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Czechoslovak Mathematical Journal, Vol. 12 (1962), No. 3, 346–355

Persistent URL: `http://dml.cz/dmlcz/100522`

Чехословацкий математический журнал, т. 12 (87) 1962, Прага

LOCALLY G_δ-SPACES

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(Received July 11, 1960)

Locally topologically complete (in the sense of E. ČECH) completely regular spaces are investigated.

The notation and terminology of J. KELLEY, General Topology, are used throughout. All spaces are supposed to be completely regular. The Stone-Čech compactification of a space P will be always denoted by $\beta(P)$.

A space P is said to be a G_δ-space (or topologically complete in the sense of E. Čech) if P is a G_δ-subset of $\beta(P)$. If a G_δ-space P is a dense subspace of a space Q, then P is a G_δ-subset of Q. In [2] the following "internal" (*i.e.* without reference to larger spaces) characterization of G_δ-spaces is given:

A space is a G_δ-space if and only if there exists a complete sequence of open coverings of the space P.

Let us recall that a sequence $\{\mathfrak{A}_n\}$ of open coverings is said to be complete if, whenever $\mathfrak{M}$ is a family of subsets of P with the finite intersection property such that $\mathfrak{M} \cap \mathfrak{A}_n \neq 0$ for all $n = 1, 2, \ldots$, then the intersection of all $\overline{M}$, $M \in \mathfrak{M}$, is non-void:

In the sequel we shall need the following two propositions (for proofs see [2]).

Proposition 1. *If P is a G_δ-space, then there exists a complete sequence $\{\mathfrak{A}_n\}$ of open coverings of P such that* (1) *every $\mathfrak{A}_n$ is additive, i.e. $A \in \mathfrak{A}_n$, $B \in \mathfrak{A}_n$ imply $A \cup B \in \mathfrak{A}_n$.*

(2) $$\mathfrak{A}_n \supset \mathfrak{A}_{n+1} \quad (n = 1, 2, \ldots)\,.$$

(3) *If an open set B is contained in some $A \in \mathfrak{A}_n$, then B belongs to $\mathfrak{A}_n$.*

Proposition 2. *If $\{\mathfrak{A}_n\}$ is a complete sequence of open coverings, $A_n \in \mathfrak{A}_n$, $A = \bigcap_{n=1}^{\infty} A_n$, then the closure of A is a compact space.*

In section 1 the concept of a locally G_δ-space is introduced and studied. Every paracompact locally G_δ-space is a G_δ-space; the assumption of paracompactness is essential. The topological product of a countably compact locally G_δ-space and a countably space (pseudocompact space) is a countably compact space (pseudocompact space, respectively). In locally G_δ-spaces the concepts of local characters (in the sense of E. Čech) and local-pseudocharacters coincide.

In section 2 the invariance under mappings of locally G_δ-spaces (and locally compact spaces, G_δ-spaces) is studied. The image Q under a closed and continuous mapping of a metrizable G_δ-space is a G_δ-space if and only if Q is metrizable.

1. LOCALLY G_δ-SPACES

Definition 1. A space P is said to be locally a G_δ-space, if every point of P is contained in an open subspace R of P which is a G_δ-space.

The following example shows that a space which is locally a G_δ-space may fail to be a G_δ-space.

Example 1. Let T be the space of all countable ordinals. Let us denote by K the set of all real numbers x, $0 \leqq x < 1$. Let us define an order $>$ for the set $R = T \times K$ such that $(\alpha, x) > (\beta, y)$ if and only if either $\alpha > \beta$ (in T) or $\beta = \alpha$ and $x > y$ (in K). The set R with the order topology is a space which will be denoted by R again. It is well-known that the space R is locally compact, countably compact and locally metrizable. Let us denote by R' the set of all points $(\alpha, 0)$ of R where α runs over all isolated ordinals in T. Evidently the set R' is discrete. The space R being countably compact, every set $M \subset R'$ closed in R is finite. Thus R' is not a F_σ-subset of R. It follows that $P = R - R'$ is not a G_δ-subset of R. Since P is dense in R, by definition P is not a G_δ-space. On the other hand it is easy to see that P is locally a G_δ-space and locally metrizable.

