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Ladislav Rieger

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A CONTRIBUTION TO GÖDEL'S AXIOMATIC SET THEORY, III

(*The axiomatic dyadic aritmetics of finite sets and their classes*)

LADISLAV RIEGER, Praha

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Axiomatic dyadic arithmetic consists of a certain elementary (i.e. first order) axiomatic description of Hensel's integral dyadic (i.e. p-adic with $p = 2$) numbers; it is built of 28 axioms concerning addition, multiplication and potentiation of 2 as the only primitive notions. In short the main result is as follows. The axiomatic dyadic arithmetic is precisely the axiomatic theory of finite sets and their classes[1]) of Bernays-Gödel, though with other primitive notions.

Contents: 1. Introductory remarks. 2. Axioms of the theory of finite sets (of Bernays-Gödel) and their reductions. 3. The axiomatic system of dyadic arithmetics. 4. The equivalence-theorem and the reproduction-theorem. 5. Conclusive remarks and open problems.

1. INTRODUCTORY REMARKS

The present paper is a selfcontained continuation of the paper [II][2]) (under the same title, in this Journal). There we had in fact constructed various particular examples of "non-normal" dyadic arithmetics (called "dyadic s-t-rings"); here we define the general notion of dyadic arithmetic by means of 27 + 1 elementary axioms; the only primitive notions are addition, multiplication and potentiation of two. (See sec. 3.)

1i. Assuming the 27 axioms of dyadic arithmetics (*i.e.* with the last axiom omitted) we can define the membership-predicate, say $\in^*$, by the formula

$$(*) \qquad x \in^* Y \underset{\text{df}}{\Leftrightarrow} [Y/2^x] - 2[Y/2^{x+1}] = 1$$

(where $[Y/2^x]$ means the integral part of the quotient $Y/2^x$), and we can prove all the 18 axioms of the axiomatic set theory of Bernays-Gödel (see [G]), though with the axiom of infinity C1 replaced by its negation, called the axiom of finity.

[1]) I.e. with the axiom of infinity replaced by its negation, called the axiom of finity.

[2]) See the references at the end of this paper; the present paper has been the subject of a seminar held in the school-year 1959—60 at Charles University, Praha.

Iii. Assuming the mentioned axioms of Bernays-Gödel (with the axiom of finity), we can define[3]) addition and multiplication for all classes as well as potentiation of two $= \{\{\emptyset\}\}$ for them – and we can prove all the 28 axioms of dyadic arithmetics.

Ii and Iii are the two parts of our so-called equivalence theorem (in the sense of the mutual interpretability of dyadic arithmetics and of the Bernays-Gödel theory of finite sets and their classes; see sec. 4).

Moreover, we prove in sec. 4 the so-called reproduction theorem IIi, IIii. IIi states that, supposing (among others) the last axiom of dyadic arithemtics and defining new addition, multiplication and potentiation of two in the sense of Iii, applied to the already introduced $\in_*$ of Ii, we obtain the originally assumed operations of dyadic arithmetics. IIii says that, conversely, defining $\in_*$ by means of the already introduced arithmetical operations upon classes (in the sense of Iii) we obtain the originally assumed membership-relation.

These results show that the nature of the axiomatic membership-relation is an arithmetical one if one assumes the axiom of finity; in this respect, it is far from the intuitive Cantorian concept. Indeed, every class in the sense of the theory of finite sets of Bernays-Gödel can be taken for a dyadic integral number (in the generalized sense of Hensel), every set in this theory can be taken for such a "finite" (nonnegative) dyadic integral number. The intuitive justification of the first fact is suggested by the observation that the relation between a Hensel dyadic integral number Y and a positive integer x, that $Y = \ldots + 2^x + \ldots$ in the dyadic expansion of Y, indeed is a membership-relation, satisfying all the axioms of the theory of finite sets (and their classes). The intuitive justification of the second fact goes back the following one-to-one correspondence between positive integers and finite sets:

$$\emptyset \leftrightarrow 0\,; \quad \{\emptyset\} \leftrightarrow 2^0 = 1\,; \quad \{\{\emptyset\}\} \leftrightarrow 2^{2^0} = 2\,; \quad \{\emptyset\{\emptyset\}\} \leftrightarrow 2^0 + 2^{2^0} = 3\,;$$
$$\{\{\{\emptyset\}\}\} \leftrightarrow 2^2 = 4\,; \quad \{\emptyset\{\{\emptyset\}\}\} \leftrightarrow 2^0 + 2^2 = 5\,; \quad \{\{\emptyset\}\{\{\emptyset\}\}\} \leftrightarrow 2^1 + 2^2 = 6\,;$$
$$\{\emptyset\{\emptyset\}\{\{\emptyset\}\}\} \leftrightarrow 2^0 + 2^1 + 2^2 = 7\,; \quad \ldots$$

which has already been realized by T. SKOLEM in [Sk*] of 1923.

The elaboration of these two hints to our equivalence-theorem and to our reproduction-theorem is not immediate; it requires some effort, not too interesting of itself, though necessary. Nevertheless, the detailed performation of the proofs suggests many subtle problems and thus perhaps it is not useless to follow them through.

In a forthcoming paper, we shall give a general algebraical method of extending dyadic arithmetics, without any countability restriction (cf. [II], where we could not proceed beyond the first uncountable ordinal, not speaking of several unnecessary complications and of some omissions; thus the forthcomming paper will include an improved reformulation of the main results of [II]).

[3]) See 4.2.

2. AXIOMS OF THE THEORY OF FINITE SETS (OF BERNAYS-GÖDEL) AND THEIR REDUCTIONS

Let us rewrite the system of axioms of the theory of finite sets and their classes of Bernays (see [B]), in the modification of Gödel (see [G]), *i.e.* the axioms A1 – A4; B1 – B8; non C1, C2, C3, C4; D; E – with some obvious minor typographical changes.

Primitive notions:[4])

$$\mathbf{Cls}(Y)\ (Y \text{ is a class}).$$
$$\mathbf{M}(X)\ (X \text{ is a set}).$$
$$X \in Y\ (X \text{ belongs to } Y).$$

The letters $X, Y, Z, \ldots$, possibly with indices, are class-variables; the letters $x, y, z, \ldots$, possibly with indices, are set-variables; general quantifiers are often omitted if possible; definitions 1.1 – 1.5 of (G) are assumed.

Axioms:

A1: $\mathbf{Cls}(x)$. (Every set is a class.)

A2: $X \in Y \Rightarrow \mathbf{M}(X)$. (Only sets become members.)

A3: $\forall u(u \in X \Leftrightarrow u \in Y) \Rightarrow X = Y$. (The axiom of extensionality.)

A4: $\forall x \forall y \exists z \forall u(u \in z \Leftrightarrow u = x \vee u = y)$. (The axiom of unordered pairs.)

B1: $\exists Z \forall x \forall y(\langle xy \rangle \in Z \Leftrightarrow x \in y)$. (The axiom of the $\in$-relation.)

B2: $\forall X \forall Y \exists Z \forall u(u \in Z \Leftrightarrow u \in X \,\&\, u \in Y)$. (The axiom of intersection.)

B3: $\forall X \exists Y \forall u(u \in Y \Leftrightarrow \neg(u \in X))$. (The axiom of complement.)

B4: $\forall X \exists Y \forall x(x \in Y \Leftrightarrow \exists y(\langle xy \rangle \in X))$. (The axiom of domain.)

B5: $\forall X \exists Y \forall x \forall y(\langle yx \rangle \in Y \Leftrightarrow x \in X)$. (The axiom of direct product.)

B6: $\forall X \exists Y \forall x \forall y(\langle xy \rangle) \in Y \Leftrightarrow \langle yx \rangle \in X)$. (The axiom of pair-conversion.)

B7: $\forall X \exists Y \forall x \forall y \forall z(\langle xyz \rangle \in Y \Leftrightarrow \langle yzx \rangle \in X)$. (The first axiom of triple-conversion.)

B8: $\forall X \exists Y \forall x \forall y \forall z(\langle xyz \rangle \in Y \Leftrightarrow \langle xzy \rangle \in X)$. (The second axiom of triple-conversion.)

Now we shall formulate the negation of the axiom of infinity C1 of [G], *i.e.* the axiom of finity $\neg$C1 ($\mathbf{Em}(z)$ states that Z is empty; $\subset$ excludes identity).

$\neg$C1: $\neg \exists z\{\neg \mathbf{Em}(z) \,\&\, \forall x[x \in z \Rightarrow \exists y(y \in z \,\&\, x \subset y)]\}$. (Explicitly: There does not exist a non-empty set z such that every member of z is included in some other member of z.)

A positive reformulation ($\mathbf{Pr}(X)$ means $\neg \mathbf{M}(X)$, *i.e.* X is a proper class):

$\neg$C1*: $\forall X(\{\neg \mathbf{Em}(X) \,\&\, \forall x[x \in X \Rightarrow \exists y(y \in X \,\&\, x \subset y)]\} \Rightarrow \mathbf{Pr}(X))$. (Explicitly: If every member of a non-empty class X is included in some other member of X, then X is a proper class (not a set).)

C2: $\forall x \exists y \forall u \forall v(u \in v \,\&\, v \in x \Rightarrow u \in y)$. (The set-sum-axiom.)

C3: ($\subseteq$ includes identity): $\forall x \exists y \forall u(u \subseteq x \Rightarrow u \in y)$. (The potency-set-axiom.)

[4]) $\mathbf{Cls}$ and $\mathbf{M}$ as well as equality $=$ can be defined; for methodical reasons, however, we assume the original version.

(Define by 1.3 of [G]:

$$\mathbf{U}n(X) \underset{\mathrm{df}}{\Leftrightarrow} \forall u \forall v \forall w(\langle vu \rangle \in X \,\&\, \langle wu \rangle \in X \;\Rightarrow v = w),$$

i.e. X is unambiguous.)

C4: $\forall x \forall U \{\mathbf{U}n(U) \Rightarrow \exists y \forall u [u \in y \Leftrightarrow \exists v(v \in x \,\&\, \langle uv \rangle \in U)]\}$. (The axiom of replacement; every set x has an image-set by means of any given unambiguous class U.)

(Define $\mathbf{E}x(X, Y) \underset{\mathrm{df}}{\Leftrightarrow} \forall u \;\neg(u \in X \,\&\, u \in Y)$ by 1.23 of [G], *i.e.* X and Y are disjoint.)

D: $\forall Z(\neg \mathbf{E}m(Z) \Rightarrow \exists u(u \in Z \,\&\, \mathbf{E}x(u, Z)))$. (The so-called Fundierungsaxiom: in every nonempty class Z there are members disjoint with Z.)

E: $\exists W\{\mathbf{U}n(W) \,\&\, \forall x[\neg \mathbf{E}m(x) \Rightarrow \exists y(y \in x \,\&\, \langle yx \rangle \in W)]\}$. (This is the Gödel's strong axiom of choice: there exists a class W which is a mapping ascribing to every nonempty set exactly one of its elements.)

Remark. Let us denote by $\cup$, $\cap$, $\dot{-}$ the set-theoretical join, meet and difference of two classes (in order to avoid collision with the arithmetical $+$, . , $-$).

As can be expected, the axioms listed are not independent. The two reductions we shall use (perhaps they are not the only possible ones) are as follows.

I. It is known that the axiom B8 can be omitted, as a consequence of the remaining axioms of the original system Σ of [G] — no matter whether the axiom of infinity is supposed or not. (See *e.g.* [H.-K.] or the axiom B6[M].)

II. It can be shown that the axioms C3 and E can be then deduced from the remaining axioms (of the theory of finite sets).

The reduction II and especially the dependence of the axiom of choice in the axiomatic theory of finite sets (and their classes) is due to P. Vopěnka [V].

Therefore we can and will assume the reduced axiomatic system of the theory of finite sets and their classes (of Bernays-Gödel), *i.e.* the axioms A1 – A4, B1 – B7, $\neg$C1, C2, C4 and D.

Now we must emphasize that these axioms of the theory of finite sets and their classes are trivially satisfied by the "model" consisting of the sole void class. (I owe this remark to V. Trnková.) In order to avoid this singular unintended interpretation, we must assume an additional existential axiom, say F:

F: $\exists x \; \mathbf{M}(x)$ (*i.e.*, there is at least one set).

Adding this axiom F to the already listed ones, we obtain the definitive list of the system Σ' of 16 axioms of the theory of finite sets and their classes, in the sense of Bernays-Gödel (perhaps it is better to say of v. Neumann-Bernays-Gödel).

3. THE AXIOMATIC SYSTEM (σ) OF DYADIC ARITHMETICS

Assume the first-order (classical) logic with the identity $=$. The letters X, Y, Z, ...; A, B, C, ..., possibly with indices, are individual variables (or individual signs) for the so-called dyadic integers, which form our universe of discourse.

There are three primitive notions of (σ):

(i) the ternary predicate of addition. Say $+(X, Y, Z)$ means that Z is the sum of X and Y;

(ii) the ternary predicate of multiplication. Say $\cdot(X, Y, Z)$ means that Z is the product of X and Y;

(iii) the binary predicate of potentiation of 2. Say $\mathbf{2}(X, Y)$ means that Y is the potency of 2 by X. (See the axioms sub (d) below.)

Like the axioms of the theory Σ' of finite sets and their classes, our axiomatic system (σ), consisting of 27 + 1 requirements describing $+(.,.,.)$, $\cdot(.,.,.)$, $\mathbf{2}(.,.)$, falls into three groups. Their formulation and discussion requires some preliminary preparations of later axioms by means of lemmas depending on the former ones; however, these lemmas will also be applied in the main section 4.

The first group (r) (ring-axioms) deals with + and $\cdot$ only. It requires first that dyadic integers form integrity domain (*i.e.* a commutative ring with unit and without divisors of zero). Further, the characteristic of this domain shall be different from $2 = = 1 + 1$. (Note, however, that the non-elementary general notion of a characteristic is not to be used). Finally, there is an important last special ring-axiom (r 13), requiring the "absence of one-half".

The second group (d) (dyadic axioms) deals with the potentiation of 2 and is divided into two subgroups. The first of these, (d′), enables us to introduce the notion of a "finite dyadic integer", or equivalently to define the property of "to be positive or zero" of any X, by means of $2^X \neq 0$. (We thus obtain a certain discretely ordered subdomain of the whole integrity domain.)

The second subgroup (d″) (of the group (d) of dyadic axioms) enables us to define the dyadic membership-relation $\in_*$ by the formula (*) of section 1 already mentioned.

The so-called integral part $[Y/2^x]$ of the quotient $Y/2^x$ is warranted by the axiom (d″3), though without using the nonelementary notion of quotient-field. The last (d)-axiom (d″4) enables to introduce the (dyadic) exponential valuation (in the sense due to W. Krull, see [K]), defined, however, for our integrity domain and thus taking on values from the additive discrete ordered commutative semigroup (with cancellation and zero) of the already defined "finite dyadic integers".

The third group (cl) ("class"-axioms) is essentielly equivalent to group B of [G] as reformulated for the above $\in_*$ by means of the solvability of suitable arithmetical equations.

There is an additional requirement, the so-called successor-principle (s); despite of [II], this axiom is not necessary in order to ensure any axiom of Σ' for $\in_*$, since the requirements mentioned previously suffice to this purpose. Neverthelless, (s) seems to be indispensable in order to ensure the converse reproduction-theorem IIi (already mentioned in section 1).

Let us turn to the axioms of (σ) themselves.

The group (r):

(r1): $\forall X \forall Y \exists Z + (X, Y, Z)$. (Axiom of unrestricted existence of the sum of dyadic integers.)

(r2): $[X_1 = X_2 \& Y_1 = Y_2 \& + (X_1, Y_1, Z_1) \& + (X_2, Y_2, Z_2)] \Rightarrow Z_1 = Z_2$. (Axiom of unicity of the sum, giving rise to the terms $X_1 + Y_1, X_2 + Y_2, \ldots$ by means of the equivalences $+ (X_1, Y_1, Z_1) \underset{\mathrm{df}}{\Leftrightarrow} X_1 + Y_1 = Z_1$, $+ (X_2, Y_2, Z_2) \underset{\mathrm{df}}{\Leftrightarrow} X_2 + Y_2 = Z_2, \ldots$,

(r3): $X + (Y + Z) = (X + Y) + Z$. (The law of associativity of addition.)

(r4): $X + Y = Y + X$. (The law of commutativity of addition.)

(r5): $\forall X \forall Y \exists Z (X + Z = Y)$. (The law of unrestricted subtraction.)

Lemma 1. and convention 1.

a) *The Z in* (r5) *is* (*up to* $=$) *uniquely determined by the given* X, Y; *thus we write* $Z = Y - X$, *introducing a further kind of term.*

b) *If* $X = Y$ *in* (r5), *then the Z is uniquely determined independently of* X; *this unique Z is the so-called zero,* $0 = X - X$ *for every* X, 0 *being an individual constant.*

c) *We write* $0 - X = - X$ *for every* X, *introducing a further* (*usual*) *kind of term.*

Proof can be omitted.

(r6): $\forall X \forall Y \exists Z \cdot (X, Y, Z)$. (The axiom of unrestricted existence of products of dyadic integers.)

