ON THE STRUCTURE OF A CLASS OF COMMUTATIVE SEMIGROUPS

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The purpose of the paper is to clarify the structure of a special type of commutative semigroups to which several authors have been led by studying decompositions of general semigroups.

1. Introduction and summary. In this paper we investigate the structure of a class of semigroups which we call N-semigroups. An N-semigroup is a commutative non-potent archimedean cancellative semigroup. E. HEWITT and H. S. ZUCKERMAN [2] have shown that if G is any commutative semigroup, then there is a maximal separative homomorphic image G' of G, and that a member H_x of the maximal semilattice decomposition of G' is either a group or an N-semigroup. T. TAMURA [5] has given a characterization of N-semigroups. T. TAMURA and N. KIMURA [4] have shown that a member of the maximal semilattice decomposition of an arbitrary commutative semigroup is archimedean and has at most one idempotent. Š. SCHWARZ [6] has established certain properties of decompositions of a semigroup similar to those already mentioned.

In section 2 we define N-semigroups and discuss some properties of commutative semigroups in connection with it. Then in section 3 we establish a property of N-semigroups with a finite number of generators. In section 4 we find the structure of N-semigroups with two generators. Finally in section 5 we give a classification and several examples of N-semigroups.

A semigroup is a non-empty set on which an associative multiplication is defined. We will discuss only commutative semigroups. Throughout the whole paper S will denote an arbitrary commutative semigroup unless stated otherwise. We follow the notation and terminology of A. H. CLIFFORD and G. B. PRESTON [1] for all concepts not defined in the paper. By $x^m y^n$ with $m = 0$ and $n > 0$, we mean y^n.

The writer wishes to thank Professor *Edwin Hewitt* for mentioning this problem to him, and Professor *Herbert S. Zuckerman* for his help in preparation of this paper.

2. Definitions and properties. We first define an N-semigroup and then discuss some properties of semigroups in connection with it.

Definition. S will be called an N-semigroup if it has the following properties:

(A) for every $x, y, z \in S$, $xz = yz$ implies $x = y$ (S is cancellative);

(B) for every $x, y \in S$, $x^n = ay$ for some $a \in S$ and some natural number n (S is archimedean);

(C) S has no idempotents (S is nonpotent).

Proposition 1. *If S satisfies condition* (B) *and*

(D) *for every $x, y \in S$, $x^2 = y^2 = xy$ implies $x = y$ (S is separative), then S also satisfies* (A).

Proof. Let S satisfy conditions (B) and (D) and suppose that $xz = yz$ for some $x, y, z \in S$. Then $x^m = az$ and $y^n = bz$ for some $a, b \in S$ and some m, n. Hence

$$x^{m+1} = (az)\,x = a(xz) = a(yz) = (az)\,y = x^m y\,, \tag{1}$$

$$y^{n+1} = (bz)\,y = b(yz) = b(xz) = (bz)\,x = y^n x\,. \tag{2}$$

If in (1) $m > 1$, then

$$x^{2m-2}xy = x^{m-2}x^{m+1}y = x^{m-2}x^m y^2 = x^{2m-2}y^2\,,$$
$$x^{2m-2}xy = x^{m-1}x^m y = x^{m-1}x^{m+1} = x^{2m}\,.$$

Consequently $(x^{m-1}y)^2 = (x^m)^2 = x^m(x^{m-1}y)$ and thus $x^m = x^{m-1}y$. After $m - 1$ steps, we obtain $x^2 = xy$. Similarly from (2), we obtain $y^2 = xy$ and therefore $x = y$.

Corollary. *In the definition of an N-semigroup, we can substitute condition* (A) *by the weaker condition* (D).

The proofs of the following statements are either contained in the works mentioned at the beginning of the paper or in [3].

Proposition 2. *If S satisfies* (B), *then it contains at most one idempotent.*

Proposition 3. *If S satisfies* (A) *and* (B) *and does not satisfy* (C), *then it is a group.*

Theorem 1. *S contains no prime (proper semiprime) ideals if and only if S satisfies* (B).

Theorem 2. *Each member of the maximal semilattice decomposition of S satisfies* (B).

Theorem 3. *S satisfies* (D) *if and only if S is a semilattice of semigroups each of which satisfies* (A).

3. Finitely generated N-semigroups. The set of positive integers under addition is an N-semigroup generated by the element 1. It is evident that this is the only cyclic N-semigroup. The following theorem establishes a property of N-semigroups with a finite number of generators.