The space P from the preceding example 1 contains a dense locally compact subspace $T \times I$ where I is the open interval $(0, 1)$ of real numbers. Generalising, we shall deduce the following .

Theorem 1. *Every space which is locally a G_δ-space contains a G_δ-space as an open and dense subspace.*

Proof. Let P be locally a G_δ-space. Let us consider the family $\mathfrak{A}$ of all open subspaces of P which are G_δ-spaces. Let $\mathfrak{B}$ be a maximal disjoint subfamily of $\mathfrak{A}$. Evidently, if an open set B is contained in an $A \in \mathfrak{A}$, then B also belongs to $\mathfrak{A}$. It follows at once that the union R of the family $\mathfrak{B}$ is a dense subset of P. Of course, R is an open subset of P. We shall prove that R is a G_δ-space. For every B in $\mathfrak{B}$ there exists a complete sequence $\{\mathfrak{A}_n(B)\}$ of open coverings of the space B. Put

$$\mathfrak{A}_n = U\{\mathfrak{A}_n(B), B \in \mathfrak{B}\} \quad (n = 1, 2, \ldots) .$$

It is easy to see that $\{\mathfrak{A}_n\}$ is a complete sequence of open coverings of the space R. Thus R is a G_δ-space.

To show that every paracompact space which is locally a G_δ-space is a G_δ-space, we shall prove the following

Theorem 2. *If there exists a locally finite open covering $\mathfrak{B}$ of the space P such that every $B \in \mathfrak{B}$ is a G_δ-space, then P is a G_δ-space.*

Proof. According to the proposition 2, for every B in $\mathfrak{B}$ then exists a complete sequence $\{\mathfrak{A}_n(B)\}$ of open coverings of the space B such that

(1) $$\mathfrak{A}_{n+1}(B) \subset \mathfrak{A}_n(B) \quad (n = 1, 2, \ldots)$$

and

(2) every $A \in \mathfrak{A}_n(B)$, $B \in \mathfrak{B}$, $n = 1, 2, \ldots$, meets only a finite number of sets from $\mathfrak{B}$ For every $n = 1, 2, \ldots$ put

$$\mathfrak{A}_n = \bigcup\{\mathfrak{A}_n(B), B \in \mathfrak{B}\}\,.$$

We shall prove that $\{\mathfrak{A}_n\}$ is a complete sequence of open coverings of P. Let $\mathfrak{M}$ be a family (of subsets of P) with the finite intersection property and such that $\mathfrak{A}_n \cap \mathfrak{M} \neq 0$ for every $n = 1, 2, \ldots$. To prove $\bigcap \overline{\mathfrak{M}} \neq 0$, it is sufficient to show that $\mathfrak{A}_n(B) \cap \mathfrak{M} \neq 0$ for some B in $\mathfrak{B}$ and all $n = 1, 2, \ldots$. Let us choose an A in $\mathfrak{A}_1 \cap \mathfrak{M}$. According to the assumption (2) there exists only a finite number of sets $B \in \mathfrak{B}$ meeting A, namely $B_1, \ldots, B_k$. It follows that for every $n = 1, 2, \ldots$,

$$\mathfrak{A}_n \cap \mathfrak{M} \subset \bigcup_{i=1}^{k} (\mathfrak{A}_n(B_i) \cap \mathfrak{M})\,.$$

Thus there exists a $B_i = B$ such that $\mathfrak{A}_n(B) \cap \mathfrak{M} \neq 0$ for an infinite number of n. According to the assumption (1) we have

$$\mathfrak{A}_n(B) \cap \mathfrak{M} \neq 0\,, \quad n = 1, 2, \ldots\,;$$

this completes the proof.

