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AN EXTENSION THEOREM FOR SEPARATELY CONTINUOUS FUNCTIONS AND ITS APPLICATION TO FUNCTIONAL ANALYSIS

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Introduction. Let S be a family of functions on a set T. The main idea of the present remark consists in considering the value $s(t)$ of an $s \in S$ at the point t as a function of two variables $\langle s, t \rangle$ on the cartesian product $S \times T$ thus making the role of s and t completely symmetric. In this manner the points $t \in T$ themselves may be considered as functions on the set S and from this point of view it becomes entirely natural to introduce linear combinations of points — a possibility which has not yet been systematically exploited. In the present paper we shall describe some of the advantages of this approach.

Let us consider, as an example, the following situation. Suppose we have a sequence $s_n \in S$ and an $s_0 \in S$ such that $\lim s_n(t) = s_0(t)$ for each $t \in T$. Suppose further that we extend the set T by adjoining to it an "ideal" point τ and ask whether $\lim s_n(\tau) = s_0(\tau)$ as well. Of course the point τ has to belong, in some sense, to the closure of T: if, e.g., all the functions s are bounded on T, we may take on T any topology which makes them continuous and consider points $\tau \in \beta T$. Our problem is to decide whether $\lim s_n(\tau) = s_0(\tau)$ as well. It is a well known fact that this will be true if the function s_0 may be arbitrarily well approximated by the s_n uniformly on T; it is, however, easy to see that the following much weaker condition is also sufficient. The function s_0 may be arbitrarily well approximated by convex combinations of the s_n, in other words: for each $\varepsilon > 0$ there exists a convex combination $\sum \lambda_i s_i$ such that $|\sum \lambda_i s_i(t) - s_0(t)| < \varepsilon$ for all $t \in T$. In our new notation, $|\langle \sum \lambda_i s_i - s_0, t \rangle| < \varepsilon$ for all $t \in T$.

An entirely analogous situation presents itself in the dual case. Suppose we have a sequence $t_n \in T$ and a point $t_0 \in T$ such that $\lim s(t_n) = s(t_0)$ for each $s \in S$. Suppose that we extend the family S by adjoining ot if a function σ which is, in some sense, a limit of functions $s \in S$ at each point $t \in T$. The question arises whether $\lim \sigma(t_n) = \sigma(t_0)$ as well. The obvious sufficient condition — for each $\varepsilon > 0$, there exists a t_n such that $|s(t_n) - s(t_0)| < \varepsilon$ for all s — is too strong. In analogy with the preceding case — treating points as functions — the following condition will be sufficient: for each $\varepsilon > 0$, there exists a convex mean such that $|\sum \lambda_i s(t_i) - s(t_0)| < \varepsilon$ for all $s \in S$. If we agree to consider $\sum \lambda_i s(t_i)$ as the value of s at $\sum \lambda_i t_i$, we may write $|\langle s, \sum \lambda_i t_i - t_0 \rangle| < \varepsilon$ for all $s \in S$ in complete analogy with the preceding case. In both cases, the approximation of limit points by convex combinations was essential.

Let us sketch briefly how these ideas could be used to obtain e.g. more profound information about the structure of the Čech-Stone compactification. This may be seen in the following manner: the topology of the compactification of a topological space T is defined, roughly speaking, by the postulate that two points τ_1, τ_2 are considered to be near each other if the difference $f(\tau_1) - f(\tau_2)$ is small for all f contained in a given finite set F of continuous functions. In this topology T is dense in βT, in other words: given $\tau \in \beta T$ and $\varepsilon > 0$, there exists a point $t \in T$ such that

$$|f(t) - f(\tau)| < \varepsilon \quad \text{for all} \quad f \in F .$$

If we admit infinite sets F as well we cannot hope to find a point $t \in T$ with $|f(t) - f(\tau)| < \varepsilon$ for all $f \in F$; in this topology T will not be dense in βT any more. Nevertheless it is possible to show that even for infinite families F of a certain type the ideal point τ may be approximated uniformly on F by a convex combination of ordinary points, i.e. it will be possible to find points $t_i \in T$ and numbers λ_i so that $|\sum \lambda_i f(t_i) - f(\tau)| < \varepsilon$ for all $f \in F$. This shows the possibility of a more refined description of the structure of βT.

