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A GENERALIZATION OF REALCOMPACT SPACES

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Almost realcompact spaces are introduced and studied. In the last section some theorems about realcompact spaces are proved.

1. NOTATION AND TERMINOLOGY

All spaces under consideration are supposed to be Hausdorff. The closure of a subset M a space P will be denoted by $\overline{M}^P$ or simply $\overline{M}$. If $\mathfrak{A}$ is a family of subsets of a space P, then the symbol $\overline{\mathfrak{A}}^P$, or simply $\overline{\mathfrak{A}}$, will be used to denote the family of all $\overline{A}^P$, $A \in \mathfrak{A}$. An almost covering of a space P is a family $\mathfrak{M}$ of subsets of P such that the union of $\mathfrak{M}$ is dense in P.

A family $\mathfrak{M}$ of sets will be called centered if $\mathfrak{M}$ has the finite intersection property. A family $\mathfrak{M}$ has the countable intersection property (in another terminology $\mathfrak{M}$ is countably centered) if the intersection of every countable subfamily of $\mathfrak{M}$ is non-void.

If $\mathfrak{M}$ is a family of subsets of a set P, and if N is a subset of P, then the symbol $\mathfrak{M} \cap N$ will be used to denote the family of all $M \cap N$, $M \in \mathfrak{M}$. The union and intersection of a family of sets $\mathfrak{A}$ will be denoted by $\bigcup\mathfrak{A}$ and $\bigcap\mathfrak{A}$, respectively.

The Čech-Stone compactification of a completely regular space P will always be denoted by $\beta(P)$.

2. INTRODUCTION

It is well-known that a completely regular space is a realcompact (see [5]; in the original terminology of E. HEWITT [6], realcompact spaces are called Q-spaces) if, and only if, the following condition is fulfilled:

(1) If the intersection of a maximal centered family $\mathfrak{Z}$ of zero-sets[1]) in P is empty, then the intersection of some countable subfamily of $\mathfrak{Z}$ is empty.

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[1]) $Z \subset P$ is a zero-set in P if there exists a real-valued continuous function f on P such that

$$Z = \{x; x \in P, f(x) = 0\}.$$

a cozero-set is the complement of a zero-set.

In the present note we shall investigate a class of spaces closely connected to realcompact spaces.

Definition 1. A space P will be called almost realcompact if the following condition is fulfilled:

(2) If $\mathfrak{A}$ is a maximal centered family of open subsets of P with $\bigcap \overline{\mathfrak{A}} = \emptyset$, then $\bigcap \overline{\mathfrak{B}} = \emptyset$ for some countable subfamily $\mathfrak{B}$ of $\mathfrak{A}$.

Every completely regular realcompact space is almost realcompact (realcompactness is defined for completely regular spaces only). Every normal almost realcompact space is realcompact. Almost realcompact spaces are invariant under perfect mappings (*i.e.* closed continuous mappings such that the inverses of points are compact). In particular, the image under a perfect mapping of a normal realcompact space is realcompact. Almost realcompact spaces P are externally characterized as intersections of special subspaces (generalization of locally compact σ-compact spaces) of Katětov's almost-compact extension of P.

3. A CHARACTERIZATION OF ALMOST REALCOMPACT SPACES BY A COMPLETENESS PROPERTY

It is well-known that a completely regular space P is realcompact if and only if the uniformity generated by the family of all continuous real-valued functions in P is complete. For our purpose we shall need the following general concept of complete collections of open coverings (see [2]).

Definition 2. Let $\alpha = \{\mathfrak{A}\}$ be a collection of open coverings of a space P. An α-Cauchy family is a centered family $\mathfrak{B}$ of open subsets of P such that for every $\mathfrak{A}$ in α there exists an A in $\mathfrak{A}$ and a B in $\mathfrak{B}$ with $B \subset A$. The collection α will be called complete if $\bigcap \overline{\mathfrak{B}} \neq \emptyset$ for every α-Cauchy family $\mathfrak{B}$.