(r7): $[X_1 = X_2 \& Y_1 = Y_2 \& \cdot (X_1, Y_1, Z_1) \& \cdot (X_2, Y_2, Z_2)] \Rightarrow Z_1 = Z_2$. (The axiom of unicity of multiplication, giving rise to the terms $X_1 . Y_1, X_2 . Y_2, \ldots$ by means of the equivalences $\cdot (X_1, Y_1, Z_1) \Leftrightarrow X_1 . Y_1 = Z_1$, $\cdot (X_2, Y_2, Z_2) \underset{\mathrm{df}}{\Leftrightarrow} X_2 . Y_2 = Z_2, \ldots$) We often write XY instead of $X . Y$.

(r8): $X . (Y . Z) = (X . Y) . Z$. (The law of associativity of multiplication.)

(r9): $X . Y = Y . X$. (The law of commutativity of multiplication.)

(r10): $(X + Y) . Z = X . Z + Y . Z$. (The law of distributivity.)

(r11): $\exists Z \forall Y (Z . Y = Y)$. (The law of the unit.)

Lemma 2 and convention 2. *The Z of* (r11) *is unique and is called the unit and denoted by* 1; *thus by convention,* $1 . Y = Y$ *for every* Y. *Further, we obtain that* $- 1 . Y = - Y$.

Proof can be omitted.

(r12): $\neg(1 + 1 = 0)$. (The characteristic of the considered ring is not 2.)

Consequence. $\neg(1 = 0)$, *i.e.* our ring cannot degenerate to zero.

Convention 3. $1 + 1 = 2$, thus 2 is a further individual constant, the so-called two, not to be confused with the predicate **2**(., .).

(r13): $\forall X \neg(X + X = 1)$. (This is the unusual "law of excluding one-half"; we also write $X + X \neq 1$.)

(r14): $\forall X \forall Y(X \,.\, Y = 0 \Rightarrow X = 0 \vee Y = 0)$. (The law of excluding divisors of zero.)

The group (d):

(d'1): $\forall X \exists Z\, \mathbf{2}(X, Z)$. (The law of unrestricted existence of the potency Z of 2 by X; see later.)

(d'2): $[X_1 = X_2 \,\&\, \mathbf{2}(X_1, Z_1) \,\&\, \mathbf{2}(X_2, Z_2)] \Rightarrow Z_1 = Z_2$. (The law of unicity of the potencies of 2, giving rise to the terms $2^X, 2^Y, \ldots$ by means of the equivalences $\mathbf{2}(X, Z) \underset{\mathrm{df}}{\Leftrightarrow} 2^X = Z$, $\mathbf{2}(Y, U) \underset{\mathrm{df}}{\Leftrightarrow} 2^Y = U$, ...).

d'3): $2^1 = 2$.

(d'4): $2^X \,.\, 2^Y \neq 0 \Rightarrow 2^X \,.\, 2^Y = 2^{X+Y}$.

(d'5): $2^{X.Y} = 0 \Rightarrow 2^X \,.\, 2^Y = 0$.

(d'6): $2^{X-Y} + 2^{Y-X} = 0 \Rightarrow 2^X \,.\, 2^Y = 0$.

(d'7): $X \,.\, (1 - X) \,.\, 2^X \,.\, 2^{1-X} = 0$.

We shall use a limited number of the following individual constants (all of them different from zero, on account of the axioms (d'3), (d'4) and (r14)): $2^{1+1} = 2^2, 2^{1+1+1} = 2^3, 2^{1+1+1+1} = 2^4, \ldots$

In order to prepare the second subgroup (d″) of the group (d), let us include some definitions and lemmas.

Definition 1. If $2^X \neq 0$, then we say that X is positive or zero, and write $0 \leqq X$; if, moreover $0 \leqq X \,\&\, X \neq 0$, then X is called positive. *E.g.*, $0 \leqq 1$ and 1 is positive by (d'3).

Lemma 3. $0 \leqq X \,\&\, 0 \leqq Y \Rightarrow 0 \leqq X + Y$.

Proof. By (d'4) and (r14).

Lemma 4. $0 \leqq X \,\&\, 0 \leqq Y \Rightarrow 0 \leqq X \,.\, Y$.

Proof. By (d'5) and (r14).

Lemma 5. $0 \leqq X \,\&\, 0 \leqq Y \Rightarrow 0 \leqq X - Y \vee 0 \leqq Y - X$.

Proof. $2^X \neq 0 \,\&\, 2^Y \neq 0 \Rightarrow 2^X \,.\, 2^Y \neq 0$ by (r14), whence $2^{X-Y} + 2^{Y-X} \neq 0$ by (d'6), *i.e.* $2^{X-Y} \neq 0 \vee 2^{Y-X} \neq 0$.

Lemma 6. We have $0 \leqq 0$, *i.e.* $2^0 \neq 0$.

Proof. Since $0 = 1 - 1$, then from $2^0 = 0$ we would obtain $2^0 + 2^0 = 0$ and thus $2^1 \,.\, 2^1 = 0$ by (d'6) (with $X = Y = 1$) – in contradiction with (r12) (on account of (r14) and (d'3)).

Lemma 7. We have $2^0 = 1$.

Proof. By (d'3), (r12), lemma 6, (r14), (d'4), we obtain $0 \neq 2^1 \,.\, 2^0 = 2^{1+0} = 2^1$, *i.e.* $2 \,.\, 2^0 = 2$; thus $2(2^0 - 1) = 0$, whence $2^0 = 1$ by (r14).

Lemma 8. $X \,.\, 2^X \,.\, 2^{-X} = 0$ for every X, *i.e.* $0 \lesssim X \,\&\, 0 \lesssim -X \Rightarrow X = 0$ (by (r14)).

Proof. Suppose $X \,.\, 2^X \,.\, 2^{-X} \neq 0$ for some X. Then $X \neq 0 \,\&\, 2^X \neq 0 \,\&\, 2^{-X} \neq 0$ by (r14), whence $2^1 \,.\, 2^{-X} = 2^{1-X} \neq 0$ by (r12), (r14), (d'3), (d'4). Further, $1 - X \neq 0$; for otherwise $X = 1$ and $2^{-X} = 2^{-1} \neq 0$ would give $2^{-1} + 2^{-1} = 2 \,.\, 2^{-1} = 2^1 \,.\, 2^{-1} = 2^0 = 1$ (by (d'3), (d'4)) – contrary to (r13).

Thus, summarizing, from $X \,.\, 2^X \,.\, 2^{-X} \neq 0$ we obtain $X \,.\, (1 - X) \,.\, 2^X \,.\, 2^{1-X} \neq 0$ (by (r14)) – in contradiction with (d'7) – proving the lemma.

Theorem 1. *Assume the axioms* (r1)–(r14), (d'1)–(d'7). *Define the predicate* $\mathbf{R}(.)$ *by the equivalence*

$$\mathbf{R}(X) \underset{\mathrm{df}}{\Leftrightarrow} 2^X + 2^{-X} \neq 0\,.$$

Then the dyadic integers with the property $\mathbf{R}$ *form a discretely ordered integrity subdomain if the ordering relation* $\prec$ *is defined thus*

$$X \prec Y \underset{\mathrm{df}}{\Leftrightarrow} 2^{Y-X} \neq 0 \,\&\, X \neq Y$$

(*i.e. leting* X *to be positive*, $0 \prec X$, *in the case* $X \,.\, 2^X \neq 0$) – *in the usual sense of elementary abstract algebra* (see *e.g.* [WI]).

More precisely: *If we relativize the quantifiers to the predicate* $\mathbf{R}$, *then* (r1)–(r14) *hold as well as the following statements* I–VII:

I: $\neg(X \prec X)$.
II: $X \neq Y \Rightarrow X \prec Y \vee Y \prec X$.
III: $(X \prec Y) \,\&\, (Y \prec Z) \Rightarrow X \prec Z$.
IV: $X \prec Y \Rightarrow X + Z \prec Y + Z$.
V: $(X \prec Y) \,\&\, (0 \prec Z) \Rightarrow X \,.\, Z \prec Y \,.\, Z$.
VI: $0 \prec 1$.
VII: $\neg\exists X\ (0 \prec X \prec 1)$.

Proof. Since $\mathbf{R}(X) \Leftrightarrow 0 \lesssim X \vee 0 \lesssim -X$ by definition, hence $\mathbf{R}(X) \Leftrightarrow \mathbf{R}(-X)$ by lemma 8 and thus $\mathbf{R}(X) \,\&\, \mathbf{R}(Y) \Rightarrow \mathbf{R}(X + Y)$ by lemma 3 and lemma 5; analogously $\mathbf{R}(X) \,\&\, \mathbf{R}(Y) \Rightarrow \mathbf{R}(X \,.\, Y)$ by lemma 4. Now, the verification of (r1)–(r14) as relativized to $\mathbf{R}$ is obvious. Further, the lemma 7, *i.e.* $1 = 2^0 = 2^{X-X} \neq 0$, states I. The definition of $\mathbf{R}$ and lemma 5 yield II. Lemma 3 states III. Point *IV* follows immediately from the definition of $\prec$ as relativized to $\mathbf{R}$. Point *V* follows from lemma 4 (and (r14)). (d'3) (together with (r12)) yields *VI*. Finaly, point *VII* (i.e. the discreteness of $\prec$) follows from (d'7).

Corrolary of theorem 1. $2^{-(X^2)} = 0$ if $X \neq 0$ *and* $2^{-(X^2)} = 1$ *if* $X = 0$ (see (r12) and V of the theorem as well as (d'1) and lemma 5).

Remark 1. It is not required that either $0 \prec X$ or $X = 0$ or $X \prec 0$ for every X, *i.e.* $\forall X \mathbf{R}(X)$ is not true in general (but not excluded by axioms sub (r), (d'), (d'')). The case of $\forall X\mathbf{R}(X)$, *i.e.* of $\forall X(2^X + 2^{-X} \neq 0)$, defines the notion of an ordered dyadic domain.

Nevertheless, further axioms of the group (cl) exclude $\forall X\mathbf{R}(X)$; it turns out that a dyadic domain satisfying the axioms of the group (cl), *i.e.* the so-called dyadic arithmetics, is not formally real (*i e.* a suitable sum of nonzero squares vanishes), so that such a dyadic domain cannot be ordered, in accordance with the standard dyadic arithmetic of Hensel's usual integral dyadic numbers.

On the other side, a suitable weakening of the axioms sub (cl) enables to preserve the property of being formally real; such dyadic domains will be used in the next paper of this series, and will be called dyadic semiarithmetics.

Remark 2. Assuming $\mathbf{R}(X)$, $\mathbf{R}(Y)$, we easily observe that

$$X \prec Y \Leftrightarrow 2^{X-Y} = 0\,.$$

Now, we are able to continue the exposition of the axioms of the group (d) of dyadic arithmetics by the second subgroup (d″), starting with the following formal simplifying convention:

Write $X = x$, $Y = y$, $Z = z$, ... instead of $0 \leqq X, 0 \leqq Y, 0 \leqq Z, \ldots$, introducing lower case letters as symbols for the positive (abstract) dyadic integers – and for zero; thus $X = x \Leftrightarrow 2^X \neq 0$. (Thus *e.g.* $\forall x\Phi$ or $\exists x\Phi$, with 'x' free in the propositional function Φ, are abbreviations of $\forall X\ (0 \leqq X \Rightarrow \Phi)$ and $\exists X\ (0 \leqq X \,\&\, \Phi)$ respectively – in the usual sense of relativizing quantifiers; relativized general quantifiers are often omitted as well as the unrelativized ones.)

(d″1): $x \prec 2^x$. (This is the so-called axiom of the dyadic exponential growth; this axiom seems unnecessary for the dyadic membership-relation $\in_*$ (as given by the formula (*) of sec. 1), if we do not insists on the "Fundierungsaxiom" D of Σ'. A closer investigation of the consequences of omitting (d″1) as well as the corresponding consistency-proof for its negation would be a future and not simple task.)

Lemma 9. a) $0 \prec 2^x$, b) $\forall X\ (0 \leqq 2^X)$.

Proof. a) is clear from (d′1) and from theorem 1.

Ad b): If $X = 0$ then $0 \leqq 2^X = 1$ by (d′1). Hence assume $X \neq 0$. If $2^X \neq 0$ then $0 \prec X = x$ and $0 \prec 2^x$ by a). Therefore $0 \leqq 2^X$ in general.

Lemma 10. a) $\forall x\forall y(2^X = 2^y \Leftrightarrow x = y)$, b) $\forall x\forall y(2^x \prec 2^y \Leftrightarrow x \prec y)$.

Proof. Ad a): Assume $x \neq y$; (by theorem 1) without loss of generality let $x \prec y$, *i.e.* $1 \leqq y - x$.

Then $2^{y-x} = 2^{1+(y-x-1)} = 2 \,.\, 2^{y-x-1} = 2^{y-x-1} + 2^{y-x-1}$, whence $2^{y-x} = 1$ is excluded by (r13). But clearly $2^{y-x} = 1 \Leftrightarrow 2^y = 2^x$ (by (d′4)), proving $2^x = 2^y \Rightarrow x = y$ and thus a). Now b) is clear by a similar argument.

(d″2): $\forall X\forall y\exists Z\exists r((X = 2^y \,.\, Z + r) \,\&\, (0 \leqq r \prec 2^y))$. (This is the so-called Euclidean axiom of division with remainder by any nonzero potency of 2.)

Lemma 11. and convention 4. *Z and therefore clearly also r of* (d″2) *are both uniquely determined by the given X and y; we shall write* $Z = [X/2^y]$ *and call this Z the "integral part of the quotient* $X/2^y$*" (not defining the quotient itself, of course), and call r the corresponding remainder; thus* $X = 2^y . [X/2^y] + r$.

Proof. Suppose $X = 2^y . Z_i + r_i$ for $i = 1, 2$ and $0 \leqq r_1 \leqq r_2 \prec 2^y$ (without loss of generality, according to theorem 1.) Then $0 \leqq r_2 - r_1 = 2^y(Z_1 - Z_2)$. If $r_1 = r_2$, then clearly $Z_1 = Z_2$. Hence suppose $0 \prec r_2 - r_1$. Then $0 \prec Z_1 - Z_2$ (by theorem 1), and moreover $1 \leqq Z_1 - Z_2$. Therefore $2^y \leqq 2^y(Z_1 - Z_2) = r_2 - r_1$, *i.e.* $2^y \leqq \leqq 2^y + r_1 \leqq r_2$, in contradiction with $r_2 \leqq 2^y$. Thus $r_1 \neq r_2$ is impossible, proving the lemma.

Remarks. (1) $r = 0$ in lemma 6 iff $X = 2^y . Z$, *i.e.* $Z = [X/2^y]$; we then say that X is divisible by 2^y (*i. e.* without remainder).

(2) For the sake of formal generality, put $[Y/2^X] = 0$ iff $2^X = 0$ (for every Y), in the case formerly excluded. (Of course, the above equality which follows from (d″2) cannot hold in the case $2^X = 0$.)

(d″3): $\forall X \exists y \exists Z(X \neq 0 \Rightarrow X = 2^y . (1 + 2 . Z))$. (This is the so-called axiom of dyadic valuation; this name may become clearer after the four following lemmas.)

Lemma 12 and convention 5. *The y and Z of* (d″3) *are uniquely determined by any given* $X \neq 0$; *we shall write* $y = \mathbf{W}(X)$ (*if* $X \neq 0$) *and call this y the dyadic (exponential) value of X.*

Proof. Suppose $0 \neq X = 2^{y_1} . (1 + 2 . Z_1) = 2^{y_2} . (1 + 2 . Z_2)$ and $y_1 \leqq y_2$, without loss of generality. The clearly $[X/2^{y_1}] = 1 + 2 . Z_1 = 2^{y_2 - y_1} . (1 + 2 . Z_2)$. Now $y_1 \neq y_2$ (and thus also $Z_1 \neq Z_2$), so that $y_1 \prec y_2$, *i.e.* $1 \leqq y_2 - y_1$, is excluded by (r13), since we would have $1 = 2^{y_2 - y_1} . (1 + 2 . Z_2) - 2 . Z_1 = 2 . (2^{y_2 - y_1 - 1} . . (1 + 2 . Z_2) - Z_2)$.

Remark to lemma 12 and convention 6. In order to obtain greater formal simplicity and generality, we could *e.g.* set -1 and, say 0, instead of y and Z respectively, in the formerly excluded case of lemma 12, *i.e.* put $\mathbf{W}(0) = -1$. Thus the identity

$$X = 2^{\mathbf{W}(X)} . (1 + 2 . Z)$$

with $\mathbf{W}(X)$ and Z uniquely determined by the given X, holds in general; this is in accordance with $2^{-1} = 0$, and this last fact is easy to see, for otherwise we would have $2^{+1} . 2^{-1} = 2^0 = 1 = 2^{-1} + 2^{-1}$, in contradiction with (r13). Hereby, the group (d) of axioms of dyadic arithmetics is closed.

We shall shortly say that the already given $21 = 14 + 7$ axioms of groups (r) and (d) describe the (elementary) notion of a dyadic domain; if, moreover, $2^X + 2^{-X} \neq 0$ holds for every X, we have the ordered dyadic domain; in this case, (d′6) holds automatically and can be omitted.

