sets of the form $A \cap B$ where $A \in \mathfrak{A}$.

Recall that the terms "set of the 1st category" and "meager set" are synonymous. Of course, the terms "non-meager set" and "set of the 2nd category" are synonymous too. The following simple theorem will be used without references

If P is a dense subspace of a space R, then a set $M \subset P$ is meager in P if and only if it is meager in R.

1. BAIRE SPACES

For convenience we recall definitions and some basic properties.

1.1. Definition. A Baire space is a topological space every non-void open subset of which is non-meager. If every closed subspace of a space P is a Baire space, then P is said to be a Baire space in the strong sense.

Note. The term "Baire space" has been introduced in [B], Chap. 9 (les espaces de Baire). It is well-known that every complete metric space is a Baire space in the strong sense. If a Baire space P is a dense subspace of a space R, then R is a Baire space. There exists a metrizable Baire space which is not a Baire space in the strong sense. Indeed, denote by π the Euclidean plane and let P be a line in π. Let R be a countable dense subspace of P. Since $\pi - P$ is a Baire space, the space $\pi - (P - R) = Q$ is a Baire space, too. However, R is a closed subspace of Q and R is of the 1st category in itself. Moreover, R is a G_δ-subspace of Q. Thus a G_δ-subspace of a metrizable Baire space may fail to be a Baire space.

The following theorem is well-known:

1.2. *A space is a Baire space if and only if the intersection of every countable family of open dense sets is a dense set.*

Proof. If an open non-void set U is of the 1st category, then $U = \bigcup_{n=1}^{\infty} F_n$, where F_n are nowhere dense. The sets $P - \overline{F}_n$ are open and dense, and their intersection is not dense in P. Conversely, if $\{U_n\}$ is a sequence of open dense sets and if $G = \bigcap_{n=1}^{\infty} U_n$ is not dense, then there exists a non-void open set U disjoint with G. Putting $F_n = U \cap (P - U_n)$, we obtain at once that U is of the 1st category.

Now we give a characterization of Baire spaces using properties of semi-continuous functions. First we shall prove two propositions concerning semi-continuous functions.

1.3. Proposition. *Let f be a semi-continuous (real valued) function on a space P. Let D be the set of all points of P at which f is not continuous. D is of the first category (in P).*

Proof. Let f be lower semi-continuous, say. Denote by R the set of all rational numbers. For every r in R let

$$G_r = \{x; x \in P, f(x) > r\}.$$

The sets G_r being open, the sets $F_r = \overline{G}_r - G_r$ are closed and nowhere dense. If is sufficient to prove

1.3.1. $$D \subset \bigcup\{F_r; r \in R\}.$$

Let x be an element of D. There exists an $\varepsilon > 0$ such that for every neighborhood U of x and every $r \in (f(x), f(x) + \varepsilon)$ the set $U \cap G_r$ is non-void. If follows

$$r \in (f(x), f(x) + \varepsilon) \Rightarrow x \in F_r.$$

Thus 1.3.1 is proved.

1.4. Proposition. *If a space P is of the first category in itself, then there exists a bounded (lower, upper) semi-continuous function which is not continuous at any point.*

Proof. There exist closed nowhere dense sets F_n, $n = 1, 2, \ldots$, such that $P = \bigcup_{n=1}^{\infty} F_n$. For each x in P put

$$f(x) = \inf\{n; x \in F_n\}.$$

Put $g = 1 - 1/f$. Evidently, g is a bounded function on P. We shall prove that g is lower semi-continuous. It is sufficient to show that f is lower semi-continuous; but this is obvious since for every $k = 1, 2, \ldots$ we have

$$\{x; f(x) \leqq k, x \in P\} = \bigcup_{i=1}^{k} F_i.$$

Moreover, f is continuous at no point of P. Indeed, if U is a non-void open subset of P, then no finite number of sets F_n cover U. In consequence, f is not bounded on U. Clearly, the function g is continuous at no point of P, too.