As an immediate consequence of the preceeding theorem we have:

Theorem 3. *Every paracompact space which is locally a G_δ-space is a G_δ-space.*

Theorem 4. *Let a space P be locally a G_δ-space. Let us suppose that a countable subset N of P has an accumulation point. Then there exists a compact subspace K of P such that $K \cap N$ is an infinite set. Moreover, if U is an open set containing an accumulation point of the set N, then K may be chosen with $K \subset U$.*

Proof. First let us suppose that P is a G_δ-space. Let x be an accumulation point of the set N. Let $\{\mathfrak{A}_n\}$ be a complete sequence of open coverings of the space P. Let us choose $A_n \in \mathfrak{A}_n$, $n = 1, 2, \ldots$, containing the point x, and put $A = \bigcap_{n=1}^{\infty} A_n$. Using the completness of the sequence $\{\mathfrak{A}_n\}$ it is easy to see that the closure of A is a compact subspace of P. Thus, if x is an accumulation point of $\bar{A} \cap N$, then we may put $K = \bar{A}$. In the other case we may assume without loss of generality that $N \cap \bar{A} = 0$. Since $x \in \bar{N}$ and the sets A_n are open, we have at once that the sets $A_n \cap N$ are infinite. Thus we can construct by induction an infiinite subset X of N such that the sets $X - A_n$ are finite. It is easy to prove that $\bar{X}$ is a compact space. Indeed, if $\mathfrak{M}$ is a maximal family of subsets of X with the finite intersection property, then the sets $A_n \cap X$ belong to $\mathfrak{M}$, and hence, by completness of the sequence $\{\mathfrak{A}_n\}$, the intersection of all $\overline{M}$, $M \in \mathfrak{M}$, is non-void. It follows that $\overline{X}$ is a compact space.

Now let the space P be locally a G_δ-space and let U be an open neighborhood of an accumulation point x of the set N. There exists an open neighborhood V of x such that V is a G_δ-space. Let us consider the G_δ-space $V \cap U$ and the infinite set $N \cap U \cap V$ having x as an an accumulation point. Applying the first part of the proof we obtain a compact set $K \subset U \cap V$ such that $K \cap N$ is an infinite set. The proof is complete.

Theorem 5. *Let a space P be locally a G_δ-space. Let us suppose that a disjoint countable infinite family $\mathfrak{N}$ of open subsets of P has an accumulation point $x \in P$ (that is, every neighborhood of x meets an infinite number of members of $\mathfrak{N}$). If U is an open neighborhood of x, then there exists a compact subspace K of U meeting an infinite number of sets from $\mathfrak{N}$.*

Proof. First let us suppose that P is a G_δ-space. Let $\{\mathfrak{A}_n\}$ be a complete sequence of open coverings of the space P. Let us choose $A_n \in \mathfrak{A}_n$, $n = 1, \ldots$ containing the point x, such that $A_n \subset A_{n+1}$, and put $A = \bigcap_{n=1}^{\infty} \overline{A}_n$. It is easy to see that the set A is compact. Thus, if x is an accumulation point of $A \cap \mathfrak{N} = \{A \cap N; N \in \mathfrak{N}\}$, then we may put $A = K$. In the other case we may assume without loss of generality that $A \cap N = 0$ for every N in $\mathfrak{N}$. The sets A_n being open, the families $\{N; N \cap A_n \neq 0, N \in \mathfrak{N}\}$ are infinite. By induction we can construct a sequence $\{N_n\}$ of distinct sets from $\mathfrak{N}$ and choose x_n such that $x_n \in N_n \cap A_n$. Let X be the set of all x_n. Clearly the sets $X - A_n$ are finite. Thus the closure K of the set X is a compact subspace of P and $K \cap N \neq 0$ for an infinite number of $N \in \mathfrak{N}$.

Now the same argument as that in the proof of the preceding theorem completes the proof of theorem 5.

As a consequence of the preceding two theorems we have the following

Theorem 6. *Let a space P be locally a G_δ-space. If P is a countably compact space, then for every countably compact space Q the topological product $P \times Q$ is countably compact. If P is a pseudocompact space, then for every pseudocompact space Q the topological product $P \times Q$ is pseudocompact.*

Proof. It is well-known that if K is a compact space and a space Q is countably compact (or pseudocompact) then the topological product $K \times Q$ is countably compact (or pseudocompact, respectively). Now let P and Q be countably compact spaces. Let N be a countably infinite subset of $P \times Q$. If the projection M of N into P is a finite set, then for some $x \in P$ the set $N \cap [(x) \times Q]$ is infinite. The space $(x) \times Q$ is countably compact and hence the set N has an accumulation point $(x) \times Q$. In the other case we may choose a compact subspace K of P such that $M \cap K$ is infinite. The product space $K \times Q$ is countably compact and hence the infinite set $N \cap \cap (K \times Q)$ has an accumulation point. Thus N has an accumulation point and the product space $P \times Q$ is countably compact. The assertion concerning pseudocompactness may be proved analogously.