In order to state the axioms of the third group (cl) of dyadic arithmetics (as a particular kind of dyadic domains), we need some more consequences of the axioms stated previously. Let us note (see later for details) that axioms (r) and (d) together (and moreover, with (d″1) omitted) just suffice to ensure the axioms A1, A2, A3, A4 (*i.e.* the axiom of extensionality and the axiom of the unordered pair) and the axiom B2 (of complement) for the dyadic membership-relation $\in_*$ (as defined by the formula (*) in sec. 1), taking positive and zero (abstract) dyadic integers for "sets" and dyadic integers in general for "classes". (Assuming (d″1) too, we obtain the "Fundierungs-axiom" D also.)

Convention 7. We shall abbreviate $\operatorname{sg}(X) = 1 - 2^{1-2^X}$.

Lemma 13. *We have* $\operatorname{sg}(X) = 1, 0, -1$ *according as there is* $0 \prec X$, $0 = X$, $X \prec 0$ *respectively.*

Proof is by immediate computation, on account of $2^X \geqq 2$ for $X \geqq 1$ (see lemma 5).

Remark. If $\neg \mathbf{R}(X)$, *i.e.* if $2^X + 2^{-X} = 0$, then $\operatorname{sg}(X) = -1$. Thus $\operatorname{sg}(X)$ is defined for every X, for the sake of formal completeness and simplicity.

Convention 8. Further, we abbreviate

$$\max(X, Y) = X \,.\, \operatorname{sg}(X - Y) + Y \,.\, \operatorname{sg}(Y - X + 1)\,,$$
$$\min(X, Y) = -(\max(-X, -Y))$$

(with the same remark as before).

Lemma 14. *Assume* $\mathbf{R}(X)$, $\mathbf{R}(Y)$. *Then* $\max(X, Y)\,(= \max(Y, X)) = X$ *iff* $Y \prec X$ *and* $\max(X, Y)\,(= \max(Y, X)) = Y$ *iff* $X \precsim Y$; *similarly,*

$$\min(X, Y) = X \quad \textit{iff} \quad X \prec Y$$

and

$$\min(X, Y) = Y \quad \textit{iff} \quad Y \precsim X\,.$$

Proof is by computing, immediate by lemma 13.

The purpose of lemmas 13 and 14 is to give explicit "arithmetical" formulas (by superposition of our three primitive operations) for such functions as *e.g.* max, min.

Lemma 15. (On the dyadic valuation-properties.) *In every dyadic domain*

(a) $\mathbf{W}(X) = \mathbf{W}(-X)$.

(b) *Assuming* $X + Y \neq 0$, *then* $\mathbf{W}(X + Y) \geqq \min(\mathbf{W}(X), \mathbf{W}(Y))$. *More precisely,* $\mathbf{W}(X + Y) = \min(\mathbf{W}(X), \mathbf{W}(Y))$ *if* $\mathbf{W}(X) \neq \mathbf{W}(Y)$ *and* $\mathbf{W}(X + Y) \geqq \mathbf{W}(X) + 1$ *if* $\mathbf{W}(X) = \mathbf{W}(Y)$.

(c) *If* $X \neq 0 \neq Y$, *then* $\mathbf{W}(X \,.\, Y) = \mathbf{W}(X) + \mathbf{W}(Y)$.

Proof. (a): If $X = 0 = -X$, there is nothing to be proved. If $X \neq 0$, then write $X = 2^y(1 + 2Z)$ (by (d″3)), whence $-X = 2^y(-1 - 2Z) = 2^y(1 + 2(-Z - 1))$.

(b): If $X \,.\, Y = 0$, then (b) is trivially true. If $X \,.\, Y \neq 0$, then $Y = 2^v(1 + 2T)$, $X = 2^u \,.\, (1 + 2 \,.\, Z)$; without loss of generality let $u \precsim v$.

In the case $u \prec v$, *i.e.* $1 \leqq v - u$, we have

$$X + Y = 2^u(1 + 2(2^{v-u-1}(1 + 2T) + Z)),$$

whence $\mathbf{W}(X + Y) = u = \min(u, v)$ by lemma 7; in the complementary case $u = v$, we have $X + Y = 2^{u+1}(1 + Z + T)$, whence $\mathbf{W}(X + Y) \geqq u + 1$ by lemma 7 again.

(c): $X \,.\, Y = 2^u(1 + 2Z)\, 2^v(1 + 2T) = 2^{u+v}(1 + 2(Z + T + 2ZT))$ by lemma 7 again.

Now let us state the axioms of the group (cl), by means of the fundamental and already mentioned

Definition 2. (Of the dyadic membership.) Define a new so-called membership-operation[5] $\in_*(.,.)$ by the following superposition of our primitive dyadic operations with the previously introduced operation of "dividing by a potency of 2, with remainder":

$$\in_*(X, Y) = \mathrm{sg}(2^X) \,.\, ([Y/2^X] - 2[Y/2^{X+1}])\,.$$

We say that X is a dyadic member of Y iff $\in_*(X, Y) = 1$; we write also $X \in_* Y$ in this case.

In this sense, we shall also call any X with $2^X \neq 0$ (*i.e.* with $0 \leqq X$) a "set" ($X = x$) – and any Y in general – a "class".

Lemma 16. $\forall X \forall Y(\in_*(X, Y) = 0 \vee \in_*(X, Y) = 1)$.

Proof (cf. the proof of lemma XVII of [II]).

(a): Prove the auxiliary identity

(i) $$[U/2^{v+1}] = [[U/2^v]/2]\,.$$

Indeed, by lemma 6 $[U/2^v] = Q \,.\, 2 + t$, $0 \leqq t \prec 2$, $Q = [[U/2^v]/2]$.

Case (1): $t = 0$. Then $[U/2^v] = Q \,.\, 2$, so that $U = Q \,.\, 2 \,.\, 2^v + r$, $0 \leqq r \prec 2^v$, *i.e.* $U = Q \,.\, 2^{v+1} + r$, $0 \leqq r \prec 2^{v+1}$, *i.e.* $Q = [U/2^{v+1}]$.

Case (2): $t = 1$. Then $[U/2^v] = Q \,.\, 2 + 1$, so that $U = (Q \,.\, 2 + 1) \,.\, 2^v + r$, $0 \leqq r \prec 2^v$, *i.e.* $U = Q \,.\, 2^{v+1} + (2^v + r)$, $0 \leqq (2^v + r) \prec 2^v + 2^v = 2^{v+1}$, so that $[U/2^{v+1}] = Q$ again.

(b): If $2^X = 0$, then clearly $\in_*(X, Y) = 0$. Hence suppose $2^X \neq 0$, *i.e.* $X = x$. Then $\mathrm{sg}(2^X) = 1$ and $\in_*(X, Y) = [Y/2^x] - 2[[Y/2^x]/2]$ by (i), whence $\in_*(X, Y)$ is the remainder on dividing $[Y/2^x]$ by $2^1 = 2$, *i.e.* $\in_*(X, Y) = 0, 1$.

Convention 9. (The "unordered" and the "ordered pair".) a): Let us write

$$\{XY\}_* = 2^X + 2^Y \,.\, (\mathrm{sg}(X - Y))^2\,;$$

thus $\{xy\}_* = 2^x + 2^y$ iff $x \neq y$ – and $\{xy\}_* = 2^x$ iff $x = y$, other cases of $\{..\}_*$ being defined also, but irrelevant.

[5]) We write $\in_*(X, Y)$ instead of C^X_Y in [II].

b): Put $\langle XY\rangle_* = \{\{XX\}_* \{XY\}_*\}_*$ as a further abbreviation; we observe that $\langle xy\rangle_* = 2^{2^x}(1 + 2^{2^y})$ if $x \neq y$ and $\langle xy\rangle_* = 2^{2^x}$ iff $x = y$, other cases being defined but irrelevant.

Lemma 17. $\langle x_1 y_1\rangle_* = \langle x_2 y_2\rangle_* \Rightarrow x_1 = x_2 \,\&\, y_1 = y_2$.

Proof. (a): Assume $x_1 = y_1$. Then $\langle x_1 y_1\rangle_* = \{\{x_1 x_1\}_* \{x_1 x_1\}_*\}_* = \{2^{x_1} 2^{x_1}\}_* = = 2^{2^{x_1}}$. Thus $\langle x_1 y_1\rangle_* = \langle x_2 y_2\rangle_*$ entails (by lemma 5) the impossibility of $x_2 \neq y_2$; thus $x_2 = y_2$ and $\langle x_2 y_2\rangle_* = 2^{2^{x_2}}$. Therefore $x_1 = x_2 = y_1 = y_2$ in this case, as well as in the case $x_2 = y_2$.

(b): Suppose $x_1 \neq y_1$, $\langle x_1 y_1\rangle_* = \langle x_2 y_2\rangle_*$. Then $x_2 \neq y_2$ (by the preceding result) so that $\langle x_i y_i\rangle_* = \{2^{x_i} 2^{x_i} + 2^{y_i}\}_* = 2^{2^{x_i}} + 2^{2^{x_i} + 2^{y_i}} = 2^{2^{x_i}}(1 + 2^{2^{y_i}})$; $i = 1, 2$. Thus $x_i = \mathbf{W}(\mathbf{W}(\langle x_i y_i\rangle_*))$ and $y_i = \mathbf{W}(\mathbf{W}([\langle x_i y_i\rangle_*/2^{2^{x_i}}] - 1))$ for $i = 1, 2$, whence $x_1 = x_2$, $y_1 = y_2$.

Convention 10. (The "first member" and the "second member" of X.) According to lemma 17 let us abbreviate

$$^1X = \mathbf{W}(\mathbf{W}(X))\,,$$

$$^2X = \mathbf{W}(\mathbf{W}([X/2^{\mathbf{W}(X)}] - 1))\,.\,\mathrm{sg}\,([X/2^{\mathbf{W}(X)}] - 1) + {}^1X(1 - \mathrm{sg}^2\,([X/2^{\mathbf{W}(X)}] - 1))\,.$$

We can then verify that $^1X = x$ and $^2X = y$ if $X = \langle xy\rangle_*$. Especially, we see that $^1X = {}^2X = x$ if $X = \langle xx\rangle_* = 2^{2^x}$. For the case of X not of the form $\langle xy\rangle_*$, ^1X and ^2X are defined, but irrelevant. In all cases, the operations ^1X and ^2X are the so-called "first member" and "second member" of X.

Now we are ready to proceed to the group (cl) (of the so-called "class"-axioms) of dyadic arithmetics. (Note that lower case letters in (cl 1)–(cl 6), (s) are used only for better readability – and could be replaced by upper case ones, due to def. 2.)

(cl 1): (The "membership"-axiom.)

$$\exists Y \forall z(\in_* (z, Y) = \in_* (^1z, {}^2z)\,.\,2^{-(z - \langle ^1z^2z\rangle_*)^2})$$

$$(i.e.\ z \in_* Y \text{ iff } z = \langle ^1z^2z\rangle_* \text{ and } {}^1z \in_* {}^2z)\,.$$

(cl 2): (The set-theoretical difference-axiom.)

$$\forall X \forall Y \exists Z \forall u(\in_* (u, Z) = 2^{\in_*(u,X) - \in_*(u,Y) - 1})\,.$$

Conventions 11. (Boolean operations.) a) Write $Z = X \mathbin{\dot{-}_*} Y$ in (cl 2), calling it a set theoretical difference of X and Y. Note that, for the moment, we do not need the unicity of the Z in question (this follows later, by the "extensionality" of $\in_*$). Thus the symbol $X \mathbin{\dot{-}_*} Y$ is preliminarily an ε-symbol (in the well known logical sense of Hilbert); the same remark holds for the terms introduced by means of the term $X \mathbin{\dot{-}_*} Y$, as well as concerning analogous ε-terms.

b) Write $-1 \mathbin{\dot{-}_*} Y = \dot{-}_* Y$ and $X \cap_* Y = X \mathbin{\dot{-}_*} (X \mathbin{\dot{-}_*} Y)$ and $X \cup_* Y = \dot{-}_* (\dot{-}_* X \cap_* \dot{-}_* Y)$, calling them a "union", and an "intersection" of the "classes" X and Y respectively.

(cl 3): (The "direct product"-axiom.)

$$\forall X \forall Y \exists Z \forall u (\in_* (u, Z) = \in_* ({}^1u, X) \,.\, \in_* ({}^2u, Y) \,.\, 2^{-(u - \langle {}^1u {}^2u \rangle_*)^2})$$

$$(i.e.\ u \in_* Z \text{ iff } u = \langle {}^1u {}^2u \rangle_* \text{ and } {}^1u \in_* X,\ {}^2u \in_* Y)\,.$$

Conventions 12. a) Write $Z = X \overset{\times}{*} Y$ for any Z given by X and Y in (cl 3).

b) Further, given X, Y, set $\mathbf{U}(Y, X) = ((-1) \overset{\times}{*} 2^Y) \overset{\cap}{*} X$, introducing a further ε-term; this term is useful in a not obvious dyadic formulation[6]) of the next axiom "of domain"; note that $t \in_* \mathbf{U}(X, u)$ iff $t = \langle {}^1t {}^2t \rangle_* \in_* X$ and ${}^2t = u$.

(cl 4): $\forall X \forall Y \forall u (\in_* (u, Y) = \mathrm{sg}\,(\mathbf{W}(\mathbf{U}(u, X)) + 1))$. (Explicitly, to every X there is an Y such that any u "belongs" (in the sense of $\in_*$) to Y iff $\mathbf{U}(u, X) = ((-1) \overset{\times}{*} 2^u) \overset{\cap}{*}$ $\overset{\cap}{*} X \neq 0$, *i.e.* iff there is an y such that $\langle yu \rangle_* \in_* X$; see convention 6 and the fact that -1 is the "universal class".

Convention 13. Write $Y = \mathbf{D}_*(X)$ for any Y given by X in (cl 4) and call $\mathbf{D}_*(X)$ a "domain" of X.

(cl 5): (The axiom of conversion in ordered pairs.)

$$\forall X \exists Y \forall u (\in_* (u, Y) = \in_* (\langle {}^2u {}^1u \rangle_*, X) \,.\, 2^{-(u - \langle {}^1u {}^2u \rangle_*)^2})\,.$$

Convention 14. Write $Y = \mathbf{C}nv_{*1}(X)$ for any Y given by X in (cl 5).

(cl 6): (The axiom of "conversion in ordered triples".)

$$\forall X \exists Y \forall u (\in_* (u, Y) = \in_* (\langle {}^1({}^2u) \langle {}^2({}^2u)\, {}^1u \rangle_* \rangle_*, X) \,.\, 2^{-(u - \langle {}^1u \langle {}^1({}^2u) {}^2({}^2u) \rangle_* \rangle_*)^2})\,.$$

Convention 15. Write $Y = \mathbf{C}nv_{*2}(X)$ for any Y given by X in (cl 6) This concludes the "class"-axioms.

Finally, let us state the singular additional so-called successor-principle:

(s): There exists a "class" S such that $\in_* (u, S) = ({}^2u - {}^1u) \,.\, 2^{-(u - \langle {}^1u {}^2u \rangle_*)^2}$. This principle requires the "class" of all "ordered pairs" such that the "second member" exceeds the "first member" by 1; thus the "class" S represents the successor-relation. As has been mentioned, the purpose of (s) is not to ensure some set-theoretical axiom for $\in_*$, but to imply the reproduction-theorem IIi of sec. 4.

The given $14 + 7 + 6 + 1 = 28$ axioms of the system (σ) (divided into the groups (r), (d), (cl) and (s), and containing several consequences and conventions) define the notion of dyadic arithmetic. It is not too difficult to prove that Hensel's integral dyadic numbers (represented *e.g.* by zero-one sequences) satisfy these axioms: the non-negative integers (in the dyadic system) are "sets", and the other dyadic integers are "proper classes". In the forthcomming paper, we illustrate the variety of other examples of dyadic arithmetics; some have already been shown in the paper [II].

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[6]) For further constructive purposes, we attempt to satisfy two requirements: (i) The "characteristic function" $\in_*(u, Y)$ of the "domain" Y of X shall be explicitely given (by a term as in other (cl)-axioms). (ii) The "domain-axiom" shall have a simple prenex normal form, with an equation as the scope. For the exceptional character of the "domaino-peration", see also 4,2 (III) below.

4.1. Our aim is to prove the two main theorems of this paper as indicated in sec. 1. Let us formulate them precisely on the basis of sections 2 and 3.

The equivalence-theorem.

Ii. (The first part of the equivalence-theorem.)

Let $(\cdot) + (\cdot), (\cdot) \cdot (\cdot), 2^{(\cdot)}$ *be the three primitive operations of dyadic arithmetic (of abstract Hensel's dyadic integers) satisfying the 27 (proper) axioms of the system* (σ) *of sec 3 (i.e. with the successor principle (s) disregarded). Define two unary predicates* $\mathbf{Cl}(\cdot)$, $\mathbf{M}(\cdot)$ *and the binary predicate* $(\cdot) \in_* (\cdot)$ *as follows*

$$\mathbf{Cl}_*(Y) \underset{\mathrm{df}}{\Leftrightarrow} Y = Y, \quad \mathbf{M}_*(X) \underset{\mathrm{df}}{\Leftrightarrow} 2^X \neq 0 ;$$

$$X \in_* Y \underset{\mathrm{df}}{\Leftrightarrow} \mathrm{sg}\,(2^X) \,.\, ([Y/2^X] - 2[Y/2^{X+1}]) = 1 \,.$$

Then $\mathbf{Cl}_*$, $\mathbf{M}_*$, $\in_*$ *satisfy the axioms of the system* Σ' *of sec. 2, of the theory of finite sets of Bernays-Gödel.*

Iii. (The second part of the equivalence-theorem.)