Note 1. Let α be a collection of open coverings of a space P and let $\mathfrak{B}$ be a maximal centered family of open subsets of P. $\mathfrak{B}$ is an α-Cauchy family if and only if $\mathfrak{A} \cap \mathfrak{B} \neq \emptyset$ for every $\mathfrak{A}$ in α.

Note 2. A completely regular space P is topologically complete in the sense of E. Čech (*i.e.* P is G_δ in $\beta(P)$) if and only if there exists a complete countable collection of open coverings of P (for proof see [2], for further information and literature see [3]).

Note 3. It is easy to see that a uniformity is complete if and only if the family of all uniform open coverings is complete in the sense of Definition 2.

We shall prove the following theorem:

Theorem 1. *A space P is almost realcompact if and only if the collection $\gamma = \gamma(P)$ of all countable open coverings of P is complete.*

Since every centered family of open sets is contained in a maximal one, the preceding Theorem 1 is an immediate consequence of the following lemma:

Lemma 1. *Let γ be the collection of all countable coverings of a space P. A maximal centered family $\mathfrak{M}$ of open subsets of P is a γ-Cauchy family if and only if the family $\overline{\mathfrak{M}}$ has the countable intersection property.*

Proof. Let $\mathfrak{M}$ be a γ-Cauchy family. Let us suppose that there exists a countable subfamily $\mathfrak{N}$ of $\mathfrak{M}$ with $\bigcap \overline{\mathfrak{N}} = \emptyset$. Put

$$\mathfrak{A} = \{P - \overline{N};\ N \in \mathfrak{N}\}\ .$$

By our assumption $\mathfrak{A}$ belongs to γ. Thus we can choose an $A = P - \overline{N}$ in $\mathfrak{A} \cap \mathfrak{M}$. We have

$$A \in \mathfrak{M}\ ,\quad N \in \mathfrak{M}\ ,\quad A \cap N = \emptyset$$

which contradicts the finite intersection property of $\mathfrak{M}$.

Conversely, suppose that $\overline{\mathfrak{M}}$ has the countable intersection property. Let $\mathfrak{A} \in \gamma$. If $\mathfrak{A} \cap \mathfrak{M} = \emptyset$, then evidently all sets of the form $P - \overline{A}$, $A \in \mathfrak{A}$, belong to $\mathfrak{M}$ and

$$\bigcap \{\overline{P - \overline{A}};\ A \in \mathfrak{A}\} \subset \bigcap \{P - A;\ A \in \mathfrak{A}\} = P - \bigcup \mathfrak{A} = \emptyset$$

which contradicts the countable intersection property of $\overline{\mathfrak{M}}$. The proof is complete.

Note 4. Let P be a completely regular space. For every continuous real-valued function f put

$$\mathfrak{W}(f) = \{\{x;\ |f(x)| < n\};\ n = 1, 2, \ldots\}\ .$$

Let α be the collection of all $\mathfrak{W}(f)$. P is realcompact if and only if the collection α is complete (for proof and further information see [4]). A centered family $\mathfrak{M}$ of open subsets of P is an α-Cauchy family if and only if every continuous real-valued function is bounded on some M in $\mathfrak{M}$.

4. EXTERNAL CHARACTERIZATION

A space P will be called almost-compact (*H-closed* in the terminology of M. KATĚTOV) if $\bigcap \overline{\mathfrak{A}} \neq \emptyset$ for every centered family $\mathfrak{A}$ of open subsets of P, or equivalently, if every open covering of P contains a finite almost covering. If P is a space, then there exists an almost-compact space νP containing P as a dense subspace and such that

(3) if R is an almost-compact space containing P as a dense subspace, then there exists a continuous mapping f of $S \subset \nu P$ onto R such that the restriction of f to P is the identity mapping.

The space νP will be called Katětov's almost-compact extension of P or simply the Katětov extension of P. The space νP has been defined and studied in [7]. Let us recall that $\nu P - P$ is a closed discrete subspace of νP.