As an immediate consequence of 1.3 and 1.4 we obtain:

1.5. Theorem. *A space P is of the 2nd category if and only if every semi-continuous function on P is continuous at least at one point of P.*

1.6. Theorem. *A space P is a Baire space if and only if the set of points of continuity of every (bounded) semi-continuous function is dense in P.*

Proof. From theorem 1.5 we obtain at once that the set of all points of continuity of a semi-continuous function on a Baire space P is dense in P. Conversely, suppose that P is not a Baire space. There exists a non-void meager open subset U of P. By 1.4 there exists a bounded semi-continuous function f on U which is continuous at no point. Since U is open, there exists a bounded semi-continuous function g on P such that f is the restriction of g to U. This g is continuous at no point of a non-void open set.

1.7. Theorem. *If P is a non-meager space, then the following assertion holds:*

1.7.1. *If Φ is a family of lower semi-continuous functions on P such that for every $x \in P$ the set of all $f(x)$ (where $f \in \Phi$) is bounded, then there exists a non-void open subset U of P and a real-number k such that*

1.7.2. $$y \in U,\ f \in \Phi \Rightarrow f(y) \leqq k\,.$$

Conversely, if P is of the 1st category in itself, then there exists a lower semi-continuous function f on P such that f is non-bounded from above on every non-void open subset of P.

Proof. First let us suppose that Φ is a family of lower semi-continuous functions on a space P of 2nd category in itself. Moreover, suppose that for each x in P the set of all $f(x)$, $f \in \Phi$, is bounded from above. For every $n = 1, 2, \ldots$ denote by F_n the set

$$\{x;\ x \in P,\ f \in \Phi \Rightarrow f(x) \leqq n\}\,.$$

Evidently the sets F_n are closed. P being of the 2nd category in itself, there exists a positive integer k such that the interior U of F_k is non-void. Evidently 2.7.2 is satisfied.

To prove the second assertion it is sufficient to consider the function f from the proof of 1.4. This function is lower semi-continuous and is bounded from above on no non-void open subset of P. The proof is complete.

From 1.7 we deduce at once:

1.8. Theorem. *If P is a Baire space, then the following assertion holds:*

1.8.1. *If Φ is a family of lower semi-continuous functions on P such that for each x in P the set $\{f(x);\ f \in \Phi\}$ is bounded from above, then for every non-void open* *subset U of P there exists a non-void open subset V of U and a positive integer k such that*

$$y \in V,\ f \in \Phi \Rightarrow f(y) \leqq k\,.$$

Conversely, if P is not a Baire space, then there exists a lower-semicontinuous function f on P and a non-void open subset U of P such that for every non-void open subset V of U

$$\sup \{f(x);\ x \in V\} = \infty\,.$$

1.9. Evidently, any open subspace of a Baire space is a Baire space. As we know from 2.2, a G_δ-subset of a Baire space may fail to be a Baire space. On the other hand, every G_δ-subset of a Baire space in the strong sense is a Baire space in the strong sense. Indeed, let G be a G_δ-subset of a Baire space in the strong sense P. Let F be a closed subspace of G. Evidently F is a dense G_δ-subset of $\overline{F}^P$ which is a Baire space in the strong sense. Thus F is a Baire space.

Invariance of the property of being a Baire space under mappings. The space of all rational numbers is a continuous image of the countable infinite discrete space. Thus, the continuous image of a Baire space may fail to be a Baire space. A mapping of a space P to a space Q is said to be open if the image of every open set is an open set. The image of a Baire space under an open mapping may fail to be a Baire space. Indeed, let us define a topology $\mathfrak{A}$ for the real line P in the following way: $A \in \mathfrak{A}$ if and only if for each x in A there exists the usual (Euclidean) neighborhood U of x such that all rational numbers of U belong to A.

Evidently, the identity mapping φ from $(P, \mathfrak{A})$ onto P (in the Euclidean topology) is continuous. Thus φ^{-1} is open. But P is a Baire space in the strong sense and $(P, \mathfrak{A})$ contains the space of rational numbers as an open subspace. Of course it is possible to give an example with $(P, \mathfrak{A})$ metrizable. For instance, let R be an open square in the Euclidean plane π and let P, $R \subset P \subset \overline{R}$, be a subspace of π such that $P - R$ is homeomorphic with the space of rational numbers. Making the topology of P finer in such a way that $P - R$ becomes open, we obtain a space $(P, \mathfrak{A})$ satisfying our requirements.