Let $\mathbf{Cl}(\cdot)$, $\mathbf{M}(\cdot)$, $(\cdot) \in (\cdot)$ *be the three primitive notions of the theory of finite sets and their classes of Bernays-Gödel as based on the axiomatic system* Σ' *(of sec. 2) Then it is possible to introduce three basic dyadic operations* $(\cdot)(+)(\cdot)$, $(\cdot)(\cdot)(\cdot)$, $(2)^{(\cdot)}$ *with* $(2) = \{\{\emptyset\}\}$ *(where* $\emptyset$ *is the void set) for all classes of* Σ' *as taken for abstract Hensel's dyadic integers in such a manner that they satisfy all the proper 27 axioms of dyadic arithmetic; the successor principle (s) is also satisfied.*

The reproduction-theorem.

IIi. (The first part of the reproduction-theorem.)

Suppose the situation of theorem Ii, and also let the successor-principle (s) hold. Let us introduce three new dyadic operations, say $(\cdot) \overset{+}{*} (\cdot)$, $(\cdot) \overset{\cdot}{*} (\cdot)$, $2_*^{(\cdot)}$, *upon "classes" in the sense of the already (in theorem Ii) defined dyadic membership-predicate* $\in_*$, *applying theorem Iii to* $\in_*$ *instead of to* $\in$.

Then $\underset{*}{2} = 2$ *and* $(\cdot) \overset{+}{*} (\cdot)$, $(\cdot) \overset{\cdot}{*} (\cdot)$, $\underset{*}{2}^{(\cdot)}$ *are identical with the supposed primitive dyadic operations* $(\cdot) + (\cdot)$, $(\cdot) \cdot (\cdot)$, $2^{(\cdot)}$ *respectively.*

IIii. (The second part of the reproduction-theorem.) *Suppose the situation of theorem Iii. Let us introduce a dyadic membership-predicate, say* $(\in)_*$, *by means of the already defined (in theorem Iii) dyadic operations* $(\cdot)(+)(\cdot)$, $(\cdot)(\cdot)(\cdot)$, $(2)^{(\cdot)}$ *applying theorem Ii to them (instead of to* $(\cdot) + (\cdot)$, $(\cdot) . (\cdot)$, $(2)^{(\cdot)}$ *respectively). Then* $(\in)_*$ *is identical with the originally supposed primitive membership-predicate* $\in$.

Remark. In [II], we have reached, in fact, a result essentially very close to Ii. Both the non-elementary formulations and the proofs of [II] will be considerably improved, clarified and elementarized in what follows. The rest of this sec. 4 appears to be new. Note again that our notion of so-called set-theoretical (s.-t.) dyadic rings of [II] is not elementary because it is formulated as a class of certain rings within the axiomatic set theory (of Bernays-Gödel, with the axiom of infinity), in the usual sense of abstract set-theoretical algebra. Here in [III], on the contrary, dyadic arithmetic appears as a self-contained elementary axiomatic theory. In [IV] we shall also generalize and improve the constructions of various concrete dyadic arithmetics in the set theory of Bernays-Gödel (as initiated in [II]).

Proof of Ii. We consider a dyadic arithmetic in the elementary sense, as a certain integrity domain (of the so-called dyadic integers also called "classes"), without one half and with characteristic not 2; there is a discretely ordered subdomain consisting of all dyadic integers X such that either $2^X \neq 0$ or $2^{-X} \neq 0$. In particular, all the X's with $2^X \neq 0$ (i.e. $0 \leqq X$) are called "naturals" or also "sets" and denoted by $x, y, z, \ldots$

The above binary predicate $\in_* (X, Y)$ clearly has the following property: X is a "set" iff there exists a "class" Y such that $X \in_* Y$. (This follows at once from the definition of $\mathrm{sg}\,(\cdot)$.)

Thus the axiom A2 holds, and the axiom A1 is satisfied trivially. Let us turn to the verification of the axiom of extensionality A3.

First note that, considering $[Y/2^X]$, we cannot speak of the quotient $Y/2^X$ itself explicitly, as in the remark on p. 9 of [II], for we have no quotient field at our disposal. (Compare the original proof of the crucial lemma XXIII of [II].)

Lemma 20. *If* $X \neq Y$ *then* $\in_* (\mathbf{W}(X - Y), X) \neq \in_* (\mathbf{W}(X - Y), Y)$. *As a consequence, we have: If* $\forall U (U \in_* X \Leftrightarrow U \in_* Y)$ *then* $X = Y$, *i.e. the axiom of extensionality* A3 *is true.*

The proof of lemma 20 requires a further lemma also useful for other purposes.

Lemma 21. *Given* X, Y, z *let us write* (by lemma 6 of sec. 3)

$$X = [X/2^z] \,.\, 2^z + x\,, \quad 0 \leqq x \prec 2^z\,,$$
$$Y = [Y/2^z] \,.\, 2^z + y\,, \quad 0 \leqq y \prec 2^z\,,$$

and let us assume further that $y \leqq x$. *Then*

$$[X - Y/2^z] \doteq [X/2^z] - [Y/2^z]\,.$$

Proof of lemma 21. From the supposition we have $X - Y = ([X/2^z] - [Y/2^z]) \,.$ $.\, 2^z + x - y$. Since $0 \leqq x - y \prec 2^z$, lemma 6 (sec. 3) implies the result.

Proof of lemma 20.

Suppose the contrary, *i.e.* suppose $X \neq Y$ and $\in_* (\mathbf{W}(X - Y), X) = \in_* (\mathbf{W}(X -$ $- Y), Y)$.

Because then $\mathbf{W}(X - Y)$ is a "set" (by the lemma 7), putting $z = \mathbf{W}(X - Y)$ we have

$$\in_*(z, X) = [X/2^z] - 2[X/2^{z+1}] = \in_* (z, Y) = [Y/2^z] - 2[X/2^{z+1}]\,.$$

Therefore $[X/2^z] - [Y/2^z] = 2 . ([Y/2^{z+1}] - [X/2^{z+1}])$. Applying the above lemma 21, we infer $\pm [(X - Y)/2^z] = 2[(Y - X)/2^{z+1}]$. On the other hand, $[(X - Y)/2^z] = 1 + 2Z$ for some Z, according to lemma 7 (sec. 3). Thus $1 = 2 . . (- Z \pm [(Y - X)/2^{z+1}])$ in contradiction with the axiom (r13) (of the absence of one-half); this proves lemma 20. – Thus the axiom of extensionality A3 is true for $\in_*$. In order to prove further axioms, let us infer some more lemmas.

Lemma 22. *If* $X \neq 0$ *then* $\mathbf{W}(X) \in_* X$.

Proof. According to lemma 16, we have $[X/2^{\mathbf{W}(X)}] = [2^{\mathbf{W}(X)} . (1 + 2Z)/2^{\mathbf{W}(X)}] = = 1 + 2Z$. Using the auxiliary identity of the proof of lemma 16, we obtain

$$[X/2^{\mathbf{W}(X)+1}] \doteq [[2^{\mathbf{W}(X)} . (1 + 2Z)/2^{\mathbf{W}(X)}]/2] = [(1 + 2Z)/2] = Z .$$

Therefore indeed $\in_* (\mathbf{W}(X), X) = (1 + 2Z) - 2Z = 1$.

Lemma 23. *If* $z \prec \mathbf{W}(X)$ *then* $\neg(z \in_* X)$.

Proof. $z \prec \mathbf{W}(X)$ implies $[X/2^z] = [2^{\mathbf{W}(X)} . (1 + 2Z)/2^z] = 2^{\mathbf{W}(X)-z}(1 + 2Z)$. Likewise (for $z + 1 \leqq \mathbf{W}(X)$) $[X/2^{z+1}] = 2^{\mathbf{W}(X)-z-1} . (1 + 2Z)$. Thus indeed

$$\in_*(z, X) = 2^{\mathbf{W}(X)-z} . (1 + 2Z) - 2.2^{\mathbf{W}(X)-z-1} . (1 + 2Z) = 0 .$$

Lemma 24. *If* $u \in_* v$ *then* $u \prec v$ *(i.e. if* $v \leqq u$ *then* $\neg(u \in_* v)$*).*

Proof. If $v \leqq u$ then $v \prec 2^u$ (by the axiom (d″1)), whence $[v/2^u] = [v/2^{u+1}] = 0$, so that indeed $\in_* (u, v) = 0 - 2 . 0 = 0$.

Remark. The following simple observation may be useful when determining whether a given "set" t is "an element" of a given "class" U:

U can be written in the form $U = Z . 2^{t+1} + r, 0 \leqq r \prec 2^{t+1}$, where the "class" Z and the "set" r are unique; thus there are exactly two cases:

Case 1. $r \prec 2^t$. In this case (because $U = 2Z . 2^t + r, 0 \leqq r \prec 2^t$) clearly $[U/2^t] = = 2Z$ and $[U/2^{t+1}] = Z$.

Therefore $[U/2^t] - 2[U/2^{t+1}] = 0$, *i.e.* $\neg(t \in_* U)$.

Case 2. $2^t \leqq r \prec 2^{t+1}$. In this case $U \doteq Z . 2^{t+1} + 2^t + \bar{r} = (2Z + 1) . 2^t + \bar{r}$, where $0 \leqq \bar{r} = r - 2^t \prec 2^t$ (because $r \prec 2^t + 2^t = 2^{t+1}$ and $\bar{r}$ is unique).

Therefore in this case $[U/2^t] = 2Z + 1$, though $[U/2^{t+1}] = Z$ as before. Thus $[U/2^t] - 2[U/2^{t+1}] = (2Z + 1) - Z = 1$, *i.e.* $t \in_* U$.

Summarising: Let $U = Z . 2^{t+1} + r$, $0 \leqq r \prec 2^{t+1}$ (with Z and r uniquely determined by the given U and t). Then $t \in_* U$ iff $t \in_* r$, *i.e.* iff $2^t \leqq r$.

Now we prove the fundamental lemma:

Lemma 25. *Assume* $\neg(t \in_* X)$. *Then* $x \in_* X + 2^t \Leftrightarrow x \in_* X \vee x = t$. *(In other terms:* $X \cup_* \{t\}_* = X + 2^t$ if $\neg(t \in_* X)$, *introducing in this case the sign of the "class-sum"* $\cup_*$, in the sense of $\in_*$.)

Proof. By the above remark, $\neg(t \in_* X)$ means that $X = Z \,.\, 2^{t+1} + r_t$, $0 \leqq r_t \prec 2^t$. Therefore $X + 2^t = Z \,.\, 2^{t+1} + (2^t + r_t)$, where $2^t \leqq 2^t + r_t \prec 2^{t+1} = 2^t + 2^t$, so that $t \in_* X + 2^t$ (on account of the above remark).

Thus in the proof of the equivalence $x \in_* X + 2^t \Leftrightarrow x \in_* X \vee x = t$ we may assume $x \neq t$, *i.e.* we only have to prove the equivalence

$$(x \neq t) \,\&\, (x \in_* X + 2^t) \Leftrightarrow x \in_* X\,.$$

The case $x \prec t$: In this case $0 \prec 2^{t-x-1}$. Assume $x \in_* X + 2^t$. Then $X + 2^t = \bar{Z} \,.\, 2^{x+1} + \tilde{r}$ with $2^x \leqq \tilde{r} \prec 2^{x+1}$. Thus $X = (X + 2^t) - 2^t = \bar{Z} \,.\, 2^{x+1} - 2^t + \tilde{r} = (\bar{Z} - 2^{t-x-1}) \,.\, 2^{x+1} + \tilde{r}$ with $2^x \leqq \tilde{r} \prec 2^{x+1}$ and therefore $x \in_* X$ (by the above remark).

Conversely, assume $x \in_* X$, *i.e.* $X = Z \,.\, 2^{x+1} + r_x$ with $2^x \leqq r_x \prec 2^{x+1}$. Then

$$X + 2^t = Z \,.\, 2^{x+1} + 2^t + r_x = (Z + 2^{t-x-1}) \,.\, 2^{x+1} + r_x$$

with $2^x \leqq r_x \prec 2^{x+1}$, *i.e.* $x \in_* X + 2^t$.

The case $t \prec x$: Then $0 \prec 2^{x-t-1}$. Assume $x \in_* X + 2^t$; then $X + 2^t = \bar{Z} \,.\, 2^{x+1} + 2^x + s$, $0 \leqq s \prec 2^x$.

Now, $t \in_* s$. (Indeed, otherwise we would have $s = u \,.\, 2^{t+1} + \tilde{r}_t$ with $\tilde{r}_t \prec 2^t$ and thus $X + 2^t = \bar{Z} \,.\, 2^{x+1} + 2^x + u \,.\, 2^{t+1} + \tilde{r}_t = (\bar{Z} \,.\, 2^{x-t} + 2^{x-t-1} + u)\, 2^{t+1} + \tilde{r}_t$, $\tilde{r}_t \prec 2^t$, *i.e.* we would have $\neg(t \in_* X + 2^t)$ contrary to the already proved result.) From $t \in_* s$ we obtain $s = u \,.\, 2^{t+1} + 2^t + \tilde{r}_t$, $2^t \leqq \tilde{r}_t$. Thus

$$X = (X + 2^t) - 2^t = \bar{Z} \,.\, 2^{x+1} + 2^x + (s - 2^t)\,, \quad 0 \leqq s - 2^t \prec 2^x\,,$$

i.e. $x \in_* X$. Conversely, assume $x \in_* X$, *i.e.* $X = Z \,.\, 2^{x+1} + 2^x + r$ with $r \prec 2^x$. For similar reasons as above, since $\neg(t \in_* X)$, then $\neg(t \in_* r)$, *i.e.* $r = v \,.\, 2^{t+1} + q$ with $q \prec 2^t$. Thus $X + 2^t = Z \,.\, 2^{x+1} + 2^x + v \,.\, 2^{t+1} + 2^t + q = (Z \,.\, 2^{x-t} + 2^{x-t-1} + v) \,.\, 2^{t+1} + 2^t + q$, $q \prec 2^t$, proving that $x \in_* X + 2^t$. This proves lemma 25.

Now, the axiom A4 (of unordered pairs) holds for $\in_*$; this is a simple consequence of lemma 25. Indeed, given the "sets" x, y, write $\{xy\}_* = 2^x + 2^y \,.\, (\mathrm{sg}\,(x - y))^2$, *i.e.* $\{xy\}_* = 2^x + 2^y$ if $x \neq y$ and $\{xx\}_* = 2^x$. We observe directly that $z \in_* \{xy\}_* \Leftrightarrow z = x \vee z = y$.

Turning to the verification of the axioms of group B (of Σ'), note that 0 indeed is the "void set" (clearly $\in_* (x, 0) = 0$ for every x since $[0/2^x] = [0/2^{x+1}] = 0$) and that -1 is the "universal class" (since for every x, $-1 = -1 \,.\, 2^x + (2^x - 1)$, $2^x - 1 \prec 2^x$, *i.e.* $[-1/2^x] = [-1/2^{x+1}] = -1$, so that $\in_* (x, -1) = -1 - 2 \,.\, (-1) = 1$.

Having the "void set" 0 and the "universal class" -1 at our disposal, we can replace axioms B2 and B3 by the single "axiom of the class-difference" – and in fact this is the axiom (cl 2). Thus in this way axioms B2 and B3 hold for $\in_*$.

Further axioms of group B of Σ' are almost immediately ensured by their dyadic "arithmetical" counterparts (cl 1)–(cl 7), so that nothing need be proved for this group of axioms.

Remark. It would be easy to maintain the original version of the axioms sub B of Σ', *i.e.* to require B2 and B3 in the dyadic version. Then B3 would have the form

$$\forall X \forall Y \exists Z \forall u (\in_* (u, Z) = \in_* (u, X) . \in_* (u, Y)) ,$$

and B2 would be written as

$$\forall X \exists Z \forall u (\in_* (u, Z) = 1 - \in_* (u, X)) .$$

Note, however, that the last statement follows from the axioms sub (r) and (d) (*i.e.* from the only axioms of dyadic domains). Namely, we observe that $Z = -1 - X$ has the property of (Boolean) complementation. Indeed, $-1 = -1 . 2^x + (2^x - 1)$, so that the remainder r of X on division by 2^x is always less or equal to the remainder $2^x - 1$ of -1 divided by 2^x. Thus we can apply lemma 21 and obtain

$$[(-1 - X)/2^x] = [-1/2^x] - [X/2^x] = -1 - [X/2^x]$$

and also

$$[(-1 - X)/2^{x+1}] = -1 - [X/2^{x+1}] .$$

Therefore indeed

$$\begin{aligned} \in_* (x, -1 - X) &= -1 - [X/2^x] - 2(-1 - [X/2^{x+1}]) = \\ &= 1 - ([X/2^x] - 2[X/2^{x+1}]) = 1 - \in_* (x, X) . \end{aligned}$$

Next, proceed to the immediate verification of the "Fundierungsaxiom" D of Σ'. Indeed, from lemmas 22, 23, 26 we conclude immediately that if $X \neq 0$, then no "set" is simultaneously a "member" of the "member" $\mathbf{W}(X)$ of the "class" X or of this "class" X itself; thus the axiom D holds for $\in_*$.