A space is said to be *countably almost-compact* (in the other terminology – *countably H-closed*) if $\bigcap\overline{\mathfrak{A}} \neq \emptyset$ for every countable centered family of open subsets, or equivalenty, if every countable open covering contains a finite almost covering.[2] Combining the above definitions we obtain

Theorem 2. *Every coutably almost-compact and almost realcompact space is almost-compact.*

It is well-known (and it is easy to prove) that every σ-compact (a union of a countable number of compact subspaces), and more generally every completely regular Lindelöf space, is realcompact. From the definition it follows at once that every Lindelöf space is almost realcompact.

Example 1. There exists a countably almost-compact space P with the following properties: P is the union of a countable number of its almost-compact subspaces, every point of P has a neighbourhood U whose closure is an almost-compact space and finally P is not almost-compact. Thus P is not realcompact (see Theorem 2).

Construction. Let N be a countable infinite discrete space, let K be a one-point compactification of N and let T be the space of all countable ordinals. The space $T \times K$ is locally compact and countably compact. Let Ω be an element with $\Omega \notin (T \times K) \cup K \cup T$. On the set $R = (T \times K) \cup (\Omega)$ let us define the topology such that $T \times K$ is an open subspace of R and the sets of the form $\{U \times N\}$ form a local base at Ω, where U runs over all sets of the form

$$U = \{\alpha;\ \alpha \in T,\ \alpha > \beta\}\,. \tag{4}$$

It is easy to see that R is a H-closed space. Indeed, if $\mathfrak{A}$ is an open covering of R, then some $A \in \mathfrak{A}$ contains Ω. The subspace

$$R_1 = \overline{R - \bar{A}} \subset R - A$$

is compact and hence some finite subfamily $\mathfrak{A}_1$ of $\mathfrak{A}$ covers R_1. Clearly, $\bar{A} \cup \bigcup\overline{\mathfrak{A}}_1 = R$. Now let

$$N = \bigcup_{n=1}^{\infty} N_n\,,$$

where N_n are disjoint infinite sets. Let $\{\Omega_n\}$ be a sequence of distinct elements, $\Omega_n \neq \Omega$, $\Omega_n \notin T \cup K \cup (K \times T)$. On the set

$$P = T \times K \cup \{\Omega_1, \Omega_2, \ldots\}$$

let us define a topology such that $T \times K$ is an open subspace of P and the family of the sets

$$(\Omega_n) \cup \{U \times N_n\}$$

where U runs over all sets of the form (4), is a local base at Ω_n.

2) For further information see [1].

Evidently, the subspaces

$$\overline{T \times N_n}^P = (\Omega_n) \cup T \times \overline{N}_n^K \tag{5}$$

are homeomorphic with R, and hence are almost compact. Thus P is the union of a countable number of almost-compact spaces.

P is locally compact at every point of $T \times K$, since $T \times K$ is open in P and locally compact. The set (5) is an almost compact neighborhood of Ω_n. Thus every point of P is contained in an almost-compact neighborhood.

Consider the family of all sets of the form

$$\{U \times \bigcup_{n=k+1}^{\infty} N_n\} \tag{6}$$

where $k = 1, 2, \ldots$ and U is of the form (4). It is easy to see that $\bigcap \overline{\mathfrak{A}} = \emptyset$. Indeed, $(T \times K) \cap \bigcap \overline{\mathfrak{A}} = \emptyset$ and Ω_k does not belong to (6). Thus P is not almost-compact.

It remains to prove that P is countably almost-compact. But this is evident, because $T \times K$ is a countably compact dense subspace of P.

It is well-known that the following conditions (a), (b) and (c) on a completely regular space P are equivalent:

(a) P is realcompact,

(b) P is the intersection of cozero-sets in $\beta(P)$,

(c) P is the intersection of σ-compact subspaces of $\beta(P)$.