However, the following theorem holds:

1.10. *Let φ be a continuous and open mapping of a space P onto a space Q. If P is a Baire space, then Q is a Baire space.*

Proof. Let f be a semi-continuous function on Q. The function $g = f(\varphi)$ is semi-continuous. By 1.6 the set F of all points at which g is continuous, is dense in P. By continuity of φ, $\varphi[D]$ is dense in Q. Since φ is open, f is continuous at every point of $\varphi[D]$. By 1.6. Q is a Baire space.

Now we proceed to investigate the topological product of two Baire spaces. I do not know if the topological product of two Baire spaces is a Baire space. It is well-know that if P and Q are Baire spaces, and if moreover P is separable and metrizable, then the topological product $P \times Q$ is a Baire space.

This theorem has the following generalization:

1.7. Theorem. *The topological product of two Baire spaces is a Baire space provided that one of them contains a countable dense subset N such that every point of N is of a countable character (t. e. every point of N posseses a countable complete neighborhood system).*

To prove 1.7 it is sufficient to prove the following theorem:

1.8. Theorem. *Let P and Q be non-meager spaces. Moreover, suppose there exists a countable dense subset N of P such that every point of N is of a countable character. Then the topological product $P \times Q$ is a non-meager space.*

Proof of 1.8. Let us suppose that there exist spaces P and Q satisfying the assumptions of 1.8, and such that $P \times Q$ is a meager space. Thus there exists a sequence $\{F_s\}$ *of closed nowhere dense subsets of* $R = P \times Q$ *such that* $\bigcup\{F_k;\ k = 1, 2, \ldots\} = R$. For each n in N let $\{U(n, k);\ k = 1, 2, \ldots\}$ be a local base at the point n. Put

$$Y(n, k, s) = \{y;\ y \in Q,\ U(n, k) \times (y) \subset F_s\},$$
$$Y = \bigcup\{Y(n, k, s);\ n \in N,\ (k, s = 1, 2, \ldots)\}.$$

The sets F_s being closed and nowhere dense (in R), the sets $Y(n, k, s)$ are closed and nowhere dense (in Q). It follows that the set Y is of the first category in Q. Therefore we may choose a point y_0 in $Q - Y$. Consider the space $P_0 = P \times (y_0)$ (homeomorphic with P). The sets $F'_s = F_s \cap P_0$ are closed and cover the non-meager space P_0. Hence there exists an s such that Int $F'_s = U \neq \emptyset$. The set $N \times (y_0)$ being dense in P_0, there exists an n in N such that $(n, y_0) \in U$. Since U is open there exists a neighborhood $U(n, k)$ of n such that $U(n, k) \times (y_0) \subset U$. It follows that $y_0 \in Y(n, k, s) \subset Y$. But y_0 has been choosen in $Q - Y$. This contradiction completes the proof.

In the third section we shall introduce a class of spaces (countably complete spaces) contained in that of Baire spaces and such that the topological product of an arbitrary indexed family spaces of this class is again a member of this class. In the third section we give a further generalization, by introducing the concept of an almost countably complete space. The class O of all almost countably complete spaces has the following properties:

1. $P \in O$, R regular, $\bar{P} = R$ imply $R \in O$.
2. $P \in O$, P regular, R a dense G_δ-subset of P imply $R \in O$.
3. If every $x \in P$ has a neighborhood belonging to O, then P belongs to O.
4. If a regular space P belongs to O, then every open subset of P belongs to O.
5. If a regular space P belongs to O, then P is a Baire space.
6. If $\{P_a;\ a \in A\}$ is a family of spaces from O, then the topological product $X\{P_a;\ a \in A\}$ belongs to O.
7. If $\{P_a;\ a \in A\}$ is a family of spaces from O, then the cartesian product $X\{P_a;\ a \in A\}$ in the box-topology is a space from O.

We have proved that the class of all Baire spaces possesses the properties (1)–(5).