Note 1. Let C be the class of all spaces P such that the topological product $P \times Q$ is countably compact for every countably compact space Q. The class C was introduced and studied in [3]. It is proved there that every countably compact space which is locally a G_δ-space belongs to C.

Let us denote by $\mathfrak{P}$ the class of all spaces P such that the topological product $P \times Q$ is pseudocompact for every pseudocompact space Q. Let us denote by $\mathfrak{P}_F$ the family of all spaces P such that every closed subspace of P belongs to $\mathfrak{P}$. The classes $\mathfrak{P}$ and $\mathfrak{P}_F$ were introduced and studied in [4]. It is proved there that every pseudocompact space which is locally a G_δ-space belongs to $\mathfrak{P}$. It is easy to see that every countably compact space which is locally a G_δ-space belongs to $\mathfrak{P}_F$. Of course there exists a space from the class $\mathfrak{P}_F$ which is not locally a G_δ-space. There exists a locally compact pseudocompact space which is not countably compact (and consequently, which does not belong to $\mathfrak{P}_F$). For example, if N is the countable infinite discrete space and if X is a countable infinite discrete subspace of $\beta(N) - N$, then the space $P = \beta(N) - (\overline{X} - X)$ is locally compact and pseudocompact (since every infinite subset of N has an accumulation point in P), but P is not countably compact, since the set X has no accumulation point in P.

Note 2. Let us recall that a space P is said to be a K-space, if $F \subset P$ is closed if and only if the set $F \cap K$ is closed for every compact subspace K of P. Equivalentely, $x \in \overline{M}$ if and only if there exists a compact subspace K of P such that $x \in \overline{M \cap K}$. Evidently every locally compact space is a K-space. The following example shows that a G_δ-space may fail to be a K-space.

Example 2. Let $\beta(N)$ be the Stone-Čech compactification of the countable infinite discrete space N. Let $\{N_n\}$ be a sequence of subsets of N such that

$$N_1 \supset N_2 \supset \ldots, \quad \bigcap_{n=1}^{\infty} N_n = 0,$$

$$N_{k+1} - N_k \quad \text{are infinite} \quad (k = 1, 2, \ldots).$$

Put

$$P = N \cap \bigcap_{n=1}^{\infty} \overline{N}_n .$$

The sets of the form $\overline{M}$, where $M \subset N$, are open and closed in $\beta(N)$. It follows that P is a G_δ-subset of the compact space $\beta(N)$, and consequently, a G_δ-space. We shall prove that there exists a point x in $P - N$ such that there exists no compact subspace K of P with $x \in \overline{K \cap N}$. Let us suppose that for every point y of $P - N$ there exists a compact subspace $K(y)$ of P such that

$$y \in \overline{N \cap K(y)} .$$

Put $C(y) = K(y) \cap N$. Thus the $C(y)$ are open and compact subspaces of P. The space

$P - N$ is compact as an intersection of compact sets $\overline{N}_n - N$. Thus there exists a finite number of elements $y \in (P - N)$, namely $y_1, \ldots, y_k$, such that the union of the corresponding $\overline{C(y)}$ covers $P - N$, *i.e.* $\bigcup_{i=1}^{k} \overline{C(y_i)} \supset P - N$. Put

$$C = \bigcup_{i=1}^{k} C(y_i) .$$

Evidently

$$\overline{C}^P = \bigcup_{i=1}^{k} \overline{C(y_i)}$$

and consequently $\overline{C}$ is a compact space. Thus we have

$$P = N \cap \overline{C}^{\beta(P)} .$$

From the inclusions $P \supset \overline{C} \supset P - N$ we deduce $\overline{N}_n \supset \overline{C}$, *i.e.* the sets $C - N_n$ are finite. The diferences $N_{n+1} - N_n$ being infinite, we may choose by induction

$$x_n \in N_{n+1} - N_n - C \quad (m = 1, 2, \ldots) .$$

The set X of all x_n is disjoint with C and hence

$$\overline{X}^{\beta(P)} \cap \overline{C} = 0 .$$

On the other hand, according to the construction of x_n, the sets $X - N_n$ are finite and consequently

$$\overline{X}^{\beta(P)} - N \subset \overline{N}_n^{\beta(P)} \quad (n = 1, 2, \ldots) .$$