The conditions (b) and (c) are examples of "external" characterizations of realcompact spaces. A natural generalization of σ-compact spaces are spaces which are the union of a countable number of almost-compact subspaces. The preceding example shows that these spaces cannot be used to characterize almost realcompact spaces. An open subspace F of a compact space is a cozero-set if and only if F is a σ-compact subspace. The preceding Example 1 shows that also the "natural" generalization of cozero-set (*i.e.* locally almost-compact unions of a countable number of almost-compact subspaces) cannot be used.

Definition. A space P will be called a generalized cozero-space if there exists a countable open covering $\mathfrak{A}$ of P such that the spaces from $\overline{\mathfrak{A}}$ are almost-compact.

It is easy to see that every generalized cozero-set is locally almost-compact and a union of a countable number of almost-compact subspaces. According to Theorem 2 and Example 1 the converse is not true because, clearly, every generalized cozero-set is almost realcompact.

Theorem 3. *The following condition is necessary and sufficient for a space P to be almost realcompact:*

(7) *P is the intersection of generalized cozero-spaces in the Katětov almost-compact extension vP of P.*

Proof. Let $\gamma = \gamma(P)$ be the collection of all countable coverings of a space P. R will be used to denote vP. First let us suppose that P is almost realcompact, *i.e.* that γ is complete. Then

$$P = \bigcap\{\bigcup\overline{\mathfrak{A}}^R;\ \mathfrak{A} \in \gamma\} \tag{8}$$

and $\bigcup\overline{\mathfrak{A}}^R$, where $\mathfrak{A} \in \gamma$, are generalized cozero-spaces.

If U is an open subset of P, then $\overline{U}^R$ is a neighborhood of every point of $\overline{U}^R - P$ (this is a property of the Katětov extension). Thus

$$\{\overline{A}^R - (\overline{A}^P - A);\ A \in \mathfrak{A}\}$$

is an open countable covering of $\bigcup\overline{\mathfrak{A}}^R$ and the closure of every member of this covering is almost-compact. Thus $\bigcup\overline{\mathfrak{A}}^R$ are generalized cozero-spaces. To prove the first assertion, let us denote by Q the right side of (8) and suppose that there exists a point x in $Q - P$. Let $\mathfrak{B}$ be the family of all open neighborhoods of x in R and put $C = \mathfrak{B} \cap P$. Since $x \in Q$, from (8) it follows at once that C is a γ-Cauchy family. Hence $\bigcap\overline{C}^P \neq \emptyset$, but this is impossible because

$$\bigcap\overline{C}^P \subset \bigcap\overline{\mathfrak{B}}^R = (x) \subset R - P\,.$$

We have proved that the condition is necessary.

Conversely, let us suppose that there exists a family $\mathfrak{B}$ of generalized cozero-spaces in R with $\bigcap\mathfrak{B} = P$. For every B in $\mathfrak{B}$ choose an open countable covering $\mathfrak{A}(B)$ of B such that the closures in B of members if $\mathfrak{A}(B)$ are almost-compact. It is easy to see that the collection

$$\beta = \{\mathfrak{A}(B) \cap P;\ B \in \mathfrak{B}\} \subset \gamma$$

is complete. Indeed, if C is a maximal centered family of open subsets of P and if C is β-Cauchy, then

$$\emptyset \neq \bigcap\overline{C}^R \subset \bigcap\mathfrak{B} = P\,,$$

and consequently $\bigcap\overline{C}^P \neq \emptyset$, which completes the proof of Theorem 3.

5. PROPERTIES OF ALMOST REALCOMPACT SPACES

An open subspace of an almost realcompact space may fail to be almost real compact. For example, it is sufficient to consider non-compact countably compact open subspaces of a compact space. A closed subspace of an almost realcompact space may fail to be almost realcompact. For example the space $F = T \times (K - N)$ from the Example 1 is not almost realcompact, although the space R is almost realcompact and F is closed in R.

Theorem 4. *A regularly closed subset of an almost realcompact space is almost realcompact.*

Proof. A regularly closed subset of a space P is a set of the form $\overline{U}$, where U is open. Let U be an open subset of an almost realcompact space P and let $\mathfrak{A}$ be a maximal centered family of open subsets of $\overline{\mathfrak{A}}$ such that the intersection of closures in $\overline{U}$ of sets from $\mathfrak{A}$ is empty.