The question concerning the topological product of two Baire spaces in the strong sense is more simple. Indeed, every regular countably compact space is a Baire space in the strong sense. According to [2], theorem 2.6, every separable metrizable space may be embedded as a closed subspace into the topological product of two completely regular countably compact spaces. Embedding the space of all rational numbers in the topological product of two completely regular countably compact spaces, we obtain at once that the topological product of two (completely regular) Baire spaces in the strong sense may fail to be a Baire space in the strong sense.

It is well-known that every complete metric space is a Baire space in the strong sense. The topological product of un uncountably infinite family of complete metric spaces may fail to be a Baire space in the strong sense. Indeed, every separable metrizable space may be embedded as a closed subspace into the topological product of an indexed family of real lines (e. g. see [4]).

2. COUNTABLY COMPLETE SPACES

In this section we shall introduce two generalizations of complete metric spaces.

2.1. Definition. Let $\{\mathfrak{A}_n\}$ be a sequence of open families[1]) in a space P. The sequence $\{\mathfrak{A}_n\}$ is said to be countably complete if for every centred sequence of sets $\{A_{n_k}\}$, where $A_{n_k} \in \mathfrak{A}_{n_k}$, the set $\bigcap_{k=1}^{\infty} \bar{A}_{n_k}$ is non-void. The sequence $\{\mathfrak{A}_n\}$ is said to be strongly countably complete if the following condition is satisfied.

2.1.1. If $\{F_n\}$ is a centered sequence of closed subsets of P and if every F_n is contained in some $A_n \in \mathfrak{A}_n$, then the set $\bigcap_{n=1}^{\infty} F_n$ is non-void.

2.2. Note. In [3], the concept of a complete sequence of open coverings was introduced. A sequence $\{\mathfrak{A}_n\}$ of open coverings of a space P is said to be complete if the following condition is satisfied:

2.2.1. If $\mathfrak{F}$ is a centered family of closed subsets of the space P such that for every $n = 1, 2, \ldots$ some $F_n \in \mathfrak{F}$ is contained in some $A_n \in \mathfrak{A}_n$, then $\bigcap \mathfrak{F} \neq \emptyset$.

Now we recall theorem from [3]:

2.2.2. Theorem. *A strongly countably complete sequence of open coverings of a regular space P is complete if and only if the following condition is satisfied*:

If F is a closed subspace of P and if for every $n = 1, 2, \ldots$ there exists a finite subfamily $\mathfrak{B}_n$ of $\mathfrak{A}_n$ such that the union of the family $\mathfrak{B}_n$ contains F, then F is a compact space.

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[1]) An open family is a family consisting of open sets.

2.3. Definition. A space P is said to be countably complete if there exists a countably complete sequence of open bases for P. P is said to be strongly countably complete if there exists a strongly countably complete sequence of open coverings of P.

It is easy to see that, every regular strongly countably complete space is a countably complete space. By [3], theorem 2.8, a completely regular space P possesses a complete sequence of open coverings if and only if it is a G_δ-space, i. e. if P is G_δ in every compactification. There exists a completely regular strongly countably complete space which is not a G_δ-space. Indeed, according to [3], example 3.10, there exists a completely regular countably compact space which is not a G_δ-space. On the other hand the following theorem is true.

2.4. Theorem. *Every metrizable countably complete space is a G_δ-space.*

Proof. Let $\{\mathfrak{A}_n\}$ be a complete sequence of open coverings of a metrizable space P. Let φ be a metric for the space P. Without loss of generality we may assume that the diameters of sets from $\mathfrak{A}_n$ are less than $1/n$. Let (P^*, φ^*) be the complete envelope of the metric space (P, φ). We may assume that $P \subset P^*$. It is sufficient to show that P is a G_δ-subset of P^*, since P^* is a G_δ-space. For every open subset A of P choose an open subset A' of P^* such that $A' \cap P = A$. Put