So far we have verified the axioms sub A, B, D of Σ' for $\in_*$.

Finally, let us turn to the verification of the axioms of group C of Σ'. We emphasize here, that, despite of our opinion put forward in [II], these axioms are satisfied without any additional requirement concerning our primitive operations of dyadic arithmetics; the proof is by very similar simple devices.

Let us elaborate the argument in the case of the axiom C4 (of replacement). In order to prove this axiom for $\in_*$, let us apply a certain particular instance of the general existence-metatheorem M1 of [G], which holds for our $\in_*$ on account of the axioms already verified. Roughly speaking, this metatheorem, say $(\mathrm{M1})_*$ with respect to our membership-predicate $\in_*$ is only a "comprehension"-statement. It enables us to define a "class", say U, by a condition of the form $u \in_* U \underset{\mathrm{df}}{\Leftrightarrow} \Phi(u)$ with u free in Φ, supposing only that in the propositional function $\Phi(u)$ there are no quantified "class" variables; ($\Phi(u)$ is "normal"); further eventual free "set"- or "class"-variables of $\Phi(u)$ are then parameters. (The definition of the notion of a propositional function is by the obvious metamathematical recursion, starting with atomic propositional functions of the forms $x \in_* Y$, $x \in_* y$, $X \in_* y$, $X \in_* y$, $x = X$, $X = Y$ and successively applying logical operations; we refer to [G] for details.) If we wish to avoid any metamathematical notion in using a particular instance of $(\mathrm{M1})_*$, we have only to follow the successive logical building up of the given condition $\Phi(u)$ by forming an explicit term for the "class" U in question, by means of the basic "class"-operations introduced in (cl), according to the proof of M1 in [G]; and this is a matter of routine which can be omitted here for the sake of better readability.

The particular instance of $(\mathrm{M1})_*$ that we use in order to prove C4 for $\in_*$ is as follows:

Let Z be a given "class". Then there exists a "class", say U_z such that $v \in_* U_z$ if $Z''_* v$ (= the "class" of all the "Z-images" of "members" of the "set" v) is a "proper class" (*i.e.* not a "set"). Clearly $U_z \neq 0$ for some Z if C4 does not hold for $\in_*$.

Thus, contrary to C4, assume $U = U_z \neq 0$ for a certain Z. Then $u = \mathbf{W}(U)$ is a "set" and $u \in_* U$, *i.e.* $Z''_* u$ is a "proper class". We have $\mathbf{W}(U) \neq 0$ because $Z''_* 0 = 0$ is a "set". Therefore $\mathbf{W}(\mathbf{W}(U)) = \mathbf{W}(u) = v$ is a "set" also (and $v \in_* u$).

Now, lemma 25 states that the "class-difference" $u \dot{-}_* \{v\}_*$ (existing as a "class" on account of (cl 2) is precisely the ring-theoretical difference $u^* = u - 2^v$ of two "sets" (where $2^v \lesssim u$), so that u^* is a "set"; furthermore, we observe $u^* \prec u$, whence $\neg(u^* \in_* U)$.

Therefore $Z''_* u^*$ is a "set". But $Z''_* 2^v = Z''_* \{v\}_* = \{Z'_* v\}_*$ is a "set" also and the "set" u is the "class-sum" of the "disjoint" "set-summands" u^* and $\{v\}_*$ (for otherwise $Z''_* u^* = Z''_* u$ would be a "set", contrary to our assumption).

Therefore the "image-class" $Z''_* u$ satisfies the equations

$$Z''_* u = (Z''_* u^*) \underset{*}{\cup} (Z''_* \{v\}_*) = Z''_* u^* + 2^{Z'_* v}$$

and thus is a "set" (according to lemma 25). From this contradiction we conclude $U_Z = 0$ for every "class" Z; therefore the replacement-axiom C4 is true for our dyadic membership-predicate $\in_*$.

The remaining needed verifications of the C-axioms of Σ' for $\in_*$ now are simple and could be performed by the argument already used. However, let us show a more simple and direct way.

According to section 2, it is sufficient to verify the "set-sum"-axiom C2 and the axiom of finiteness $\neg$(C1).

As to C2, we already have the "sum-class" $\mathbf{S}(x)$ for every "set" x as the "class" Y such that $u \in_* Y$ precisely iff there is a "set" v with $u \in_* v \,\&\, v \in_* x$, by a further particular instance of the mentioned $(\mathrm{M1})_*$. Thus by lemma 24, we conclude $u \prec v \,\&$ $\&\, v \prec x$, whence $u \prec x$ for every $u \in_* Y$.

Now, there is a "set" q such that $\forall t (t \prec x \Leftrightarrow t \in_* q)$; namely $q = 2^{x+1} - 1$. (Indeed, if $t \prec x$ then

$$2^{x+1} - 1 = 2^{x+1} - 2^{t+1} + 2^{t+1} - 1 = (2^{x-t} - 1) \,.\, 2^{t+1} + 2^t + (2^t - 1)\,,$$

whence $t \in_* 2^{x+1} - 1$.) We observe that the "sum-class" $\mathbf{S}(x)$ (over the "set" x) is "included" in the "set" $2^{x+1} - 1$ and therefore $\mathbf{S}(x)$ itself is a "set"; this is a simple consequence of the already verified "replacement-axiom" C4 (see [G], 4.31 and 5.11).

Finally, as to the axiom $\neg(C1)$ of finiteness, we proceed as in the above verification of C4. On account of a further instance of $(M1)_*$, define the "class" U so that $v \in_* U$ iff v violates the axiom of finity. Assume $U \neq 0$, *i.e.* that $\neg(C1)$ does not hold. Then $u = \mathbf{W}(U)$ is the lowest "set" in U and clearly $u \neq 0$. Thus $\mathbf{W}(u) = w$ is a "set" also and $u = (u - 2^w) \underset{*}{\cup} 2^w$ with "disjoint" "set-summands". Here $0 \leqq u - 2^w \prec u$, so that $u - 2^w$ does not violate $\neg(C1)$. This is a contradiction, since then clearly u itself could not violate $\neg(C1)$.

Hereby the axiom of finiteness $\neg(C1)$ is also verified, and the whole first half Ii of the equivalence-theorem proved.

4.2. Proof of both the Iii and IIii:

(I) *The idea of* the following *simultaneous proof* of Iii and IIii can be traced thus:

We shall introduce the so-called Hensel's dyadic integers over ordinals of Σ', as zero-one ordinal functions. We hereby imitate Hensel's original immediate introduction of this intuitive dyadic integers rather than the usual definition of them (by completing rationals in the sense of a dyadic metric). In this part of the proof then only remains to ensure that the elementary logical tools of Σ' just suffice to this purpose. Then we shall verify the axioms of dyadic domains for dyadic integers over ordinals of Σ' and introduce the dyadic binary membership-predicate $\in_*(.,.)$ to dyadic integers, by the well-known formula. On the other hand, we observe that the so-called finite dyadic integers (with ultimately constant zero-value) can be isomorphically replaced by ordinals, using their dyadic digital images; thus we transfer $\in_*$ (between finite dyadic integers) into the dyadic membership-relation, say $\in^*(.,.)$, between ordinals. ($\in^*$ is defined in the sense of the usual Peanian arithmetic of ordinals of Σ' as of naturals, by the formula (*) again.)

Finally, $\in^*$ itself will be transferred into the primitive $\in$ between sets, by an isomorphism F (as a mapping of $\mathbf{O}n$ onto V) the inductive construction of which is a particular instance of an idea of Mostowski [MII] (see also Shepherdson [Sh]). These results finally yield the binary one-to-one correspondence-predicate $\mathbf{F}(.,.)$ between our dyadic integers over ordinals of Σ' and all classes of Σ', $\mathbf{F}$ being an isomorphism with respect to $\in_*$ and $\in$. At the same time, of course, $\mathbf{F}$ induces to all classes the three basic operations of dyadic arithmetic as satisfying all the 28 axioms of the system (σ) of section 3; these induced operations reproduce the primitive $\in$ as an arithmetical dyadic membership-predicate, in the sense of the known formula (*).

In an appendix of this proof of both Iii and IIii – we give a more direct and more intuitive description of the just mentioned "natural" arithmetical operations on classes of Σ', in terms of $\in$.

Many lengthy but obvious or essentialy well known details of the proof may be omitted; the main points of the proof (according to the just traced scheme) consist in warranting the normality of some used notions (in the sense of [G], as applied to the system Σ'), in order to ensure the existence of some needed classes. (However, a knowledge of Hensel's intuitive dyadic numbers is desirable, of course.)

(II) *The construction of Hensel's dyadic integers over ordinals of the system Σ'.*

Notation. We assume the axioms of Σ' (see section 2) and we shall use the following notations. By μ (possibly with indices), we denote zero-one functions from an ordinal v into the set $\{01\}$. (We take the values 0, 1 for "dyadic numerals" in Σ' and the μ form a numerical (digital) image of a certain ordinal in the dyadic digital system, as usual; on the other hand, recall that $\{01\} = 2$.)

We write the ordinal corresponding to a given μ thus

$$\alpha_\mu = \sum_{\xi=0}^{v-1} \mu(\xi) \cdot 2^\xi$$

in the usual arithmetical sense of the symbols, as applied to ordinals of Σ'.

Conversely, if an ordinal α is given, then there are many zero-one functions μ with $\alpha = \alpha_\mu$ (they form a proper class, of course). Exactly one μ of them is defined on the lowest ordinal, say $\bar{v}$, such that $\alpha = \alpha_{\bar{v}}$. This zero-one function called the shortest dyadic digital image of the given ordinal α – and we denote it by $\vec{\alpha}$.

For any zero-one function μ (as defined on an ordinal v) and for any given ordinal β, let us define another zero-one function $\mu \mid \beta$ thus:

If $\beta \leqq v$, then $\mu \mid \beta$ is the function μ partialized to β (in the usual sense, cf. [G]); if otherwise $v < \beta$, then $\mu \mid \beta$ is the μ extended by means of zero-values, *i.e.*

$$(\mu \mid \beta)(\xi) = 0 \quad \text{if} \quad \xi \in \beta \dot{-} v \quad \text{and} \quad (\mu \mid \beta)(\xi) = \mu(\xi) \quad \text{if} \quad \xi \in v\,.$$

Definition 3. (Dyadic integers.) a) A Hensel's dyadic integer over ordinals of Σ', in short, a dyadic integer, is a zero-one ordinal function; thus Γ is a dyadic integer iff $\Gamma \subseteq \mathbf{On} \times \{01\}$ and Γ maps the class $\mathbf{On}$ of ordinals into the set $\{01\}$.

Greek capitals will denote variables for dyadic integers. If Λ is a dyadic integer and α is an ordinal, then $\Lambda \mid \alpha$ denotes, as usual, the Λ partialized to the ordinal α.

b) A dyadic integer Λ is said to be finite iff it is ultimately a zero-function, *i.e.* if $\exists\bar{\xi}(\bar{\xi} \leqq \xi \Rightarrow \Lambda(\xi) = 0)$.

For every finite Λ, let us write $\Lambda = \Lambda_\alpha$ where $\alpha = \sum_{\xi=0}^{\bar{\xi}} (\Lambda \mid (\bar{\xi} + 1))(\xi) \cdot 2^\xi$ with the uniquely determined lowest possible $\bar{\xi}$.

Conversely, given any nonzero ordinal $\alpha = 2^{\alpha_1} + 2^{\alpha_2} + \ldots + 2^{\alpha_\kappa}$ (with $\alpha_1 < \alpha_2 < \ldots < \alpha_\kappa$), there is exactly one Λ such that $\Lambda = \Lambda_\alpha$; this Λ_α simply is the sum-class of the class of all the functions of the form $\vec{\alpha} \mid \beta$ with $\beta > \alpha_\kappa$. In the case of $\alpha = 0$, Λ_0 is the zero-function (with $\forall\xi(\Lambda_0(\xi) = 0)$).

c) Thus there is a one-to-one correspondence, given by a binary predicate, say $\mathbf{C}orr(.,.)$, between ordinals and finite dyadic integers. In normal terms, we write

$$\mathbf{C}orr(\alpha, \Gamma) \Leftrightarrow \Gamma = \Lambda_\alpha \quad (\text{in the sense of b})\,.$$

The normality of the predicate $\mathbf{C}orr(.,.)$ is warranted by the equivalence

$$\mathbf{C}orr(\alpha, \Gamma) \Leftrightarrow \exists\bar{\beta}\forall\beta(\bar{\beta} \leqq \beta \Rightarrow \sum_{\xi=0}^{\beta-1} (\Gamma \mid \beta)(\xi) \cdot 2^\xi = \alpha)\,;$$

on the other hand, of course, this correspondence is not a function in Σ' since finite dyadic integers are proper classes in Σ'. (A dyadic integer which is not finite may be called infinite.) Thus we write $\Lambda_0 = \bar{0}$, $\Lambda_1 = \bar{1}$, $\Lambda_2 = \bar{2}$.

Definition 4. *Addition, multiplication and potentiation of two for dyadic integers.*)

Let Λ, Γ be dyadic integers. If $\beta_1 < \beta_2$, then abbreviate

$$\Lambda \mid \beta_1 = \mu_1 , \quad \Gamma \mid \beta_1 = \mu_1^* , \quad \Lambda \mid \beta_2 = \mu_2 , \quad \Gamma \mid \beta_2 = \mu_2^*$$

and set

$$\alpha_{\mu_i} = \sum_{\xi=0}^{\beta_i - 1} \mu_i(\xi) \,.\, 2^{\xi} , \quad \alpha_{\mu_i^*} = \sum_{\xi=0}^{\beta_i - 1} \mu_i^*(\xi) \,.\, 2^{\xi} \quad \text{for} \quad i = 1, 2 .$$

a) (*Addition.*) Realizing the usual addition of ordinals by means of their dyadic digital images, we easily observe that

$$\overrightarrow{\alpha_{\mu_1} + \alpha_{\mu_1^*}} \mid \beta_1 = \overrightarrow{\alpha_{\mu_2} + \alpha_{\mu_2^*}} \mid \beta_1 .$$

This is the so-called stability-condition of dyadic digits for addition. (In nonelementary terms of intuitive dyadic integers, this is a quite strong uniform-continuity condition, which is fullfiled by the addition of naturals in the sense of their dyadic metric.[7]) Thus we can and will define the dyadic integer $\Lambda + \Gamma$ as the sum-class of the class of all the zero-one-functions on ordinals of the form $\overrightarrow{\alpha_\mu + \alpha_{\mu^*}} \mid \beta$, where $\mu = \Lambda \mid \beta$, $\mu^* = \Gamma \mid \beta$ and $\beta \in \mathbf{On}$ is arbitrary.

Hereby we introduce a ternary normal (operational) predicate, say $+(.,.,.)$, so that

$$+ (\Lambda, \Gamma, \Delta) \underset{\text{df}}{\Leftrightarrow} \Delta = \Lambda + \Gamma .$$

b) (*Multiplication.*) Analogously, observing the multiplication of ordinals of Σ' in their dyadic digital images, we see an analogous stability-condition (in the above symbols):

$$\overrightarrow{\alpha_{\mu_1} \,.\, \alpha_{\mu_1^*}} \mid \beta_1 = \overrightarrow{\alpha_{\mu_2} \,.\, \alpha_{\mu_2^*}} \mid \beta_1 \quad \text{if} \quad \beta_1 < \beta_2 .$$

Thus we can and will define the dyadic integer $\Lambda \,.\, \Gamma$ as the sum-class of the class of all the zero-one functions on ordinals of the form $\overrightarrow{\alpha_\mu \,.\, \alpha_{\mu^*}} \mid \beta$ with $\mu = \Lambda \mid \beta$, $\mu^* = \Gamma \mid \beta$ and with an arbitrary $\beta \in \mathbf{On}$.

Hereby we introduce a further normal ternary (operational) predicate $.(.,.,.)$ so that $.(\Lambda, \Gamma, \Delta) \underset{\text{df}}{\Leftrightarrow} \Delta = \Lambda \,.\, \Gamma$.

c) (*Potentiation of two.*) Denote by $\bar{2}$ the dyadic integer Λ_2 (with $\bar{2}(0) = 0$, $\bar{2}(1) = 1$, $\bar{2}(\tau) = 0$ for every $\tau > 1$).

[7]) The dyadic metric is defined *e.g.* thus: (i) if $0 \leqq m_1 < m_2$ are naturals and $m_2 - m_1 = 2^n(1 + 2q)$, then the dyadic distance is $d(m_1, m_2) = 2^{-n}$; (ii) if otherwise $m_1 = m_2$, then $d(m_1, m_2) = 0$, of course.

We see immediately the corresponding stability-condition (in the sense of the above symbols) thus

$$\overrightarrow{2^{\alpha_{\mu_1}}} \mid \beta_1 = \overrightarrow{2^{\alpha_{\mu_2}}} \mid \beta_1$$

whenever $\mu_1 = \Lambda \mid \beta_1$, $\mu_2 = \Lambda \mid \beta_2$, $\beta_1 < \beta_2$.