Theorem 4. *A finitely generated N-semigroup S satisfies the following condition:*

(E) *for every $x, y \in S$, there are natural numbers p and q such that $x^p = y^q$.*

Proof. Let $a_1, a_2, \ldots, a_n$ be the set of generators of S. We first show that condition (E) holds for $a_1, a_2, \ldots, a_n$. We do this by mathematical induction on the number k defined as follows. A power of a fixed generator can be written as a product of powers of any $n - k$ of the remaining generators with the power of any specified generator positive, $1 \leqq k < n$.

The proof for $k = 1$. Consider the generator a_1; the other cases are similar. For $m > 1$, we have

$$a_1^t = (a_1^{t_1} a_2^{t_2} \ldots a_n^{t_n})\, a_m$$

for some $t, t_1, t_2, \ldots, t_n$ with $t > 1$ and $\sum_{i=1}^{n} t_i > 0$. Here $t > t_1$, for otherwise we would arrive at a contradiction after cancellation. Hence

$$a_1^{t-t_1} = a_2^{t_2} \ldots a_m^{t+1} \ldots a_n^{t_n} .$$

Suppose now that the condition stated at the beginning of the proof is satisfied for some k, $1 \leqq k < n$. We again consider only the case of the generator a_1, the other cases being similar. We show that the condition in question is also valid for $k + 1$. By hypothesis we have

(1)
$$a_1^p = a_{k+1}^{p_{k+1}} a_{k+2}^{p_{k+2}} \ldots a_n^{p_n},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_{k+1}^q = a_1^{q_1} a_{k+2}^{q_{k+2}} \ldots a_n^{q_n}$$

where $p_n > 0$. We obtain

$$a_1^{pq} = a_{k+1}^{p_{k+1}q} a_{k+2}^{p_{k+2}q} \ldots a_n^{p_n q} = a_1^{q_1 p_{k+1}} a_{k+2}^{q_{k+2}p_{k+1}} \ldots a_n^{q_n p_{k+1}} a_{k+2}^{p_{k+2}q} \ldots a_n^{p_n q}$$

whence

$$a_1^{pq - q_1 p_{k+1}} = a_{k+2}^{q_{k+2}p_{k+1} + p_{k+2}q} \ldots a_n^{q_n p_{k+1} + p_n q}$$

since necessarily $pq > q_1 p_{k+1}$ and also $p_n q > 0$. The general case is proved by considering in (1) any $n - k$ generators different from a_1 which merely amounts to a change of notation. This concludes the proof of induction.

We have in particular $a_1^{m_i} = a_i^{s_i}$ for $i = 2, 3, \ldots, n$ where $m_i, s_i > 1$. Let $a_1^{k_1} a_2^{k_2} \ldots \ldots a_n^{k_n}$, $a_1^{l_1} a_2^{l_2} \ldots a_n^{l_n} \in S$ be arbitrary. Then

$$(a_1^{k_1} a_2^{k_2} \ldots a_n^{k_n})^{s_2 s_3 \ldots s_n (l_1 s_2 s_3 \ldots s_n + m_2 l_2 s_3 \ldots s_n + \ldots + m_n s_2 s_3 \ldots s_{n-1} l_n} =$$
$$= a_1^{(k_1 s_2 s_3 \ldots s_n + m_2 k_2 s_3 \ldots s_n + \ldots + m_n s_2 s_3 \ldots s_{n-1} k_n)(l_1 s_2 s_3 \ldots s_n + m_2 l_2 s_3 \ldots s_n + \ldots + m_n s_2 s_3 \ldots s_{n-1} l_n)} =$$
$$= (a_1^{l_1} a_2^{l_2} \ldots a_n^{l_n})^{s_2 s_3 \ldots s_n (k_1 s_2 s_3 \ldots s_n + m_2 k_2 s_3 \ldots s_n + \ldots + m_n s_2 s_3 \ldots s_{n-1} k_n},$$

which completes the proof.

4. N-semigroups with two generators. In this section we find the structure and give a concrete realization of N-semigroups with two generators. We do not consider N-semigroups with more than two generators because the characterization given in this paper becomes too involved in such a case. We first introduce some notations.

Notation. Let S be an N-semigroup with two generators, say a_1 and a_2. Let m_1 and m_2 be the smallest positive integers such that $a_1^{m_1} = ua_2$ and $a_2^{m_2} = va_1$ for some $u, v \in S$. We suppose that $m_1 \leqq m_2$ and denote S by $N(m_1, m_2)$.