Let $\mathfrak{B}$ be a maximal centered family of open subsets of P with $\mathfrak{B} \supset \mathfrak{A} \cap U$. Clearly, $\bigcap \overline{\mathfrak{B}}^P = \emptyset$, and consequently, P being almost realcompact, $\bigcap \overline{C}^P = \emptyset$ for some countable subfamily of $\mathfrak{B}$. It follows that $C \cap U \subset \mathfrak{A}$ and $\bigcap \overline{C \cap U} = \emptyset$. Thus $\overline{U}$ is almost realcompact.

Theorem 5. *Closed subspaces of a regular almost realcompact space are almost realcompact.*

The proof follows at once from the following simple lemma:

Lemma 2. *If $\alpha = \{\mathfrak{A}\}$ is a complete collection of open coverings of a regular space P, then the following condition is fulfilled:*

(9) *If $\mathfrak{M}$ is a centered family of sets and if for every $\mathfrak{A}$ in α there exists a M in $\mathfrak{M}$ and an A in $\mathfrak{A}$ with $M \subset A$, then $\bigcap \overline{\mathfrak{M}} \neq \emptyset$.*

Proof. Let $\mathfrak{B}$ be the family of all open subsets B of P such that $B \supset M$ for some M in $\mathfrak{M}$. Clearly, $\mathfrak{B}$ is an α-Cauchy family. Thus $\bigcap \overline{\mathfrak{B}} \neq \emptyset$. On the other hand, according to the regularity of P we have $\bigcap \overline{\mathfrak{B}} = \bigcap \overline{\mathfrak{M}}$.

Proof of Theorem 5. If $\alpha = \{\mathfrak{A}\}$ is a complete collection of open coverings of a regular space P and if F is a closed subspace of P, then from Lemma 2 it follows at once that the collection $\alpha \cap F = \{\mathfrak{A} \cap F\}$ of open coverings of F is complete.

Theorem 6. *If a subspace R of a space P is the intersection of almost realcompact subspaces of P, then R is almost realcompact.*

Proof. Let $\mathfrak{A}$ be a maximal centered family of open subsets of R with $\bigcap \overline{\mathfrak{A}}^R = \emptyset$. Since $\bigcap \mathfrak{A}^P$ is at most a one-point set, we can choose an almost realcompact space $S \supset R$ such that $\bigcap \overline{\mathfrak{A}}^S = \emptyset$. Let $\mathfrak{B}$ be a maximal centered family of open subsets of S with $\mathfrak{B} \cap R \supset \mathfrak{A}$. Evidently $\mathfrak{B} \cap R = \mathfrak{A}$. Since

$$\emptyset = \bigcap \overline{\mathfrak{A}}^S = \bigcap \overline{\mathfrak{B}}^S,$$

according to almost realcompactness of S there exists a countable subfamily C of $\mathfrak{B}$ with $\bigcap \overline{C}^S = \emptyset$. Clearly $C \cap R \subset \mathfrak{A}$ and $\bigcap \overline{C \cap R}^S = \emptyset$. Thus R is an almost realcompact space.

Note. Theorem 5 is an immediate consequence of Theorems 4 and 6, since in a regular space every closed subspace is the intersection or regular closed subspaces.

Theorem 7. *The topological product of an arbitrary family of almost realcompact spaces is an almost realcompact space.*

Proof. Let P be the topological product of a family $\{P_a;\ a \in A\}$ of almost realcompact spaces. Let $\mathfrak{A}$ be a maximal centered family of open subsets of P with the countable intersection property. For every a in A let $\mathfrak{A}_a$ be the family of projections onto P_a of all sets from $\mathfrak{A}$. It is easy to see that $\mathfrak{A}_a$ is a maximal centered family of open subsets of P_a with the countable intersection property. Thus

$$\bigcap \overline{\mathfrak{A}}_a^{P_a} = (x_a) .$$

It is easy to see that

$$\bigcap \overline{\mathfrak{A}}^P = (\{x_a; a \in A\}) .$$

The proof is complete.