It follows that

$$\overline{X}^{\beta(P)} - N \subset \bigcap_{n=1}^{\infty} \overline{N}_n^{\beta(P)} \subset P .$$

But this is impossible, since $\overline{C} \cap \overline{X} - N = 0$ and $P = N \cap \overline{C}$. The proof is complete.

Note 3. From the preceding example the following result may be obtained: If $N \subset P \subset \beta(N)$, then P is locally compact if an only if P is a K-space. Of course, this result may be obtained for every discrete space N.

Now we shall recall the definition of characters and pseudocharacters of points of a space. The character of a point x in a space P is the least cardinal $\mathfrak{m}$ for which there exists a local base at x of potency $\mathfrak{m}$. The pseudocharacter of x in P is the least cardinal $\mathfrak{m}$ for which x is the intersection of $\mathfrak{m}$ open sets. It is well-known that for compact spaces the characters and pseudocharacters coincide. We shall prove the following generalization of this result.

Theorem 7. *If a space P is locally a G_δ-space, then the characters and pseudo-characters of all points coincide.*

Proof. Let $\{\mathfrak{A}_n\}$ be a complete sequence of open coverings of the space P. If a point x of P is isolated then there is nothing to prove. To prove our theorem it is sufficient to show that if $\mathfrak{B}$ is a family of open neighborhoods of a $x \in P$ such that

(*) $\bigcap\{\bar{B};\, B \in \mathfrak{B}\} = (x)$,

(**) $\mathfrak{B}$ is multiplicative (*i. e.* $B_1, \ldots, B_n \in \mathfrak{B}$ implies $\bigcap_{i=1}^{n} B_i \in \mathfrak{R}$),

(***) $\mathfrak{A}_n \cap \mathfrak{B} \neq 0 \quad (n = 1, 2, \ldots)$,

then $\mathfrak{B}$ is a local base at x. Let us suppose that there exists an open neighborhood A of x such that $B - \bar{A} \neq 0$ for every $B \in \mathfrak{B}$. It is easy to see from (**) that the family $\mathfrak{M}$ of all sets of the form $B - \bar{A}$, $B \in \mathfrak{B}$, has the finite intersection property. According to the completness of $\{\mathfrak{A}_n\}$ (by (***)), we have

$$\bigcap\{\overline{B - A};\, B \in \mathfrak{B}\} \neq 0\,.$$

But this is impossible, since A is a neighborhood of the point x and the assumption (*) holds. The proof is complete.

Note. Of course, the preceding theorem may be easily proved using the direct definition of G_δ-spaces as a G_δ-subspace of the Čech-Stone compactification.

In the second section, from the preceding theorem we shall deduce that the image Q of a metrizable G_δ-space under a continuous and closed mapping is a G_δ-space if and only if Q is metrizable, or equivalently, if an only if the boundaries of inverses of points are compact.

2. INVARIANCE UNDER MAPPINGS

In this section we shall prove two theorems concerning open and perfect mappings.

A mapping f of a space P into a space Q is said to be open if the images under f of open subsets of P are open subsets of Q.

Theorem 8. *Let f be a continuous and open mapping of a space P onto a space Q. If P is locally compact, then Q is a locally compact space. If P is a G_δ-space, then Q is a G_δ-space. If P is locally a G_δ-space, then Q is locally a G_δ-space.*

Proof. The first assertion is obvious, the second assertion was proved in [5] and the third is an immediate consequence of the second.

Theorem 9. *Let f be a closed and continuous mapping of a space P onto a space Q such that the inverses (i. e. inverse images) of points are compact. Then*

(a) *P is locally compact if and only if Q is such.*

(b) *P is a G_δ-space if and only if Q is such.*

(c). *P is locally a G_δ-space if and only if Q in such.*

Proof. The "if" parts of (a) and (b) were proved in [5]. The "if" part of the assertion (c) is an immediate consequence of that of the assertion (b).