Thus again, we can and will define the dyadic integer $\bar{2}^\Lambda$ as the sum-class of the class of all the zero-one functions on ordinals of the form $\overrightarrow{2^{\alpha_\mu}} \mid \beta$, where $\mu = \Lambda \mid \beta$, $\beta \in \mathbf{On}$.

It is easy to see that if $\Lambda = \Lambda_\varrho$ is a finite dyadic integer, then $\bar{2}^\Lambda = \Gamma_{2^\varrho}$, so that $\bar{2}^\Lambda(\xi) = 0$ everywhere except for $\xi = \varrho$. Thus in particular $\bar{2}^{\bar{1}} = \bar{2}$.

If otherwise Λ is an infinite dyadic integer, then we observe that $\bar{2}^\Lambda = \bar{0}$.

Hereby we introduce a binary (operational) normal predicate, say $\bar{2}(.,.)$, with

$$\bar{2}(\Lambda, \Delta) \underset{\text{df}}{\Leftrightarrow} \Delta = \bar{2}^\Lambda .$$

The definitions of dyadic integers and their basic operations in Σ' are then complete.

Note that if $\Lambda = \Lambda_\lambda$, $\Gamma = \Gamma_\gamma$ are finite dyadic integers, then

$$\Lambda_\lambda + \Gamma_\gamma = \Delta_{\lambda+\gamma}, \quad \Lambda_\lambda \,.\, \Gamma_\gamma = \Delta_{\lambda.\gamma}, \quad \bar{2}^{\Gamma_\gamma} = \Delta_{2^\gamma},$$

carrying thus the above addition, multiplication and the potentiation of two finite dyadic integers isomorphically into the usual addition, multiplication and potentiation of two respectively, corresponding ordinals in the sense of the Peanian arithmetic. We shall use this isomorphism later.

Now, analysing the above definitions 3 and 4, we find that dyadic integers form an associative and commutative semigroup with cancellation and with the neutral $\bar{0}$ with respect to the operation $+$. Further, the nonzero dyadic integers form an associative and commutative semigroup with cancellation and with $\bar{1}$ (where $\bar{1}(0) = 1$, $\bar{1}(\xi) = 0$ if $\xi > 0$) as the neutral unit with respect to the operation . . Moreover, it is not difficult to prove the distributive law $\Delta \,.\, (\Lambda + \Gamma) = \Delta \,.\, \Lambda + \Delta \,.\, \Gamma$.

Next, consider the dyadic integer, say Ω, with $\forall\xi(\Omega(\xi) = 1)$.

By the definition 4 a), it is easily seen that Ω can be taken for $-\bar{1}$, *i.e.* $\Omega + \bar{1} = \bar{0}$.

Thus, on account of the distributive law, we conclude that dyadic integers form an integrity domain with the unit $\bar{1}$ and the zero $\bar{0}$. In addition, from the definition of $+$, we observe almost immediately that $\Delta + \Delta \neq \bar{1}$ for every dyadic integer Δ, showing the absence of "one-half" in dyadic integers. Likewise, it is clear that "the characteristic of our integrity domain" is not two, *i.e.* that $\bar{1} + \bar{1} \neq \bar{2}$. (The general notion of the characteristic of an integrity domain or of a field is not elementary, of course.)

In this way, we have verified the axioms of the group (r) of dyadic arithmetics for our dyadic integers of Σ'.

The verification of the axioms of the next group (d) (of dyadic arithmetics) is now almost immediate, in view of the above one-to-one correspondence $\mathbf{Corr}(.,.)$ (between ordinals and finite dyadic integers, which is an isomorphism with respect to $+, ., 2^{(\cdot)}$).

As to the subgroup (d′), we first observe that the normal predicate, say $\mathbf{R}(.)$, with $\mathbf{R}(\Lambda) \underset{\mathrm{df}}{\Leftrightarrow} \exists\gamma(\Lambda = \Gamma_\gamma \vee \Lambda = -1 \,.\, \Gamma_\gamma)$ defines a discretely ordered integrity subdomain such that the corresponding ordering-predicate, say $\prec$, is characterized by the equivalence $\Lambda \prec \Delta \Leftrightarrow \Lambda \neq \Delta \;\&\; \bar{2}^{\Delta-\Lambda} \neq \bar{0}$; and this is exactly the statement of the axioms of subgroup (d′).

Further, the axiom (d″1) is true on account of the above def. 4c) and by the corresponding law for ordinals (applying $\mathbf{C}orr(.,.)$).

As to the axiom (d″2): Given dyadic integer Λ and a finite dyadic integer $\Gamma = \Gamma_\gamma$, we see that $\Lambda = \Delta \,.\, \bar{2}^{\Gamma_\gamma} + \Theta_\varrho$, where $\Delta(\xi) = \Lambda(\xi + \gamma)$, $\varrho < 2^\gamma$, *i.e.* $\bar{0} \leqq \Theta_\varrho \prec \bar{2}^{\Gamma_\gamma}$.

Therefore $[\Lambda/\bar{2}^{\Gamma_\gamma}](\xi) = \Lambda(\xi + \gamma)$ and $[\Lambda/\bar{2}^{\Gamma_\gamma}]$ is the "integral part of the quotient $\Lambda/\bar{2}^{\Gamma_\gamma}$" in dyadic integers.

As to the axiom (d″3): If a dyadic integer $\Lambda \neq \bar{0}$ is given and σ is the lowest ordinal with $\Lambda(\sigma) = 1$, when we observe that $\Lambda = \bar{2}^{\Gamma_\sigma}(\bar{1} + \bar{2} \,.\, \Delta)$, where $\Delta(\xi) = \Lambda(\xi + \sigma + 1)$. Therefore $\mathbf{W}(\Lambda) = \Gamma_\sigma$ (where $\Lambda(\sigma) = 1 \;\&\; \forall\xi(\xi < \sigma \Rightarrow \Lambda(\xi) = 0)$) is the dyadic value of the dyadic integer Λ.

So far we have proved that dyadic integers over ordinals of Σ' form a dyadic domain (in the sense of section 3). Thus we are able to introduce the dyadic membership-operation (as a normal ternary predicate in Σ') for dyadic integers in the sense of section 3 thus:

$$\in_*(\Gamma_\gamma, \Lambda) = [\Lambda/\bar{2}^{\Gamma_\gamma}] - \bar{2} \,.\, [\Lambda/\bar{2}^{\Gamma_\gamma + \bar{1}}]$$

and $\in_*(\Gamma, \Lambda) = 0$ iff Γ is an infinite dyadic integer.

(Note that if we define, for the sake of formal completeness, $[\Lambda/\bar{0}]$ in any manner and if we use the sg-operation (with $\mathrm{sg}(\Gamma) = \bar{1} - \bar{2}^{\bar{1} - \bar{2}^{\Gamma}}$), then we can include the case of an infinite Γ by writing

$$\in_*(\Gamma, \Lambda) = \mathrm{sg}(\bar{2}^{\Gamma}) \,.\, ([\Lambda/\bar{2}^{\Gamma}] - \bar{2} \,.\, [\Lambda/\bar{2}^{\Gamma + \bar{1}}]) \,.$$

Thus the dyadic membership-predicate $(\cdot) \in_* (\cdot)$ is as follows:

$$\Gamma \in_* \Lambda \underset{\mathrm{df}}{\Leftrightarrow} \in_*(\Gamma, \Lambda) = \bar{1} \,;$$

of course, $\neg(\Gamma \in_* \Lambda)$ whenever Γ is an infinite dyadic integer.

Let us emphasize that, in fact, $\in_*(\Gamma_\gamma, \Lambda) = \Lambda(\gamma)$; this is easily verified using the already proved results; this shows the normality of the predicate $(.) \in_* (.)$.

(III) *The isomorphism between ordinals and sets of Σ' with respect to their membership-relations.* Now, it turns out to be incovenient to continue the verification of axioms of dyadic arithmetics in our dyadic integers; moreover, the immediate verification of the (cl)-axioms in dyadic integers fails in the case of the "domain-axiom" (cl 4), for a certain needed ordinal function cannot be warranted because its immediate description is non-normal. (For intuitive dyadic integers, this state of affairs consists in that the "domain-operation" is continuous but not uniformly continuous in the dyadic metric of naturals, whereas other "set-theoretical" operations in dyadic integers are uniformly continuous in this metric.)

Thus let us now turn to the above mentioned construction of a one-to-one function F, as mapping the class $\mathbf{On}$ of ordinals onto the universal class V, isomorphicaly with respect to the dyadic membership-relation $\in^*$ in ordinals – and the primitive membership-relation $\in$ in sets.

We assume the usual elements and terms of the (Peanian) arithmetics of ordinals of Σ'. Especially, $\alpha + \beta$, $\alpha . \beta$, 2^α have their usual sense; $\alpha - \beta$ is the "arithmetical" difference, *i.e.* $(\alpha - \beta) + \beta = \alpha$ if $\beta < \alpha$, and $\alpha - \beta = 0$ if $\alpha \leqq \beta$. Likewise $[\alpha/\beta]$ (the result of the "arithmetical" division with remainder) has its obvious sense: we have (uniquely for every α, $\beta \neq 0$)

$$\alpha = [\alpha/\beta] . \beta + \varrho , \quad 0 \leqq \varrho < \beta .$$

The dyadic membership-operation $\in^*$ in ordinals thus is again (see section 3)

$$\in^* (\alpha, \beta) = [\alpha/2^\beta] - 2[\alpha/2^{\beta+1}] .$$

We often write $\alpha \in^* \beta$ instead of $\in^* (\alpha, \beta) = 1$. We observe that

$$\alpha \in^* \beta \Leftrightarrow \beta = 2^{\beta_1} + 2^{\beta_2} + \ldots + 2^{\beta_\kappa} ,$$

where $\beta_1 < \beta_2 < \ldots < \beta_\kappa$ and $\alpha = \beta_\tau$ with a unique τ, $1 \leqq \tau < \kappa$. Of course, the relation $(.) \in^* (.)$ is "extensional", *i.e.*

$$\forall \alpha (\alpha \in^* \beta_1 \Leftrightarrow \alpha \in^* \beta_2) \Rightarrow \beta_1 = \beta_2 .$$

In order to construct a $(\in^* - \in)$-isomorphism F, first recall the notion of type (of a set, in Σ').

The α-th type $t'\alpha$ is a set defined by induction thus

$$t'0 = \emptyset , \quad t'(\alpha + 1) = \mathbf{P}(t'\alpha)$$

(*i.e.* the next type is the potency-set of the given type).

It is well known and easy to prove that the class-sum of the class of all types is the universal class V; we write $\bigcup_{\alpha \in \mathbf{On}} t'\alpha = V$. Of course, $t'\alpha \subset t'(\alpha + 1)$; note also that $x \in t'(\alpha + 1) \dot{-} t'\alpha$ iff every element of x is of type $t'\alpha$.

Analogously, denote by $\mathbf{P}^*(\alpha)$ the "potency-set" of the "set" α in the sense of the above relation $\in^*$ between ordinals, *i.e.* if $\alpha = 2^{\alpha_1} + 2^{\alpha_2} + \ldots + 2^{\alpha_\kappa}$, then let $\beta = \mathbf{P}^*(\alpha)$ iff $\beta = 2^{\beta_1} + 2^{\beta_2} + \ldots + 2^{\beta_\rho}$ with $\beta_\nu = 2^{\beta_{\nu 1}} + 2^{\beta_{\nu 2}} + \ldots + 2^{\beta_{\nu \rho_\nu}}$ $(\nu = 1, \ldots, \kappa)$, where $\beta_{\nu 1} < \beta_{\nu 2} < \ldots < \beta_{\nu \varrho_\nu}$ is any increasing subsequence of the sequence $\alpha_1 < \alpha_2 < \ldots < \alpha_\kappa$.

We infer easily that $\mathbf{P}^*(2^\gamma - 1) = 2^{2^\gamma} - 1$ for every $\gamma \in \mathbf{On}$.

Indeed, it suffices to realize the following: First, $0 \leqq \nu \leqq 2^\gamma - 1 \Leftrightarrow \nu \subseteq^* 2^\gamma - 1$, denoting by $\subseteq^*$ the "inclusion" in the sense of $\in^*$, *i.e.*

$$\alpha \subseteq^* \beta \Leftrightarrow \xi \in^* \alpha \Rightarrow \xi \in^* \beta .$$

Second, $2^\gamma - 1 = 2^0 + 2^1 + \ldots + 2^\xi + \ldots + 2^{\gamma-1}$ $(0 \leqq \xi \leqq \gamma - 1)$ and

$$2^{2^\gamma} - 1 = 2^0 + 2^{2^0} + 2^{2^1} + 2^{2^0+2^1} + \ldots + 2^\nu + \ldots + 2^{2^0+2^1+\ldots+2^{\gamma-1}} \quad (0 \leqq \nu \leqq 2^\gamma - 1)\,.$$

Thus the above formula is true in view of the definition of $\mathbf{P}^*(\alpha)$ for $\alpha = 2^\gamma - 1$.

Next, define the ordinal function τ as follows:

$$\tau'0 = 0\,, \quad \tau'(\alpha + 1) = \mathbf{P}^*(\tau'\alpha)\,,$$

$\tau'\alpha$ could be called the α-th "type" in the sense of $\in^*$ in ordinals. If we want to determine the value of $\tau'\alpha$ more explicitely, then denote by $h(\alpha)$ the α-times iterated potency of 2, i.e. set $h'0 = 1$, $h'(\alpha + 1) = 2^{h(\alpha)}$. Then the just proved formula $\mathbf{P}^*(2^\gamma - 1) = 2^{2^\gamma} - 1$ yields the simple result

$$\tau'\alpha = h(\alpha) - 1 \quad \text{for every ordinal} \quad \alpha \in \mathbf{O}n\,.$$

(The proof by induction is almost immediate.)

These preliminaries enable us to construct the so-called natural mapping F of $\mathbf{O}n$ onto the universal class V as an isomorphism with respect to $\in^*$ and $\in$ (cf. sec. 1).

Define by induction a sequence $\{f_\alpha\}_{\alpha\in\mathbf{O}n}$ of mappings as follows: f_0 is void; f_1 is defined on the ordinal $\tau'1 = 1 = \{\emptyset\}$, and we set $f_1'\emptyset = \emptyset$. Clearly $f_0 \subset f_1$ and f_1 is a one-to-one map of the set $\tau'1 = \{\emptyset\}$ onto the set $t'1 = \{\emptyset\}$, so that $\alpha \in^* \beta \Leftrightarrow f_1'\alpha \in f_1'\beta$ for every $\alpha \in \tau'1$, $\beta \in \tau'1$. (The corresponding statement is trivially true for f_0, of course.)

Suppose f_γ is defined as a one-to-one mapping of the set $\tau'\gamma$ onto the set $t'\gamma$ such that $\alpha \in^* \beta \Leftrightarrow f_\gamma'\alpha \in f_\gamma'\beta$ whenever $\alpha \in \tau'\gamma$, $\beta \in \tau'\gamma$.

Take an arbitrary ϑ with $\tau'\gamma = h(\gamma) - 1 \leqq \vartheta < h(\gamma + 1) - 1 = \tau'(\gamma + 1)$.

By the above definition of the "type" $\tau'(\gamma + 1)$, $\vartheta \in \tau'(\gamma + 1) \dot{-} \tau'\gamma$. Thus if $\nu \in^* \vartheta$ then $\nu \in \tau'\gamma$ so that $\nu < \tau'\gamma$. Therefore $f_\gamma'\nu$ is defined for every ν with $\nu \in^* \vartheta$. Thus we can and will define $f_{\gamma+1}'\vartheta$ as the set of all the images $f_\gamma'\nu$ where $\nu \in^* \vartheta$.

Putting $f_{\gamma+1}'\sigma = f_\gamma'\sigma$ for every $\sigma \in \tau'\gamma$, we define the mapping $f_{\gamma+1}$ of the "type" $\tau'(\gamma + 1)$ into the type $t'(\gamma + 1)$.

First, we have to show that $f_{\gamma+1}$ is a one-to-one mapping of $\tau'(\gamma + 1)$ onto the whole $t'(\gamma + 1)$.

In fact it is easy to see from the above definition of $f_{\gamma+1}$, from the "extensionality" of $\in^*$ and from the inductive assumption that $f_{\gamma+1}$ is one-to-one.

That $f_{\gamma+1}$ is onto the whole $t'(\gamma + 1)$ is also clear; for if $y \in t'(\gamma + 1)$ then let $f_\gamma'\alpha_1, f_\gamma'\alpha_2, \ldots, f_\gamma'\alpha_\kappa$ (with $\alpha_1 < \alpha_2 < \ldots < \alpha_\kappa$) be the different elements of y, according to the inductive assumption. Then

$$f_{\gamma+1}'(2^{\alpha_1} + 2^{\alpha_2} + \ldots + 2^{\alpha_\kappa}) = y$$

by the above definition of $f_{\gamma+1}$.