A mapping f from a space P to a space Q will be called perfect if f is continuous, closed (the images of closed sets are closed) and if the inverses of points are compact.

Theorem 8. *Let f be a perfect mapping of a space P onto a space Q. If P is an almost realcompact space, then Q is also almost realcompact. Conversely, if Q is almost realcompact and P regular, then P is an almost realcompact space.*

Proof. First let us suppose that P is an almost realcompact space. Let $\mathfrak{A}$ be a maximal centered family of open subsets of Q such that $\overline{\mathfrak{A}}$ has the countable intersection property. Let $\mathfrak{B}$ be a maximal centered family of open subsets of P with

$$\mathfrak{B} \supset f^{-1}[\mathfrak{A}] ,$$

i.e. every $f^{-1}[A]$, $A \in \mathfrak{A}$, belongs to $\mathfrak{B}$. We shall prove that $\mathfrak{B}$ has the countable intersection property. Let us suppose that there exists a countable subfamily C of $\mathfrak{B}$ with $\bigcap \overline{C} = \emptyset$. The mapping f being closed and the inverses of points compact, the family

$$\mathfrak{M} = \{Q - f[\overline{C}];\ C \in \mathfrak{C}\}$$

is an open countable covering of Q. Hence there exists a C in $\mathfrak{C}$ with

$$Q - f[\overline{C}] \in \mathfrak{A} .$$

It follows that

$$f^{-1}[Q - f[\overline{C}]] \in \mathfrak{B} .$$

But this is impossible, since $C \in \mathfrak{B}$ and

$$C \cap f^{-1}[Q - f[\overline{C}]] = \emptyset .$$

The space P being almost realcompact, we have $\bigcap \overline{\mathfrak{B}} \neq \emptyset$, and in consequence $\bigcap \overline{\mathfrak{A}} \neq \emptyset$. This proves Q is almost realcompact. The proof of the second part of Theorem 8 it follows at once from the following theorem:

Theorem 9. *Let f be a perfect mapping of a regular space P onto a space Q. If $\alpha = \{\mathfrak{A}\}$ is a complete collection of open coverings of Q, then the family $f^{-1}[\alpha]$ of all coverings*

$$f^{-1}[\mathfrak{A}] = \{f^{-1}[A];\ A \in \mathfrak{A}\}$$

where $\mathfrak{A}$ runs over all $\mathfrak{A} \in \alpha$, is a complete collection.

Proof. From the regularity of P it follows at once that Q is also regular. From Lemma 2 it follows that $\bigcap \overline{\mathfrak{M}} \neq \emptyset$ for every centered family of subsets of Q such that $\mathfrak{M} \cap \mathfrak{A} \neq \emptyset$ for all $\mathfrak{A} \in \alpha$. Let $\mathfrak{N}$ be a maximal centered family of subsets of P such that $\mathfrak{N} \cap \mathfrak{B} \neq \emptyset$ for every $\mathfrak{B}$ in $f^{-1}[\alpha]$. Clearly, the family $\mathfrak{M}$ of all $f[N]$, $N \in \mathfrak{N}$, is centered and $\mathfrak{M} \cap \mathfrak{A} \neq \emptyset$ for all $\mathfrak{A}$ in α. Thus $\bigcap \overline{\mathfrak{N}} \neq \emptyset$. Choose a point y in this intersection. Clearly $f^{-1}[y] \cap \overline{\mathfrak{M}}$ is a centered family. The space $f^{-1}[y]$ being compact, we have

$$f^{-1}[y] \cap \bigcap \overline{\mathfrak{M}} \neq \emptyset ,$$

which proves that $f^{-1}[\alpha]$ is complete.

6. RELATIONS BETWEEN REALCOMPACT AND ALMOST REALCOMPACT SPACES

In this section all spaces under consideration are supposed to be completely regular.