Now we shall prove the "only if" part of the assertion (b). Let us suppose that P is a G_δ-space. There exists a complete sequence $\{\mathfrak{A}_n\}$ of open coverings of the space P such that the families $\mathfrak{A}_n$ are additive (see Proposition 1). For every $n = 1, 2, \ldots$, let $\mathfrak{B}_n$ be the family of all sets of the form $Q - f[P - A]$ where $A \in \mathfrak{A}_n$, *i. e.*

$$\mathfrak{B}_n = \{Q - f[P - A]\,;\quad A \in \mathfrak{A}_n\}\,.$$

The mapping f being closed, the sets from $\mathfrak{B}_n$ are open in Q. We shall prove that the families $\mathfrak{L}_n$ are coverings of Q. Indeed, if $y \in Q$, then $K = f^{-1}[y]$ is a compact subspace of K and hence a finite subfamily $\{A_1, \ldots, A_k\}$ of $\mathfrak{A}_n$ covers K. But $\mathfrak{A}_n$ is additive, and consequently, $A = \bigcup_{i=1}^{k} A_i \in \mathfrak{A}_n$. It follows that

$$y \in (Q - f[P - A]) \in \mathfrak{B}_n .$$

To prove that Q is a G_δ-space it is sufficient to show that the sequence $\{\mathfrak{B}_n\}$ is complete. Let $\mathfrak{M}$ be a family (of subsets of Q) with the finite intersection property and such that $\mathfrak{M} \cap \mathfrak{B}_n \neq 0$ for every $n = 1, 2, \ldots$. For every $n = 1, 2, \ldots$ let us choose a $B_n \in \mathfrak{M} \cap \mathfrak{B}_n$ and a $A_n \in \mathfrak{A}_n$ with

$$B_n = Q - f[P - A_n] .$$

Evidently the family of all $f^{-1}[M]$, $M \in \mathfrak{M}$, has the finite intersection property and

$$A_n \supset f^{-1}[B_n] \in \mathfrak{N} \quad (n = 1, 2, \ldots) .$$

Thus

$$\bigcap\{\overline{N}; N \in \mathfrak{N}\} \neq 0 ,$$

and by continuity of f

$$\bigcap\{\overline{M}; M \in \mathfrak{M}\} \neq 0$$

which establishes the completness of the sequence $\{\mathfrak{B}_n\}$ and completes the proof of the "only if" part of the assertion (b).

It remains to prove the "only if" parts of (a) and (c). First let P be a locally compact space and let $y \in Q$. From the compactness of $f^{-1}[y]$ it follows at once that there exists a compact neighborhood K of $f^{-1}[y]$. By continuity of f, the set $f[K]$ is compact, and from the fact that f is closed we deduce at once that $f[K]$ is a neigborhood of the point y. Finally let us suppose that the space P is locally a G_δ-space and $y \in Q$. Since the set $f^{-1}[y]$ is compact and P is locally a G_δ-space, there exists a finite number of open sets $U_1, \ldots, U_k$ such that every U_i is a G_δ-space and $f^{-1}[y] \subset \bigcap_{i=1}^{k} U_i$.

Put

$$U = \bigcup_{i=1}^{k} U_i , \quad V = Q - f[P - U] .$$

Since f is closed, V is an open neighborhood of the point y. Evidently

$$H = f^{-1}[V] \subset U$$

is an open set. Thus H is a G_δ-space. Applying the "only if" part of (b) we obtain that V is a G_δ-space. Indeed, the restriction of f to H is a continuous and closed mapping and the inverses of point are compact. The proof of the theorem is complete.

From the preceding theorem we deduce at once the following

Theorem 10. *Let f be a continuous and closed mapping of a space P onto a space Q such that the boundaries of inverses of points are compact. Then if P is locally compact, locally a G_δ-space or a G_δ-space, then Q has the corresponding property.*

Proof. If the boundary $B(y)$ of $f^{-1}[y]$ is nonvoid put $K(y) = B(y)$. If $B(y) = 0$, then let $K(y)$ be a one-point set, $K(y) \subset f^{-1}[y]$. Put

$$R = \bigcup\{K(y);\, y \in Q\}\,.$$

R being a closed subset of P, if P is locally compact, locally a G_δ-space or a G_δ-space, then so is R. Moreover the restriction of f to R is a closed and continuous mapping of R onto Q such that the inverses of points are compact. On applying theorem 9 we obtain theorem 10.