Now, we have to prove by induction the main statement:

$$\alpha \in^* \beta \Leftrightarrow f'_{\gamma+1}\alpha \in f'_{\gamma+1}\beta$$

whenever $\alpha \in \tau'(\gamma + 1)$, $\beta \in \tau'(\gamma + 1)$.

Indeed, if $\beta \in \tau'\gamma$ then $\alpha \in \tau'\gamma$ and there is nothing to prove, by the inductive assumption. Thus assume $\beta \in \tau'(\gamma + 1) \dot{-} \tau'\gamma$. Then according to the inductive definition of $f_{\gamma+1}$, $f'_{\gamma+1}\beta$ is the set of all the sets of the form $f'_\gamma\beta_\lambda$ where $\beta = \ldots + 2^{\beta_\lambda} + \ldots$ *i.e.* where $\beta_\lambda \in^* \beta$. Therefore if $\alpha \in^* \beta$ then $\alpha = \beta_\lambda$ with a suitable λ and $f'_{\gamma+1}\alpha \in \in f'_{\gamma+1}\beta$ by the definition of $f_{\gamma+1}$.

Conversely, assume $f'_{\gamma+1}\alpha \in f'_{\gamma+1}\beta$. Then $f'_{\gamma+1}\alpha = f'_\gamma\alpha$ and $\alpha \in \tau'\gamma$, $\alpha \in^* \beta$, by the definition of $f_{\gamma+1}$ again.

Therefore indeed $\alpha \in^* \beta \Leftrightarrow f'_{\gamma+1}\alpha \in f'_{\gamma+1}\beta$ if the same equivalence holds for γ instead of $\gamma + 1$.

Next, since clearly $f_\gamma \subset f_{\gamma+1}$, we can and will form F as the sum-class of the class of all the already defined f_γ's, *i.e.* $F = \bigcup\limits_{\gamma \in \mathbf{On}} f_\gamma$.

It is easy to see that F is a one-to-one map defined on the whole class $\mathbf{On}$ of ordinals of Σ', and maps onto the whole universal class V of all sets. Clearly $\alpha \in^* \beta$ (*i.e.* $\beta = = \ldots + 2^\alpha + \ldots$, *i.e.* $[\beta/2^\alpha] - 2[\beta/2^{\alpha+1}] = 1$) iff $F'\alpha \in F'\beta$. Conversely, we thus have the equivalence $x \in y \Leftrightarrow F^{-1\prime}(x) \in^* F^{-1\prime}(y)$.

(IV) *The natural ordering, addition, multiplication and potentiation of two for sets of* Σ'. Next, we shall introduce the basic operations of dyadic arithmetics (as well as the corresponding ordering-relation) from ordinals to sets, by means of the already determined one-to-one mapping F. (Let us systematically denote the introduced notions by inserting the usual symbols in brackets.)

(i) (*Ordering.*) $x\,(\prec)\,y \underset{\text{df}}{\Leftrightarrow} F^{-1\prime}(x) \prec F^{-1\prime}(y)$.

(ii) (*Addition.*) $x\,(+)\,y = z \underset{\text{df}}{\Leftrightarrow} F^{-1\prime}(x) + F^{-1\prime}(y) = F^{-1\prime}(z)$.

(iii) (*Multiplication.*) $x\,(.)\,y = z \underset{\text{df}}{\Leftrightarrow} F^{-1\prime}(x) \,.\, F^{-1\prime}(y) = F^{-1\prime}(z)$.

(iv) (*Potentiation of two.*) Set $z = (0) \underset{\text{df}}{\Leftrightarrow} F^{-1\prime}(z) = 0$, *i.e.* $(0) = \emptyset$; $u = (1) \underset{\text{df}}{\Leftrightarrow} \underset{\text{df}}{\Leftrightarrow} F^{-1\prime}(u) = 1$, *i.e.* $(1) = \{\emptyset\} = 1$; $v = (2) \Leftrightarrow F^{-1\prime}(v) = 2$, *i.e.* $(2) = \{\{\emptyset\}\} \neq 2$; we observe $(1)\,(+)\,(1) = (2)$.

Then $y = (2)^x \underset{\text{df}}{\Leftrightarrow} F^{-1\prime}(y) = 2^{F^{-1\prime}(x)}$. We have $(2)^x = \{x\}$ because

$$z \in (2)^x \Leftrightarrow F^{-1\prime}(z) \in^* 2^{F^{-1\prime}(x)} \Leftrightarrow F^{-1\prime}(z) = F^{-1\prime}(x) \Leftrightarrow z = x\,.$$

(v) $$y = ([u/(2)^v]) \underset{\text{df}}{\Leftrightarrow} F^{-1\prime}(y) = [F^{-1\prime}(u)/2^{F^{-1\prime}(v)}]\,.$$

Now, evidently, since $x \in y \Leftrightarrow F^{-1\prime}(x) \in^* F^{-1\prime}(y)$, hence

$$x \in y \Leftrightarrow ([y/(2)^x])\,(-)\,(2)\,_{(.)}\,([y/(2)^{x(+)1}]) = 1\,.$$

Abbreviating $\in(x, y) = ([y/(2)^x])\,(-)\,(2)\,_{(\cdot)}\,([y/(2)^{x(+)1}])$, we have, of course,

$$\in(x, y) = 1 \Leftrightarrow \in^*(F^{-1'}(x), F^{-1'}(y)) = 1 .$$

(V) *Extending the natural arithmetical operations from sets to classes.* Our final task is to extend the already introduced operations $(+)$, $(\cdot)$, $(2)^{(\cdot)}$ to classes of Σ' in such a manner that the primitive membership-predicate becomes the dyadic membership-predicate.

To every dyadic integer A (of Σ'), there is a unique class, say $Y = \mathbf{F}(A)$, defined (in view of a particular instance of M3 of [G]) thus:

$$y \in \mathbf{F}(A) \underset{\mathrm{df}}{\Leftrightarrow} A(F^{-1'}(y)) = 1 .$$

($\mathbf{F}(A) = Y$ is the class of all sets whose natural ordinal numbers $F^{-1}(y)$ determine the ordinal place of a dyadic digit 1, if A is taken for a dyadic infinite digital image). Of course, the term $\mathbf{F}(A)$ is normal.

Conversely, to every given class Y, we can form the unique dyadic integer A such that $Y = \mathbf{F}(A)$; namely, we set $A(\xi) = 1 \Leftrightarrow F'\xi \in Y$. Thus we can write $A = \mathbf{F}^{-1}(Y)$ in this sense, introducing $\mathbf{F}^{-1}(Y)$ as a further normal term.

Now, suppose $A = A_\alpha$ is a given finite dyadic integer. Then

$$y \in \mathbf{F}(A_\alpha) \Leftrightarrow A_\alpha(F^{-1'}(y)) = 1 .$$

The right side of this equivalence means that $F^{1'}(y) \in^* \alpha$ (because $\xi \in^* \alpha \Leftrightarrow A_\alpha(\xi) = 1$). Thus, supposing $F'\alpha = x$ (with an x uniquely determined by α), we observe

$$y \in \mathbf{F}(A_\alpha) \Leftrightarrow F^{-1'}(y) \in^* F^{-1'}(x) = \alpha .$$

We thus conclude that $\mathbf{F}(A_\alpha) = F'\alpha$, $\mathbf{F}^{-1}(x) = A_{F^{-1'}(x)}$ for every ordinal α.

Definition 5. (*Natural aritmetical operations for classes of Σ'.*) Put

($\bar{\text{i}}$) (*Addition.*) $X\,(+)\,Y = Z \underset{\mathrm{df}}{\Leftrightarrow} \mathbf{F}^{-1}(X) + \mathbf{F}^{-1}(Y) = \mathbf{F}^{-1}(Z)$.

($\bar{\text{ii}}$) (*Multiplication.*) $X\,_{(\cdot)}\,Y \underset{\mathrm{df}}{\Leftrightarrow} \mathbf{F}^{-1}(X)\,.\,\mathbf{F}^{-1}(Y) = \mathbf{F}^{-1}(Z)$.

($\bar{\text{iii}}$) (*Potentiation of two.*) $\{\{\emptyset\}\} = (2)$, $(2)^X = Z \underset{\mathrm{df}}{\Leftrightarrow} \bar{2}^{F^{-1}(X)} = \mathbf{F}^{-1}(Z)$.

First, in view of the above identity $\mathbf{F}(A_\alpha) = F'\alpha$ and on account of the $(+, ., 2^{(\cdot)})$-isomorphical one-to-one correspondence $\mathbf{C}orr(.,.)$ between ordinals and finite dyadic integers, the new operations $(+)$, $(\cdot)$, $(2)^{(\cdot)}$ on classes according to ($\bar{\text{i}}$), ($\bar{\text{ii}}$), ($\bar{\text{iii}}$) respectively are indeed extensions of the operations sub (ii), (iii), (iv) respectively as formerly introduced in sets.

Second, the operations $(+)$, $(\cdot)$, $(2)^{(\cdot)}$ clearly satisfy the axioms sub (r) and (d) of dyadic arithmetics, since the dyadic integers do so.

Thus (denoting again the dyadic arithmetical operations on classes systematically by brackets) we easily obtain the desired fundamental equivalence

$$z \in Y \Leftrightarrow ([Y/(2)^z])\,(-)\,(2)\,_{(\cdot)}\,([Y/(2)^{z(+)1}]) = 1 ,$$

in other terms

$$z \in Y \Leftrightarrow \mathbf{F}^{-1}(z) \in_* \mathbf{F}^{-1}(Y).$$

In fact, setting $\mathbf{F}^{-1}(Y) = \Lambda$, then $z \in Y$ iff $\Lambda(F^{-1\prime}(z)) = 1$; thus $z \in Y$ iff $\Gamma_\gamma \in_* \Lambda$, where $F'\gamma = z$; and thus

$$\mathbf{F}^{-1}(z) = \Gamma_\gamma, \quad \Gamma_\gamma \in_* \Lambda \Leftrightarrow z \in Y.$$

Thus we have almost completed the proof of Theorems Iii and IIii.

The remaining verifications of the axioms sub (cl) and (s) of dyadic arithmetics for the already defined "natural" operations $(+)$, $(\cdot)$, $(2)^{(\cdot)}$ are now evident; note that $\in$ satisfies the axioms sub B of Σ' and the (cl)-axioms are merely the B-axioms as restated in terms of the dyadic membership-operation. Likewise the successor-principle (s) only requires the existence of a function $S \subset V \times V$ such that $S'x = F(F^{-1\prime}(x) + 1)$; such a function clearly exists since $F \subset \mathbf{O}n \times V$ is a function in Σ'.

(VI) *A more direct description of the natural ordering and of the natural arithmetical operations on classes of Σ'.* The given characterization of dyadic arithmetical operations on classes serving to represent the primitive membership-predicate as an arithmetical dyadic membership-predicate (though performed in set-theoretical terms of Σ') is quite complicated; a more direct description may be desirable.

This is not difficult to give if we translate into $\in$ (in sets) the description of $<, +, .,$ $2^{(\cdot)}$ (given directly in terms of $\in^*$ in ordinals). We do not go into obvious details — and state the results of the mentioned translation already in terms of $\in$.

First, the natural ordering $(\prec)$ of sets of Σ': Following the above F and considering the characterization of the ordering-relation $<$ for ordinals by means of their dyadic form, we can characterize $(\prec)$ thus:

Set $\prec_0 = \emptyset$. Let $\prec_\alpha$ be defined in such a manner that $\prec_\alpha \subset t'\alpha \times t'\alpha$ and that $\prec_\alpha$ orders the type $t'\alpha$. Then define $\prec_{\alpha+1}$ thus:

a) $x \prec_{\alpha+1} y \Leftrightarrow x \prec_\alpha y$ in the case $\langle xy \rangle \in t'\alpha \times t'\alpha$.

b) Put $u \prec_{\alpha+1} v \underset{\text{df}}{\Leftrightarrow} \max\limits_{\prec_\alpha}((u \dot{-} v) \cup (v \dot{-} u)) \in v$ in the opposite case (denoting by $\max\limits_{\prec_\alpha}(y)$ the greatest element of the set y as ordered by means of $\prec_\alpha$).

Clearly $\prec_{\alpha+1}$ is irreflexive and trichotomic; the transitivity of $\prec_{\alpha+1}$ follows from the transitivity of $\prec_\alpha$ by a not difficult inductive argument. Clearly $\prec_\alpha \subset \prec_{\alpha+1}$. Thus we can form $\prec = \bigcup\limits_{\alpha \in \mathbf{O}n} \prec_\alpha$ as a (well) ordering relation of the universal class V. On account of the above isomorphism F (between ordinals and sets, with respect to $\in^*$ and $\in$) we easily conclude that $\prec = (\prec)$.

Now, in order to give a direct definition of the natural addition, multiplication and potentiation of two for classes (by means of recursions in the sense of the natural well ordering $\prec$ of sets already introduced directly), we need the following notations:

To any given y define the class $\lessdot y$ and $y \gtrdot$ by setting $x \in \lessdot y \underset{\text{df}}{\Leftrightarrow} y \lesssim x$, $x \in y \gtrdot \underset{\text{df}}{\Leftrightarrow}$ $\underset{\text{df}}{\Leftrightarrow} x \lesssim y$ with the normal terms $\lessdot y$ and $y \gtrdot$. Clearly $y \gtrdot$ is always a set, $\lessdot y$ is always a proper class.

Let X be any class, y a given set. Denote by X_y the set-sum of all the "intervals" as sets of the form $(\lessdot y) \cap (z \gtrdot)$ that are subclasses of X. (More precisely, define

$$v \in U_y \underset{\text{df}}{\Leftrightarrow} \exists z((v = (\lessdot y) \cap (z \gtrdot)) \,\&\, (v \subseteq X))$$

and set $X_y = \mathbf{S}(U_y)$ with y as a "parameter" ($\mathbf{S}(A)$ is the sum-class of A).

Remark. $X_y = \emptyset$ if not $y \in X$; otherwise $\{y\} \subseteq X_y \neq \emptyset$.

Define a further (normal) mapping predicate denoting by $Z = \mathbf{M}ax(X)$ the unique y maximal in X is the sense of $\prec$ – if such a y exists – and putting $\mathbf{M}ax(X) = V$ in the opposite case.

Remark. By the preliminary argument of the proof of lemma 19 above, we observe that $\mathbf{M}ax(X) = V$ iff either $X = \emptyset$ or X is a proper class.

(Otherwise, the required y as a maximal element of the set $x = X$ exists in Σ', by $\neg$(C1). We write $y^* = y \dot{-} \{\mathbf{M}ax(y)\}$.

Definition 5. (*The dyadic addition of classes.*)

($\overline{5\alpha}$) Define first the dyadic addition of sets as follows:

(i) $\emptyset \oplus x = x \oplus \emptyset = x$ for every x.

(ii) $x \oplus \{z\} = ((x \cup \{z\}) \dot{-} x_z) \cup \{(\mathbf{M}ax(x_z))'\}$

for every x and z, writing a' the successor of a in $\prec$ and with $V' = V$.

For the general case of $y = y^* \cup \{\mathbf{M}ax(y)\}$ in $x \oplus y$, by induction (in the sense of $\prec$) put

(iii) $x \oplus (y^* \cup \{\mathbf{M}ax(y)\}) = (x \oplus y^*) \oplus \{\mathbf{M}ax(y)\}$.

Remark. The formally correct but less intuitive reformulation of this inductive definition may be omitted. Note that $x \oplus \{z\} = x \cup \{z\}$ iff $\neg(z \in x)$, by definition $\overline{5\alpha}$(ii). We shall use the fact that $y^* \prec y$ as a consequence of (iii) of lemma 19. Analogously later in definitions $\overline{6}$, $\overline{7}$. It is not difficult to prove $x \oplus y = x\,(+)\,y$ (by means of the above F).

In order to extend $\oplus$ to classes in general, let us abbreviate $\mu_X = (X \times \{1\}) \cup$ $\cup ((\dot{-}X) \times \{\emptyset\})$, forming μ_X as the so-called characteristic function of the class X (defined on V and mapping into $\{\emptyset 1\}$).

Remark. Conversely, if μ is a function from V into $\{\emptyset 1\}$ then clearly $\mu = \mu_X$, where the unique $X = \mu^{-1}{}''\{1\}$. (Of course, X need not be a proper class.)

Further, with every class Y we can associate the class of characteristic functions μ_x of their elements $x \in Y$ and we can try to define the "limit" characteristic function of this class of characteristic functions (if it exists).