Theorem 10. *Every realcompact space is almost realcompact.*

Proof. If P is realcompact, then the collection $\{\mathfrak{W}(f); f \in C(P)\}$ of coverings $\mathfrak{W}(f)$ from Note 5 is complete and hence the collection of all open coverings is complete.

Two subsets M and N of a space P will be called completely separated if there exists a real-valued continuous function f on P with $f[M] \subset (0)$ and $f[N] \subset (1)$.

Lemma 3. *Let U be an open subset of a space P and let M be a subset of P such that M and $P - U$ are completely separated. If V is a maximal open subset of $\beta(P)$ with $V \cap P = U$, then the closure of M in $\beta(P)$ is contained in V.*

Lemma 4. *Let α be a complete collection of open coverings of a space P, such that*

(10) *For every $\mathfrak{A}$ in α there exists a $\mathfrak{B}$ in α such that for every B in $\mathfrak{B}$ there exists an A in $\mathfrak{A}$ such that B and $P - A$ are completely separated.*

Then

$$\bigcap \{\bigcup \overline{\mathfrak{A}}^{\beta P}; \mathfrak{A} \in \alpha\} = P . \tag{11}$$

Proof. Let us denote by R the left side of (11). Suppose that there exists a point x in $R - P$. Let $\mathfrak{M}$ be the family of open neighborhoods of x in $\beta(P)$. Put $\mathfrak{N} = \mathfrak{M} \cap P$. It is easy to prove that $\mathfrak{N}$ is an α-Cauchy family. Indeed, if $\mathfrak{A}$ is an open covering and $\mathfrak{B}$ is the covering satisfying (10), then there exists a $B \in \mathfrak{B}$ with $x \in \overline{B}^{\beta(P)}$. If A is the set corresponding to B in accordance with (10), then $A \in \mathfrak{N}$. The collection α being complete, the intersection of $\overline{\mathfrak{N}}^P$ is non-void. But this is impossible, since $\bigcap \overline{\mathfrak{N}}^{\beta(P)} = (x) \subset R - P$.

If P is a normal space, then the family γ of all countable open coverings does have the property (10). By Lemma 4, if the space P is almost realcompact, then P is the intersection of σ-compact subspaces of $\beta(P)$, and hence, P is realcompact. Thus we have proved the following theorem:

Theorem 11. *Every normal almost realcompact space is realcompact.*

I do not know any example of an almost realcompact space which is not realcompact. Evidently, the image under a closed continuous mapping of a normal space is a normal space. Thus from Theorems 8 and 11 there follows at once the following theorem:

Theorem 12. *The image under a perfect mapping of a realcompact normal space is a realcompact space.*

Note. It is easy to prove that the image under an open perfect mapping of a realcompact space is a realcompact space. I do not know whether the assumption of normality may be omitted in Theorem 12.

Notes. In section 3 only complete collections of open coverings were defined. For completely regular spaces it is useful to define complete collections (of not necessarily open) coverings.

Definition 3. Let $\alpha = \{\mathfrak{A}\}$ be a collection of coverings of a space P. An α-Cauchy family is a centered family $\mathfrak{M}$ of subsets of P such that for every $\mathfrak{A}$ in α there exists an M in $\mathfrak{M}$ and an A in $\mathfrak{A}$ with $A \supset M$. The collection α will be called complete if $\bigcap \overline{\mathfrak{M}} \neq \emptyset$ for every α-Cauchy family $\mathfrak{M}$.