The assumption of compactness of boundaries of inverses of points is essential. This follows from the theorem 12. First we shall state the following well-known:

Theorem 11. *The following conditions on a closed and continuous mapping f of a metrizable space P onto a space Q are equivalent*:

(a) *Q is metrizable*;
(b) *every point of Q has a countable character*;
(c) *the boundaries of inverses of points are compact.*

Now we are prepared to prove the following theorem:

Theorem 12. *Let f be a closed and continuous mapping of a metrizable G_δ-space P onto a space Q. The following assertions are equivalent*:

(1) *Q is metrizable*;
(2) *Every point of Q has a countable character*;
(3) *The boundaries of inverses of points are compact*;
(4) *Q is a G_δ-space.*

Proof. According to the preceding theorem, the assertions (1), (2) and (3) are equivalent. Now let us suppose that Q is a G_δ-space. The mapping f being closed, every point of Q has a countable pseudocharacter. Applying theorem 9, we obtain that every point of Q is of a countable character. Conversely, if the assertion (3) holds, then by theorem 10 we have that Q is a G_δ-space. The proof is complete.

Let R be a line of the Euclidean plane P. Identifying the points of R, we obtain the quotient space Q, which is not a G_δ-space.

References

[1] *E. Čech:* On Bicompact Spaces. Ann. of Math. *38* (*1937*), 823–844.
[2] *Z. Frolík:* Generalisations of G_δ-property of Complete Metric Spaces. Czech. Math. J. *10* (*85*) 1960, 359–379.
[3] *Z. Frolík:* Topological Product of two Pseudocompact Spaces. Czech. Math. *J. 10* (*85*), 1960, 339–349.
[4] *Z. Frolík:* Topological Product of Two Countably Compact Spaces. Czech. Math. J. *10* (*85*), 1960, 329–338.
[5] *Z. Frolík*: Invariance of G_δ-spaces under mappings. Czech. Math. J. *10* (*86*), 1961, 259–261.

Резюме

ЛОКАЛЬНЫЕ G_δ-ПРОСТРАНСТВА

ЗДЕНЕК ФРОЛИК (Zdeněk Frolík), Прага

Рассматриваются только вполне регулярные топологические пространства. Пространство P называется локальным G_δ-пространством, если для каждого $x \in P$ существует открытая окрестность U, являющаяся G_δ-пространством (см. статью [2], терминология и результаты которой употребляются в данной заметке).

Теорема. *Каждое локальное G_δ-пространство содержит открытое плотное подмножество, являющееся G_δ-пространством.*

Теорема. *Паракомпактное локальное G_δ-пространство является G_δ-пространством.*

Теорема. *Пусть P — локальное G_δ-пространство. Если $N \subset P$ счетно, $U \subset P$ открыто и содержит точку сгущения множества N, то существует компактное $K \subset U$ такое, что множество $K \cap N$ бесконечно.*

Замечание. Можно построить G_δ-пространство, не являющееся k-пространством.

Теорема. *При совершенном отображении сохраняется (в обоих направлениях) свойство быть G_δ-пространством, равно как и свойство быть локальным G_δ-пространством.*

Теорема. *Пусть f — замкнутое непрерывное отображение P и Q. Если f периферически компактно (т. е. границы прообразов точек компактны) и P локально (соответственно, является G_δ-пространством, локальным G_δ-пространством), то также Q локально компактно (является G_δ-пространством, локальным G_δ-пространством). Если P является метризуемым G_δ-пространством, то следующие условия эквивалентны:* (1) *Q метризуемо,* (2) *каждое $y \in Q$ имеет счетный характер,* (3) *f периферически компактно,* (4) *Q является G_δ-пространством.*

Теорема. *Писть P — локальное G_δ-пространство. Если P и Q счетно компактны (соответственно, псевдокомпактны), то также $P \times Q$ счетно компактно (соответственно, псевдокомпактно).*