More precisely, to every class Y and to every $u \in V$ let us define the normal term $(\lim_{x \in Y} \mu_x)(u)$ thus

$$(\lim_{x \in Y} \mu_x)(u) = 0 \underset{\mathrm{df}}{\Leftrightarrow} \exists x \forall y (y \in Y \,\&\, x \lessapprox u \Rightarrow \mu_y(u) = 0),$$
$$(\lim_{x \in Y} \mu_x)(u) = 1 \underset{\mathrm{df}}{\Leftrightarrow} \exists x \forall y (y \in Y \,\&\, x \lessapprox u \Rightarrow \mu_y(u) = 1),$$
$$(\lim_{x \in Y} \mu_x)(u) = 2 \underset{\mathrm{df}}{\Leftrightarrow} \neg \Phi_1 \,\&\, \neg \Phi_2,$$

where Φ_1, Φ_2 abbreviate the first and second conditions respectively on the above left sides, *i.e.* we also define $(\lim_{x \in Y} \mu_x)(u)$ if neither of these disjoint conditions is satisfied. In this way, for every Y, we have defined a function, say $\lim_{x \in Y} \mu_x$, from V into $\{012\}$ (on account of M5 of [G]). Finally, introduce an auxiliary class $U_{X \oplus Y}$ to any given classes X, Y thus

$$z \in U_{X \oplus Y} \Leftrightarrow \exists x_1 \exists y_1 (z = (X \cap (x_1 \geqslant)) \oplus (Y \cap (y_1 \geqslant))).$$

($\overline{5\beta}$) Now, the dyadic addition for classes can be defined as follows

$$X \oplus Y = (\lim_{z \in U_{X \oplus Y}} \mu_z)^{-1\prime\prime} \{1\}$$

i.e. as the class of counterimages of 1 of the mapping $\lim_{y \in U_{X \oplus Y}} \mu_z$.

Remark. It is not difficult to see that $X \oplus Y$ of ($\overline{5\beta}$) indeed generalizes $x \oplus y$ of ($\overline{5\alpha}$). On the other hand, in our definition of $X \oplus Y$, we take proper classes for "limits" of sets in generalizing the sense in which proper Hensel's dyadic integers appear as dyadic limits of naturals. Likewise for the multiplication and for the potentiation of (2). It is not difficult to prove that $X \oplus Y = X \,(+)\, Y$ again.

Having defined the dyadic addition of classes, let us define their multiplication.

Definition 6. (*Dyadic multiplication of classes.*)

($\overline{6\alpha}$) *The dyadic multiplication of sets*:

(i) $x \odot \emptyset = \emptyset \odot x = \emptyset$ for every x.

(ii) $\{x\} \odot \{y\} = \{x \oplus y\}$ for every x, y.

(iii) Assume $x = x^* \cup \{\mathbf{M}ax(x)\}$.

As the inductive assumption (in the sense of $\prec$), let $x^* \odot \{y\}$ be defined. Then put $x \odot \{y\} = (x^* \odot \{y\}) \oplus (\{\mathbf{M}ax(x)\} \odot \{y\})$ (on account of (ii) above).

(iv) Assume $y = y^* \cup \{\mathbf{M}ax(y)\}$ and, as the inductive assumption, let $\{x\} \odot y^*$ already be defined. Then put

$$\{x\} \odot y = (\{x\} \odot y^*) \oplus (\{x\} \odot \{\mathbf{M}ax(y)\})$$

(on account of (ii) above).

(v) In the most general case, assume $x = x^* \cup \{\mathbf{M}ax(x)\}$, $y = y^* \cup \{\mathbf{M}ax(y)\}$. Let $x^* \odot y^*$, $\{\mathbf{M}ax(x)\} \odot y^*$, $x^* \odot \{\mathbf{M}ax(y)\}$ all be defined, as the inductive assumption. Then put

$$x \odot y = (x^* \odot y^*) \oplus (\{\mathbf{M}ax(x)\} \odot y^*) \oplus (x^* \odot \{\mathbf{M}ax(y)\}) \oplus \\ \oplus (\{\mathbf{M}ax(x)\} \odot \{\mathbf{M}ax(y)\})$$

using((ii), (iii), (iv)).

($\overline{6}\beta$) Now, define the *dyadic multiplication of classes*. Analogously to ($\overline{5}\beta$), introduce the auxiliary class $U_{X \odot Y}$ thus:

$$z \in U_{Y \odot X} \underset{\text{df}}{\Leftrightarrow} \exists x_1 \exists y_1 (z = (X \cap (x_1 \geqslant)) \odot (Y \cap (y_1 \geqslant)))$$

and then put

$$X \odot Y = (\lim_{z \in U_{X \odot Y}} \mu_z)^{-1''} \{1\} .$$

We can again prove $X \odot Y = X\,(.)\,Y$.

Definition 7. (*Potentiation of* (2).) Set $(2)^X = \{X\}$ $(= \{XX\})$ for every class X, with regard to the definitions 3.1 and 3.11 of [G]. Therefore $(2)^X$ is the singleton $\{x\}$ iff $X = x$; and $(2)^X = \emptyset$ iff X is a proper class; therefore $\{\emptyset\} = (2)^\emptyset = (1)$, $\{\{\emptyset\}\} = (2)^{(1)} = (2)$.

Clearly again this is the former potentiation of two (of def 5iii).

4.3. The proof of IIi. Let us recall our last task: Suppose the dyadic membership-predicate $\in_*$ as given by the primitive dyadic operations $+$, $\cdot$, $2^{(\cdot)}$. Then (according to the theorem proved Ii), $\in^*$ satisfies the axioms of Σ' (of sec. 2), together with the notions

$$\mathbf{M}_*(X) \underset{\text{df}}{\Leftrightarrow} 2^X \neq 0 , \quad \mathbf{C}ls_*(Y) \underset{\text{df}}{\Leftrightarrow} Y = Y .$$

Following the above proof of both theorems Iii and IIii, we can introduce three new dyadic operations, say $\underset{*}{+}, \underset{*}{\cdot}, \underset{*}{2}^{(\cdot)}$, on our (abstract) dyadic integers, defining them in terms of $\in_*$ (instead of in terms of $\in$), as in the proof of Ii; moreover, according to IIii, the dyadic membership-predicate $\in_*$ is reproduced by the new (undenoted) dyadic membership-predicate as determined by the new operations $\underset{*}{+}, \underset{*}{\cdot}, \underset{*}{2}^{(\cdot)}$ (using the known formula).

The question is whether also the $\underset{*}{+}$ is the same as the $+$, the $\underset{*}{\cdot}$ is the same as the . and whether $\underset{*}{2} = 2$ and $\underset{*}{2}^{(\cdot)}$ is the same as $2^{(\cdot)}$.

In view of the familiar basic Peanian recursions, this question is answered in the affirmative if we can prove (1) $\underset{*}{0} = 0$ (the new zero equals to the old zero); (2) $\forall x (x \underset{*}{+} \underset{*}{1} = x + 1)$ (the new successor is the old one).

Since (1) is clear by the reproduction-theorem IIii, there remains to prove (2), *i.e.* to prove that the successor in the sense of the given ordering predicate $\prec$ is the same as the successor in the new sense of, say $\prec_*$.

It is difficult to imagine another method for proving the identity (2) than the "induction", whether according to $\prec$ or according to $\prec_*$; (for futher comments as to the induction in dyadic aritmetics, cf. section 5). But in order to prove this identity by "induction" in the sense of the dyadic $\in_*$, we must know that the term $x + 1$ gives the values of a true "function" of x in the sense of the dyadic membership-predicate $\in_*$. And precisely this is warranted by the successor-principle (s). (Note that the new ordering $\prec_*$, as given by the equivalence $X \preceq_* Y \Leftrightarrow 2^{Y \underset{*}{-} X} \underset{*}{\neq} 0$, is in fact a "relation" in the sense of $\in_*$, whereas the old $\prec$ seems to be a given predicate only, with respect to $\in_*$; *i.e.* $\prec$ seems not to be definable in terms of the predicate $\in_*$ alone, thus a fortiori $\prec$ seems not to be necessarily representable by an infinite dyadic integer as a "class" (*i.e.* by a "relation"), in the sense of the dyadic membership-predicate $\in_*$. Therefore we see no possibility of proving that the new and the old ordering of finite (abstract) dyadic integers are identical without using some further principle, e.g. (s).)

Thus assume the successor-principle (s) and let us prove (2) by induction (following $\prec_*$).

First, note that clearly $0 \underset{*}{+} 1 = 0 + 1 = 1$. Therefore our identity is true for $x = 0$; it remains to prove it for $x \neq 0$.

To this purpose, we apply the reproduction-theorem IIi to the dyadic membership-predicate $\in_*$, according to the above theorem Ii.

Applying these two theorems in our particular case of $\in_*$, we first observe that $\underset{*}{2} = 2 = \{\{0\}_*\}_*$. Second, we see that $\underset{*}{2}^x = 2^x = \{x\}_*$ (*i.e.* the "singletons" in the old dyadic-membership sense, and in the new dyadic-membership sense, are formed with the same result).

This noted, assume $y \underset{*}{+} 1 = y + 1$ by induction. Then $\{y \underset{*}{+} 1\}_* = \{y + 1\}_*$, *i.e.* $2^{y \underset{*}{+} 1} = 2^y \underset{*}{+} 2^y = 2^{y+1} = 2^y + 2^y$. Thus also $2^y \underset{*}{+} 2^y \underset{*}{+} 2^y \underset{*}{+} 2^y = 2^y + 2^y + 2^y + 2^y$. Therefore

$$2^{(y+1) \underset{*}{+} 1} = 2^{(y \underset{*}{+} 1) \underset{*}{+} 1} = (2^y \underset{*}{+} 2^y) \underset{*}{+} (2^y \underset{*}{+} 2^y) = (2^y + 2^y) + (2^y + 2^y) = 2^{(y+1)+1}\,.$$

Thus $\{(y + 1) \underset{*}{+} 1\} = \{(y + 1) + 1\}$ and finally $(y + 1) \underset{*}{+} 1 = (y + 1) + 1$, q. e. d.

Since the remaining statements used to prove that the new $\underset{*}{+}, \underset{*}{\cdot}, \underset{*}{2}^{(\cdot)}$ resp. are also the same as the old $+, \cdot, 2^{(\cdot)}$ resp. in "classes" (*i.e.* in infinite (abstract) dyadic integers) now are obvious (in view of the above considerations; *cf.* (v) of the above 4.2), hence also the remaining first half IIi of our reproduction-theorem may be seen to be proved; thus the proofs of our two main theorems are complete.

5. CONCLUDING REMARKS AND SOME OPEN PROBLEMS

(I) *Dyadic arithmetics and the axiom of infinity.* We have shown that the axiomatic theory of finite sets and their classes (of Bernays-Gödel) is nothing but axiomatic dyadic arithmetic, where the so-called finite dyadic integers are sets, general dyadic integers (in the sense of Hensel) are classes. Thus the real nature of the axiomatic membership-predicate is an arithmetical one, if we assume the axiom of finiteness.

A natural question arises, whether our arithmetical approach applies to the general axiomatic set-theory (of Bernays-Gödel, with the axiom of infinity).

In an immediate sense, this question already has been answered in the negative, by the above proof of Ii. Indeed, the "class" axioms sub (cl) of dyadic arithmetic just ensure the axiom of finiteness for the dyadic membership-predicate, on account of the strong arithmetical properties of $+, \cdot, 2^{(\cdot)}$; the root of the above verification of the C-axioms springs from the basic property of the dyadic valuation (as yielding the "lowest element" in every "nonvoid class", in an arithmetically effective way). Whether an appropriate weakening of this property (of dyadic valuation) would be consistent with the axiom of infinity for the dyadic membership-predicate or not, still remains an open question. Another possibility of connecting the dyadic membership-predicate with the axiom of infinity perhaps is in trying to partialize (*i.e.* to relativize) the dyadic membership-relation in a suitable "very nonnormal" concrete dyadic arithmetic (constructed in the theory of Bernays-Gödel), so as to obtain a model with the axiom of infinity satisfied.

In this connection, a further note perhaps may be of interest: If the successor principle (s) is not assumed (in the axioms of dyadic arithmetic), then the corresponding dyadic membership-predicate $\in_*(.,.)$ in general seems to behave very curiously with respect to the ordering-predicate $\prec$ of the dyadic arithmetic. The first half IIi of our reproduction theorem then perhaps need not hold (*cf.* its proof in 4.3) and it seems to be well possible that a "set", though "finite", has no greatest (in the sense of $\prec$) "element". (This is in accordance with the possibility that the ordering predicate $\prec$ need not be given by a "class" (*i.e.* by a "relation") in the sense of the dyadic membership $\in_x$, *i.e.* there is no contradiction with the axiom of finiteness).

Another question connected with these possibilities is noteworthy: *The question of the so-called logarithmicity of dyadic arithmetics.* (The "greatest element" of an abstract finite dyadic integer – in the sense of the dyadic membership – is nothing but the (abstract) integral part of the dyadic logarithm of that integer; a dyadic arithmetic is called logarithmic if every finite dyadic integer possesses such an arithmetical dyadic logarithm.) The successor-principle entails logarithmicity, but I do not know whether the converse is true. The successor-principle further implies that $x \subset_* y \Rightarrow x \prec y$, whereas the assumption of logarithmicity only ensures $x \subset_* y \Rightarrow \Rightarrow x \prec 2y$ (*cf.* [II]).

One problem more may be noted. It is natural to ask whether a dyadic domain (or perhaps what kind of dyadic domain) can be enlarged so as to fulfill the (cl)-axiom (or also the (s)-axiom) of dyadic arithmetic (this is meant, of course, in the sense of a set-theoretical realization – and then this question is a particular interesting problem of the General Theory of Valuation (of algebraical fields)).

Further investigations and experiences with various nonnormal models of dyadic arithmetics may clarify the situation.

(II) *Peanian arithmetic vs. dyadic arithmetic; induction in dyadic arithmetic.* Peanian arithmetic (as a basic theory of the successor-predicate, with the inductive scheme) stans in the same relation to the (finitely axiomatized) dyadic arithmetic exactly as the Zermelo-Fraenkel axiomatic set theory to the theory of v. Neumann-Bernays-Gödel.

In the (formalized) Peanian arithmetic, one does not intend to give an implicit axiomatic definition of the notion of naturals, but on the contrary, one assumes (in the metatheory) the absolute naturals as intuitively clear and one gives rules only how to define and perform proofs in a recursive (finitary) manner. (In the Zermelo-Fraenkel system, one likewise does not intend to give an implicit definition of the membership-predicate, but one only gives rules, how to define sets, by means of a metalanguage involving absolute naturals.) On the other hand, both dyadic arithmetic and the Bernays-Gödel theory of sets and their classes need no special assumptions for the corresponding metalanguage – and, moreover, they do not require any metamathematics at all; the price of this "strict finitism" in the metatheory is the admission of certain infinite ideal objects (infinite dyadic integers and proper classes respectively), as forming a closure of the system. Thus the formerly logical objects (statements and concepts) become purely mathematical objects – a situation we often encounter in the evolution of mathematics. We do not intend to apologize dyadic arithmetic, but rather wish to emphasize a certain incomparability of Peanian arithmetic with dyadic arithmetic. Despite the fact that every particular arithmetical (Peanian) statement (primitive recursive definition or argument) can be imitated, word for word, in dyadic arithmetic with the successor-principle (for the concrete predicates in question are represented by suitable infinite dyadic integers as "classes", in the sense of the dyadic membership), dyadic arithmetic cannot be said to be stronger than Peanian arithmetic. This could be stated only if we strengthen dyadic arithmetic by assuming the presence of absolute naturals in the metamathematics of dyadic aritmetic; but, disregarding other difficulties, this would be contrary to our proper intention.

On the other side, of course, Peanian arithmetic cannot be said to be stronger than dyadic arithmetic, for the notion of infinite dyadic integers (of a"proper class" in the sense of the dyadic membership) cannot be defined by means of the only primitive notion of successor in Peanian arithmetic.

Concerning induction in dyadic arithmetic, we have to distinguish between dyadic arithmetic with and without the successor-principle. The latter are considerably poorer in this respect, for we can only perform inductive arguments involving exclusively predicates represented by "classes" in the sense of the dyadic membership (there *e.g.* even addition is not representable in this sense). The former, however (as has been mentioned) practically yield the same as usual Peanian arithmetic; it is, in fact, the theory of finite sets and their classes (of Bernays-Gödel). In dyadic arithmetic, inductive arguments (and definitions) rather resemble the original "naive" Peanian manner (in the sense of *e.g.* E. LANDAU: Grundlagen der Analysis, Leipzig 1930).

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Резюме

О ГЕДЕЛЕВСКОЙ АКСИОМАТИЧЕСКОЙ ТЕОРИИ МНОЖЕСТВ, III

(Аксиоматическая диадическая арифметика конечных множеств и их классов)

ЛАДИСЛАВ РИГЕР (Ladislav Rieger), Прага

Аксиоматическая диадическая арифметика конечных множеств и их классов — это (в логическом смысле элементарная) теория целых диадических (p-адических, для $p = 2$) чисел Генселя (Hensel), которая основана на 28 аксиомах, касающихся сложения, умножения и возведения числа 2 в степень (как основных понятий).

Основным результатом статьи является доказательство полной эквивалентности упомянутых 28 аксиом и системы 19 аксиом теории конечных множеств Бернаися-Геделя из [G], но где аксиома бесконечности C1 заменяется аксиомой non C1 (конечности) — см. теоремы Ii, Iii, IIi, IIii.

При этом классы $Y, Z, \ldots$ появляются в качестве целых диадических чисел вообще, множества $x, y, \ldots$ в качестве неотрицательных целых чисел, и отношение принадлежности определяется формулой

$$x \in Y \Leftrightarrow [Y/2^x] - 2[Y/2^{x+1}] = 1 ,$$

где $[Y/2^x]$ — целая часть диадического числа $Y/2^x$.

Работа является независимым продолжением работы [II] (под тем же названием, в том же журнале, *84* (1959), 1—49).