Since S is cancellative, a_2 is a generator of S, and m_1 is minimal, we have $u = a_2^{k_2}$ for some $k_2 > 0$, and thus $a_1^{m_1} = a_2^{k_2+1}$. Similarly $a_2^{m_2} = a_1^{k_1+1}$ for some $k_1 > 0$. By minimality of m_1 and m_2, $m_1 \leqq k_1 + 1$ and $m_2 \leqq k_2 + 1$. If $k_2 + 1 > m_2$, then

$$a_1^{m_1} = a_2^{k_2+1} = a_2^{m_2} a_2^{k_2+1-m_2} = a_1^{k_1+1} a_2^{k_2+1-m_2}$$

which is impossible since $m_1 \leqq k_1 + 1$. Thus $m_2 = k_2 + 1$, that is, $a_1^{m_1} = a_2^{m_2}$.

Notation. Let m_1 and m_2 be integers such that $2 \leqq m_1 \leqq m_2$. A set S will be denoted by (m_1, m_2)-s.g. if

$$S = \{(k_1, k_2) \mid k_1 = 0, 1, 2, \dots, \quad k_2 = 0, 1, 2, \dots, m_2 - 1, \quad k_1 + k_2 > 0\}$$

with multiplication

$$(k_1, k_2)(l_1, l_2) = (k_1 + l_1 + jm_1,\ k_2 + l_2 - jm_2)$$

where j is the integer such that $0 \leqq k_2 + l_2 - jm_2 < m_2$, and $(k_1, k_2) = (l_1, l_2)$ implies $k_1 = l_1$ and $k_2 = l_2$.

The following theorem gives a simple characterization of N-semigroups with two generators:

Theorem 5. *Let S be a set. Then $S = N(m_1, m_2)$ if and only if $S = (m_1, m_2)$-s.g.*

Proof. We first prove *necessity*. Thus let $S = N(m_1, m_2)$ with the generators a_1 and a_2 such that $a_1^{m_1} = a_2^{m_2}$. If x is an element of S, then $x = a_1^{k_1} a_2^{k_2}$ for some non-negative integers k_1 and k_2 such that $k_1 + k_2 > 0$. We have $0 \leqq k_2 - jm_2 < m_2$ for some non-negative integer j, and hence

$$a_1^{k_1} a_2^{k_2} = a_1^{k_1} a_2^{jm_2 + (k_2 - jm_2)} = a_1^{k_1 + jm_1} a_2^{k_2 - jm_2}.$$

Thus every element of S can be written in the form of an element of (m_1, m_2)-s.g. with a suitable change of notation. One checks similarly that the multiplication of S coincides with that of (m_1, m_2)-s.g. under the restriction that $0 \leqq k_2 < m_2$ where $a_1^{k_1} a_2^{k_2} \in S$. Suppose that $a_1^{k_1} a_2^{k_2} = a_1^{l_1} a_2^{l_2}$ with $0 \leqq k_2,\ l_2 < m_2$. If $k_1 > l_1$, then $a_1^{k_1 - l_1} a_2^{k_2} = a_2^{l_2}$ and thus necessarily $k_2 < l_2$. Consequently $a_1^{k_1 - l_1} = a_2^{l_2 - k_2}$ and thus $l_2 - k_2 \geqq m_2$ by minimality of m_2. But this contradicts the hypothesis that $l_2 < m_2$. The case $k_1 < l_1$ is symmetric. Hence $k_1 = l_1$ and consequently $k_2 = l_2$. Therefore $S = (m_1, m_2)$-s.g.

We next prove *sufficiency*. Let $S = (m_1, m_2)$-s.g. It is clear that S is closed under its multiplication and is commutative. We verify the postulates for $N(m_1, m_2)$.

Associativity. It is easily seen that both $[(k_1, k_2)(l_1, l_2)](r_1, r_2)$ and $(k_1, k_2) \cdot [(l_1, l_2)(r_1, r_2)]$ are equal to $(k_1 + l_1 + r_1 + im_1, k_2 + l_2 + r_2 - im_2)$ where i is the integer such that $0 \leqq k_2 + l_2 + r_2 - im_2 < m_2$. Hence the associative law holds.