If $\alpha = \alpha(P)$ is the collection of all countable coverings of P consisting zero-sets, then a maximal centered family $\mathfrak{Z}$ of zero-sets is an α-Cauchy family if and only if $\mathfrak{Z}$ has the countable intersection property. If $\mathfrak{M}$ is a maximal centered α-Cauchy family, then the zero-sets from $\mathfrak{M}$ form an α-Cauchy family $\mathfrak{Z}$, and the space being completely regular, $\bigcap \overline{\mathfrak{M}} = \bigcap \mathfrak{Z}$. Thus we have proved

Theorem 13. *A space P is realcompact if and only if the collection of all countable coverings consisting of zero-sets is complete.*

Let us denote by δ the collection of all countable closed (*i.e.* consisting of closed sets) coverings of a space P. Since $\alpha \subset \delta$, if α is complete, then δ is complete. If P is normal and δ complete, then it is easy to prove that P is the intersection of σ-compact subspaces of $\beta(P)$, more precisely

$$P = \bigcap \{\bigcup \overline{\mathfrak{A}}^{\beta(P)};\ \mathfrak{A} \in \delta\}\ .$$

Let R denotes the right side of the preceding equality. Let us suppose that there exists a point x in $R - P$. Let $\mathfrak{F}$ be the family of all closed subsets F of P with $x \in \overline{F}^{\beta(P)}$. P being normal, $\mathfrak{F}$ is a δ-Cauchy family. Since, clearly, $\bigcap \mathfrak{F} = \emptyset$, δ is not a complete collection. Thus we have proved the following

Theorem 14. *If P is a realcompact space, then the collection δ of all closed coverings is complete. If P is normal and δ complete, then P is realcompact.*

Let $\mathfrak{F}$ be a maximal centered family of closed sets and let us suppose that the intersection of every countable subfamily of $\mathfrak{F}$ is non-void. Then, clearly, $\mathfrak{F}$ is a δ-Cauchy family. Thus, if δ is a complete collection, then the intersection of $\mathfrak{F}$ is non-void. Conversely, let us suppose that the intersection of every maximal centered family of closed sets with the countable intersection property is non-void. If the following condition (12) is fulfilled, then the collection δ is complete.

(12) If $\mathfrak{F}$ is a maximal centered family of closed sets and if $\mathfrak{F}$ does not have the countable intersection property, then there exist $F_n \in \mathfrak{F}$ and open U_n such that $U_n \supset F_n$ and $\bigcap_{n=1}^{\infty} U_n = \emptyset$.

Indeed, from (12) it follows at once that if $\mathfrak{F}$ is a maximal centered family of closed sets and if $\mathfrak{F}$ does not have the countable intersection property, then $\mathfrak{F}$ is not a δ-Cauchy family. For example, every countably paracompact space, in particular, every perfectly normal space, has the property (12).

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Резюме

ОБОБЩЕНИЕ Q-ПРОСТРАНСТВ

ЗДЕНЕК ФРОЛИК (Zdeněk Frolík), Прага

Пространство P называется почти *Q-пространством*, если выполнено следующее условие:

Если пересечение замыканий множеств из некоторой максимальной центрированной системы $\mathfrak{A}$ открытых множеств пусто, то пусто также пересечение замыканий множеств из некоторой счетной $\mathfrak{B} \subset \mathfrak{A}$.

Оказывается, что всякое Q-пространство явяетсля почти Q-пространством и что всякое нормальное почти Q-пространстов является Q-пространством.

Для почти Q-пространств имеют место теоремы, аналогичные теоремам о Q-пространствах. Но их доказательства обычно проще.

Теорема. *Пусть P является пространством Хаусдорфа. Следующие условия эквивалентны*:

(1) *P является почти Q-пространством.*

(2) *Система всех счетных открытых покрытий полна в смысле* [2].

(3) *P является пересечением таких подпространств Q H-замкнутой оболочки Катетова* [7] *пространства P, что для всякого Q существует счетное открытое покрытие $\mathfrak{A}$ пространства Q, что $\bar{A}^Q$, $A \in \mathfrak{A}$, являются H-замкнутыми.*

Теорема. *Пусть f — совершенное отображение пространства P на пространство Q. Для того, чтобы Q было почти Q-пространством, также достаточно, а в случае регулярного P также и необходимо, чтобы P было Q-пространством.*

Из последней теоремы вытекает, что образ нормального Q-пространства при совершенном отображении является Q-пространством.

В последней части дается новая характеризация нормальных и счетно паракомпактных Q-пространств.