2.4.1. $$U_n = \bigcup\{A'; A \in \mathfrak{A}_n\},$$

2.4.2. $$G = \bigcap\{U_n; n = 1, 2, \ldots\}.$$

To prove 2.4, it is sufficient to show that $G = P$. By 2.4.1 and 2.4.2 we have $P \subset G$. Let us suppose that there exists a point x in $G - P$. Choose A_n in $\mathfrak{A}_n$ $(n = 1, 2, \ldots)$ such that $x \in A'_n$. The diameters of A'_n being less than $1/n$ $(n = 1, 2, \ldots)$ we have at once that

2.4.3. $$\bigcap\{A'_n; n = 1, 2, \ldots\} = (x).$$

Since the point x is an accumulation point of P, the sequence $\{A_n\}$ is centered. In consequence

2.4.4. $$\bigcap\{\overline{A}_n^P; n = 1, 2, \ldots\} \neq \emptyset.$$

Since $x \notin P$, 2.4.4 contradicts 2.4.3. Thus the proof is complete.

Evidently:

2.5. Locally countably H-closed spaces are countably complete; locally countably compact spaces are strongly countably complete. Every closed subspace of a strongly countably complete space is strongly countably complete.

2.6. Proposition. *Every open subset of a regular countably complete space is a countably complete space.*

Proof. Let $\{\mathfrak{A}_n\}$ be a countably complete sequence of open bases for a regular space P. Since U is a non-void open subset of P, for every $n = 1, 2, \ldots$ denote by $\mathfrak{B}_n$ the family of all $A \in \mathfrak{A}_n$ for which $\overline{A} \subset U$. It is easy to show that $\{\mathfrak{B}_n\}$ is a countably complete sequence of open basis for U.

2.7. Proposition. *Every G_δ-subspace of a regular strongly countably complete space is a strongly countably complete space.*

Proof. Let $\{\mathfrak{A}_n\}$ be a strongly countably complete sequence of open coverings of a regular space P. Let U_n, $n = 1, 2, \ldots$, be open subsets of P and let

2.7.1. $$\emptyset \neq G = \bigcap\{U_n;\ n = 1, 2, \ldots\}.$$

Without loss of generality we may assume

$$A \in \mathfrak{A}_n,\ B \text{ open},\ B \subset A \Rightarrow B \in \mathfrak{A}_n .$$

Denote by $\mathfrak{A}'_n$ the set of all $A' \in \mathfrak{A}_n$ such that for some A in $\mathfrak{A}_n$, $\bar{A}' \subset A \cap U_n$. Put

$$\mathfrak{B}_n = \mathfrak{A}'_n \cap G \quad (n = 1, 2, \ldots).$$

We shall prove that $\{\mathfrak{B}_n\}$ is a strongly countably complete sequence of open coverings of G. Suppose that $\{F_n\}$ is a centered sequence of closed subsets of G such that for every $n = 1, 2, \ldots$ some $B_n \in \mathfrak{B}_n$ contains F_n.

Choose A'_n in $\mathfrak{A}'_u$ and A_n in $\mathfrak{A}_n$ such that

$$B_n = G \cap \bar{A}'_n, \quad A'_n \subset A_n \cap U_n .$$

The sequence $\{\bar{F}^P_n\}$ is centered and $\bar{F}^P_n \subset A_n$. Thus

$$F = \bigcap\{\bar{F}^P_n;\ n = 1, 2, \ldots\} \neq \emptyset .$$

Since $\bar{F}_n \subset U$, by 2.7.1 we have $F \subset G$. Also F is closed in G, so that $F \subset F_n$, $n = 1, 2, \ldots$ Thus $F = \bigcap_{n=1}^{\infty} F_n \neq \emptyset$, which establishes 2.7.

2.8. A space P is said to be locally countably complete if and only if every $x \in P$ has a neighborhood which is a countably complete space.