Condition (A). If $(k_1, k_2)(r_1, r_2) = (l_1, l_2)(r_1, r_2)$, then

$$(k_1 + r_1 + im_1,\ k_2 + r_2 - im_2) = (l_1 + r_1 + jm,\ l_2 + r_2 - jm_2)$$

where i and j are the integers such that $0 \leqq k_2 + r_2 - im_2 < m_2$ and $0 \leqq l_2 + r_2 - jm_2 < m_2$. Hence

$$k_1 + r_1 + im_1 = l_1 + r_1 + jm_1\,, \tag{1}$$

$$k_2 + r_2 - im_2 = l_2 + r_2 - jm_2\,. \tag{2}$$

From (2) we obtain $k_2 - l_2 = (i - j)\, m_2$ which implies that $i = j$ and consequently $k_2 = l_2$. But $i = j$ in (1) yields $k_1 = l_1$. Therefore $(k_1, k_2) = (l_1, l_2)$ and the cancellation law holds in S.

Condition (B). Now let (k_1, k_2) and (l_1, l_2) be any elements of S. If $k_1 > 0$, then let $n = m_1 + l_1 + 1$ and q be the non-negative integer such that $0 \leqq nk_2 - m_2 q < m_2$. If $k_1 = 0$, then let $q = m_1 + l_1 + 1$ and n be the integer satisfying the inequality $m_2 q/k_2 \leqq n < (m_2 q/k_2) + 1$ (in this case $k_2 > 0$). Moreover let

$$j = \begin{cases} 1 & \text{if} \quad nk_2 - m_2 q - l_2 < 0\,, \\ 0 & \text{if} \quad nk_2 - m_2 q - l_2 \geqq 0\,. \end{cases}$$

It follows easily from the definitions of n, q, and j that

(a) $nk_1 + m_1 q - l_1 - jm_1 > (m_1 + l_1) - l_1 - jm_1 \geqq 0$;
(b) $nk_1 + m_1 q > 0$;
(c) $0 \leqq nk_2 - m_2 q < m_2$;
(d) $0 \leqq nk_2 - m_2 q - l_2 + jm_2 < m_2$.

We therefore have

$$(nk_1 + m_1 q - l_1 - jm_1,\ nk_2 - m_2 q - l_2 + jm_2)(l_1, l_2) = \\ = (nk_1 + m_1 q,\ nk_2 - m_2 q) = (k_1, k_2)^n\,.$$

Condition (C). Suppose now that $(k_1, k_2)^2 = (k_1, k_2)$ for some $(k_1, k_2) \in S$. Then

$$(2k_1 + jm,\ 2k_2 - jm_2) = (k_1, k_2)$$

where j is the integer such that $0 \leqq 2k_2 - jm_2 < m_2$. Hence $2k_1 + jm_1 = k_1$ and $2k_2 - jm_2 = k_2$. Consequently $k_2 - jm_2 = 0$ which implies that $j = 0$ and hence also $k_2 = 0$. But $j = 0$ in the first equation yields $k_1 = 0$, which is impossible. Therefore S has no idempotents.

$S = N(m_1, m_2)$. We have already proved that S is an N-semigroup. It is clear that (1,0) and (0,1) are the generators of S. Let n be the smallest positive integer such that $(1,0)^n = (k_1, k_2)(0, 1)$ for some $(k_1, k_2) \in S$. Then $k_1 = 0$ by minimality of n. Hence $(1, 0)^n = (0, k_2)(0, 1)$ and thus $(n, 0) = (jm_1, k_2 + 1 - jm_2)$ where j is the integer such that $0 \leqq k_2 + 1 - jm_2 < m_2$. Therefore $n = jm_1$ and $0 = k_2 + 1 - - jm_2$. Consequently $j = 1$ and thus n $= m_1$ and $k_2 = m_2 - 1$.

Similarly we see that if n is the smallest positive integer such that $(0, 1)^n = = (k_1, k_2)(1, 0)$ for some $(k_1, k_2) \in S$, then $n = m_2$. Since $m_1 \leqq m_2$, we have proved that $S = N(m_1, m_2)$.

Remark. From the definition of (m_1, m_2)-s.g. it follows easily that (m_1, m_2)-s.g. is isomorphic to the free commutative semigroup on two generators, say a_1 and a_2, with the defining relation $a_1^{m_1} = a_2^{m_2}$. This furnishes a second characterization of $N(m_1, m_2)$ by virtue of Theorem 5.