2.9. Proposition. *Let P be a locally countably complete regular space. There exists an open dense countably complete subspace of P.*

Proof. Let $\mathfrak{A}$ be the family of all non-void open countably complete subspaces of P. By 2.6 we have

2.9.1. $$B \neq \emptyset \text{ open},\ B \subset A \in \mathfrak{A} \Rightarrow B \in \mathfrak{A} .$$

Let $\mathfrak{B}$ be a maximal disjoint subfamily of $\mathfrak{A}$. Let R be the union of $\mathfrak{B}$. From the maximality of $\mathfrak{B}$ and from 2.9.1 we have at once that R is a dense subspace of P. Evidently R is an open subset of P. We shall prove that R is a countably complete space. For every B in $\mathfrak{B}$ let

$$\{\mathfrak{A}_n(B);\ n = 1, 2, \ldots\}$$

be a countably complete sequence of open bases of B. Evidently

$$\{\bigcap\{\mathfrak{A}_n(B);\ B \in \mathfrak{B}\};\ n = 1, 2, \ldots\}$$

is a countably complete sequence of open bases of R.

2.10. Theorem. *The topological product of an arbitrary indexed family of countably complete spaces is a countably complete space.*

Proof. Let $\{P_a;\ a \in A\}$ be a non-void indexed family of countably complete spaces and let
$$\Phi \neq P = X\{P_a;\ a \in A\}\ .$$
For every a in A let $\{\mathfrak{A}_n^a;\ n = 1, 2, \ldots\}$ be a countably complete sequence of open bases of P_a. Denote by π_a the projection map of P onto P_a. Let $\mathfrak{A}_n\,(n = 1, 2, \ldots)$ be the family of all sets of the form
$$\pi_{a_1}^{-1}[B_1] \cap \ldots \cap \pi_{a_k}^{-1}[B_k]\ ,$$
where $k = 1, 2, \ldots$, a_i are element of A, $a_i \neq a_j$, and $B_i \in \mathfrak{A}_n^{a_i}$. Evidently every family $\mathfrak{A}_n$ is an open base for P. We shall prove that the sequence $\{\mathfrak{A}_n\}$ is countably complete. Let $\{n_k\}$ be an increasing sequence of positive integers. Let $\{C_{n_k}\}$, where $C_{n_k} \in \mathfrak{A}_{n_k}$, be a centred sequence. We have
$$C_{n_k} = X\{C_{n_k}^a;\ a \in A\}$$
where either $C_{n_k}^a = P_a$ or $C_{n_k}^a \in \mathfrak{A}_n^a$. By countably completness of $\{\mathfrak{A}_n^a\}$, for each a in A we have
$$\bigcap\{\bar{C}_{n_k}^a;\ k = 1, 2, \ldots\} \neq \emptyset\ .$$
Choose a point x_a in this intersection.

The point $\{x_a;\ a \in A\}$ belongs to $\bigcap_{k=1}^{\infty} \bar{C}_{n_k}$. Thus the proof is complete.

2.11. Definition. Let $\{P_a;\ a \in A\}$ be an indexed family of spaces. Let $P = X\{P_a;\ a \in A\}$ be the product of the indexed family of sets $\{P_a;\ a \in A\}$. Then the family of all sets of the form $X\{U_a;\ a \in A\}$ where U_a are open in P_a, is an open base for the box-topology for P.

By the similar arguments as in 2.10 it can be proved that

2.12. Box-toplogy product of an indexed family of countably complete spaces is an countably complete space.

2.13. Topological product of two strongly countably complete spaces may fail to be strongly countably complete. Moreover, by [2], Theorem 2.5, every separable metrizable space may be embedded as a closed subspace in the topological product of two countably compact completely regular spaces.

3. ALMOST COUNTABLY COMPLETE SPACES

3.1. Definition. An open almost-base for a space P is a family $\mathfrak{A}$ of open subsets of P such that every non-void open subset of P contains some non-void $A \in \mathfrak{A}$. A space is said to be almost countably complete if there exists a countably complete sequence of open almost-bases for P.

By a similar argument as in 2.6 we may prove

3.2. Proposition. *Every open subspace of a regular almost countably complete space is an almost countably complete space.*

3.3. Proposition. *Suppose that R is an almost countably complete dense subspace of a space P. Then P is an almost countably complete space.*

Proof. Let $\{\mathfrak{A}_n\}$ be a countably complete sequence of open almost-bases of R. For every $n = 1, 2, \dots$ denote by $\mathfrak{B}_n$ the family of all open subsets U of P with $U \cap R \in \in \mathfrak{A}_n$. If $V \neq \emptyset$ is an open subset of P, there exists an A in $\mathfrak{B}_n$ such that $U \subset V \cap R$. By definition of $\mathfrak{B}_n$, there exists a non-void set B in $\mathfrak{B}_n$ such that $A \subset B \subset V$. Thus $\mathfrak{B}_n$ is an open almost-base for P. Since R is a dense subspace of P we obtain at once that a family $\mathfrak{A}$ of open subsets of P is centered if and only if the family $R \cap \mathfrak{A}$ is centered. Thence we conclude that $\{\mathfrak{B}_n\}$ is a countably complete sequence. The proof is complete.