We next give a concrete realization of (m_1, m_2)-s.g. First we introduce some notation.

Notation. For integers m_1 and m_2 such that $2 \leqq m_1 \leqq m_2$, let $C(m_1, m_2)$ be the subsemigroup of the group of non-zero complex numbers, generated by the two elements

$$a_1 = 2^{1/m_1} e^{(2\pi i)/m_1} \quad \text{and} \quad a_2 = 2^{1/m_2} e^{(4\pi i)/m_2}.$$

Then we have the following result:

Theorem 6. *The semigroups $C(m_1, m_2)$ and (m_1, m_2)-s.g. are isomorphic.*

Proof. First note that $a_1 \neq a_2$ because $2 \leqq m_1 \leqq m_2$. Since $C(m_1, m_2)$ is a subsemigroup of a commutative group (non-zero complex numbers), by the remark above it suffices to show that $a_1^{k_1} = a_2^{k_2}$ implies $k_1/k_2 = m_1/m_2$ and $k_1 \geqq m_1$. If $a_1^{k_1} = a_2^{k_2}$, that is,

$$2^{k_1/m_1} e^{[(2\pi i)/(m_1)]k_1} = 2^{k_2/m_2} e^{[(4\pi i)/(m_2)]k_2},$$

then by equating moduli, we obtain $k_1/m_1 = k_2/m_2$. But then

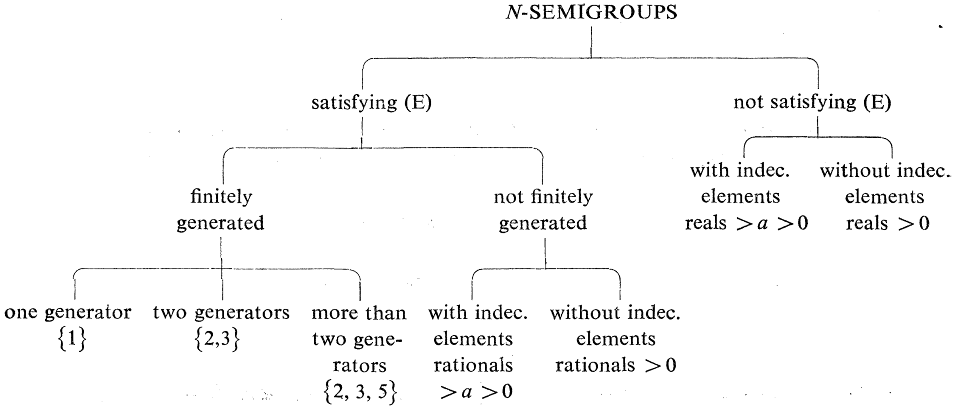
$$e^{2\pi i[2(k_1/m_1) - (k_1/m_1)]} = 1$$

and consequently $k_1 \geqq m_1$. The theorem follows.

The following is essentially a resume of some results of this section:

Theorem 7. *The semigroup $C(m_1, m_2)$ is an N-semigroup with two-generators and $C(m_1, m_2) = C(m_1', m_2')$ only if $m_1 = m_1'$ and $m_2 = m_2'$. Conversely, every N-semigroup with two generators is isomorphic to the semigroup $C(m_1, m_2)$ for some (unique) integers m_1 and m_2, $2 \leqq m_1 \leqq m_2$.*

5. Classification and examples of N-semigroups. We classify N-semigroups according to whether they satisfy condition (E) of Theorem 4, whether they are finitely generated, and whether they contain indecomposable elements. We say that an element x of S is indecomposable if $x \neq yz$ for all $y, z \in S$. We also give an example for each group of N-semigroups. All the examples given are subsemigroups of the additive semigroup of positive real numbers. The numbers in brackets denote the generators of S; the letter a denotes any positive real number.



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Резюме

О СТРОЕНИИ ОПРЕДЕЛЕННОГО КЛАССА КОММУТАТИВНЫХ ПОЛУГРУПП

МАРИО ПЕТРИХ (Mario Petrich), Мариланд (США)

Коммутативная полугруппа S называется N-полугруппой, если 1. В S имеет место правило сокращения; 2. для каждой пары $x, y \in S$ существует $a \in S$ и целое число $n > 0$ так, что $x^n = ay$; 3. S не имеет идемпотентов. Работа посвящена изучению строения N-полугрупп. Именно, описана структура всех N-полугрупп, обладающих двумя генераторами.