By a similar argument as in 2.9 we may deduce from 3.2 that a locally almost countably complete space contains an open dense subspace which is almost countably complete. From this fact and from 3.3 we have at once:

3.4. Theorem. *Every regular locally almost countably complete space is an almost countably complete space.*

3.5. Theorem. *Every dense G_δ-subspace of a regular almost countably complete space is an almost countably complete space.*

Proof. Let $\{\mathfrak{A}_n\}$ be a countably complete sequence of open almost-bases of a regular space P. Let U_n $(n = 1, 2, \dots)$ be open subsets of P and let

$$G = \bigcap\{U_n;\ n = 1, 2, \dots\}$$

be a dense subset of P. Denote by $\mathfrak{B}_n$ $(n = 1, 2, \dots)$ the family of all sets of the form $A \cap G$, where $A \in \mathfrak{A}_n$ and $\bar{A} \subset U_n$. Evidently every $\mathfrak{B}_n$ is an open almost base of G. It is easy to show that $\{\mathfrak{B}_n\}$ is a countably complete sequence.

As in 2.11 and 2.13 we can prove:

3.6. Theorem. *The topological product of almost countably complete spaces is an almost countably complete space. The Cartesian product in the box-topology of almost countably complete spaces is an almost countably complete space.*

3.7. Theorem. *Every regular almost countably complete space is a Baire space.*

Proof. Let P be a regular almost countably complete space and let U_n $(n = 1, 2, \dots)$ be an open dense subset of P. Put $G = \bigcap_{n=1}^{\infty} U_n$. It remains to prove that $U \cap G \neq \emptyset$ for every non-void open subset U of P. Let $\{\mathfrak{A}_n\}$ be a countably complete sequence of open almost bases of P. The space P being regular, there exist $A_n \in \mathfrak{A}_n$ $(n = 1, 2, \dots)$ such that $\bar{A}_1 \subset U \cap U_1$ and $\bar{A}_k \subset A_{k-1} \cap U_k$ $(k = 2, 3, \dots)$. By countable completeness of $\{\mathfrak{A}_n\}$ we have

$$F = \bigcap\{\bar{A}_k;\ k = 1, 2, \dots\} \neq \emptyset\,.$$

But $F \subset U \cap G$. Thus the proof is complete.

3.8. Theorem. *Every metrizable almost countably complete space P contains a dense countable complete space R. (R being metrizable, R is a G_δ-space).*

Proof. Let $\{\mathfrak{A}_n\}$ be a countably complete sequence of almost bases of a metrizable space P. Let φ be a metric for P. For every $k, n = 1, 2, \ldots$ denote by $\mathfrak{A}_n^k$ the family of all $A \in \mathfrak{A}_n$ such that the diameter of A is $\leqq 1/k$. Put

$$\mathfrak{A}'_1 = \mathfrak{A}_1^1, \quad U_1 = \bigcap \{A;\ A \in \mathfrak{A}'_1\}.$$

For every $p \geqq 2$ denote by $\mathfrak{A}'_p$ the family of all $A \in \mathfrak{A}_p^p$ for which

$$\bar{A} \subset \Big[\bigcap_{k=1}^{p-1} \bigcup\{B;\ B \in \mathfrak{A}_k^p\}\Big] \cap U_{p-1},$$

and put

$$U_p = \bigcup\{A;\ A \in \mathfrak{A}'_p\}.$$

By induction it is easy to prove that U_p, $p = 1, 2, \ldots$, are dense subspaces of P. Since U_p is open, from theorem 3.7 we have that

$$G = \bigcap\{U_p;\ p = 1, 2, \ldots\}$$

is a dense subspace of P. Put

$$\mathfrak{B}_n = \mathfrak{A}'_n \cap G \quad (n = 1, 2, \ldots).$$

By construction, for every n the family $\mathfrak{B}_n$ is an open base for G. Indeed, the sets $A \in \mathfrak{A}'_n$ of diameter less than $1/k$ cover G for $k = 1, 2, \ldots$

It remains to prove that the sequence $\mathfrak{B}_n$ is countably complete. Suppose $B_{n_k} \in \mathfrak{B}_{n_k}$ $(k = 1, 2, \ldots)$ and let the sequence $\{B_{n_k}\}$ be centered. Choose A_{n_k} in $\mathfrak{A}'_{n_k}$ such that $B_{n_k} = A_{n_k} \cap G$. The sequence $\{\mathfrak{A}_n\}$ being countably complete (and $\mathfrak{A}'_n \subset \mathfrak{A}_n$), we have

$$\Phi \neq F = \bigcap\{\bar{A}_{n_k}^P;\ k = 1, 2, \ldots\}.$$

But for every $k \geqq 2$ $\bar{A}_{n_k}^P \subset U_{n_k - 1}$, and consequently $F \subset G$. Since G is a dense subset of P and A_{n_k} open, B_{n_k} is a dense subset of A_{n_k}. It follows that for every k,

$$F \subset \bar{A}_{n_k} \subset \bar{\bar{B}}_{n_k} = \bar{B}_{n_k}.$$

Since $F \subset G$ and $B_{n_k} \subset G$, F is contained in $\bar{B}_{n_k}^G$. Thus G is a countably complete space.

3.9. Corollary. *There exists a metrizable Baire space which is not almost countably complete.*

Proof. There exists a metrizable Baire space which contains no dense G_δ-space.

Added in proof: See the paper of Oxtoby, Fund. Math. 1961, Nr. 1.

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Резюме

ПРОСТРАНСТВА БЭРА И НЕКОТОРЫЕ ОБОБЩЕНИЯ ПОЛНЫХ МЕТРИЧЕСКИХ ПРОСТРАНСТВ

ЗДЕНЕК ФРОЛИК (Zdeněk Frolík), Прага

Топологическое пространство P называется пространством Бэра, если всякое непустое открытое $U \subset P$ является множеством второй категории.

В первой части дается определение пространств Бэра при помощи полунепрерывных функций. Автору неизвестно, является ли топологическое произведение двух пространств Бэра опять пространством Бэра. В статье доказываетя следующее обобщение классической теоремы:

Если P и Q — пространства Бэра и если P содержит плотное счетное множество N такое, что всякая точка множества N имеет счетный характер, то топологическое произведение $P \times Q$ является пространством Бэра.

Во второй части определены *счетно полные пространтва*. Оказывается, что метризуемое пространство P счетно полно тогда и только тогда, если оно является G_δ-пространством, (иначе говоря, если оно полно при некоторой метризации). Всякое регулярное счетно полное пространство является пространством Бэра и топологическое произведение счетно полных пространств является счетно полным пространством.

В третьей чати определены т. наз. *почти счетно полные пространства*. Семейство O всех почти счетно полных пространств имеет следующие свойства:

1. *Всякое регулярное пространство из O является пространством Бэра.*

2. *Топологическое произведение любой системы пространств из O принадлежит O.*

3. *Всякое открытое подпространство всякого регулярного пространства из O принадлежит семейству O.*

4. *Если некоторое пространство из O является плотным в пространстве P, то P тоже принадлежит семейству O.*

5. *Если для всякой точки x регулярного пространства P некоторая окрестность точки x принадлежит семейству O, то само пространство P принадлежит семейству O.*

6. *Всякое плотное G_δ-подпространство регулярного пространства из O принадлежит семейству O.*

7. *Если $P \in O$ метризуемо, то существует плотное $\boldsymbol{R} \subset P$, являющееся G_δ-пространством, т. е., существует метрика φ так, что метрическое пространство $(\boldsymbol{R}, \varphi)$ полно.*