For the purpose of investigating the approximation of ideal points by convex combinations of real points we have given in [8] a combinatorial lemma which essentialy describes the degree of approximation of a measure by a convex combination of points and which contains everything essential; all the results on weak compactness follow from it just by pure logic. This lemma forms the basis of the present paper.

To return to our original program of treating families of functions as a function of two variables suppose we have a completely regular topological space T and a family S of continuous functions on T which generates a function f on $S \times T$ by $f(s, t) = s(t)$. If s is fixed, $f(s, t)$ is continuous as a function of t. Suppose further that the family S is also given a topology, e.g. the topology of pointwise convergence or some finer topology. Then f will be separately continuous on $S \times T$. This explains why we study separately continuous functions and not continuous ones: it is easy to see that continuity of f imposes a rather heavy restriction on the family S. We shall restrict our attention to bounded f only.

Consider now a function σ on T which belongs to the closure of S in the topology of pointwise convergence. To ask whether σ will also be continuous amounts to the same as to ask whether f will stay separately continuous also on $S \cup (\sigma) \times T$. We are thus led to the problem of investigating conditions under which f may be extended as a separately continuous function if S and T are imbedded in larger spaces. As we have seen, it will be useful to adjoin to the domain of f also linear combinations of functions s and points t.

We intend to show that it is advantageous to consider together with each topological space a certain superstructure which may be described as follows. First make an algebraic extension by adding to the space all formal convex combination of its points, further, a topological extension obtained in very much the same way as βX is obtained from X. The ideal points obtained in this manner could, of course, be interpreted – roughly speaking – as measures; this can be avoided by using the combinatorial lemma which not only eliminates the machinery of integration theory but at the same time yields deeper results.

There is, of course, an obvious way of obtaining, for each completely regular space T, an extension of the type described above: the natural imbedding of T in the dual of the Banach space $C_\beta(T)$ of all bounded continuous functions on T (taken in the weak * topology). In this manner the topological space T is imbedded in a topological linear space where algebraic operations are defined as well. It follows that the product $S \times T$ may be considered as a subset of $C_\beta(S)' \times C_\beta(T)'$ and the study of convex combinations of functions and points is then made possible if we treat f as a bilinear form on the linear extensions of S and T.

Now we may formulate the main problem.

Let f be a bounded and separately continuous function on $S \times T$. Under what conditions does there exist a separately continuous bilinear form on $C_\beta(S)' \times C_\beta(T)'$ which extends f?

The main result consists in showing that such an extension exists if and only if f satisfies the double limit condition: if $s_i \in S$ and $t_j \in T$ are such that both $\lim_i \lim_j f(s_i, t_j)$ and $\lim_j \lim_i f(s_i, t_j)$ exist then they have to be equal to each other. The essential idea of this condition appears first in the work of S. BANACH [1] as a condition for weak convergence and was later brought to its full generality by A. GROTHENDIECK [5].

The extension theorem that we intend to discuss here is thus connected with the fundamental question of Analysis, the question of inverting two limit processes.

An interesting feature of this result is the fact that it enables us to conclude the continuity of a function in a topology which has an uncountable character from purely sequential assumptions.

The main (and only) tool is the combinatorial lemma obtained in [8]. The first section contains some definitions and conventions, section two is devoted to the proof of the theorem. A weaker version of the theorem is already contained in the author's paper [9]. To show how it can be applied we give, in section three, a proof of a result which includes both the Krein theorem and the Eberlein theorem. This result is an immediate consequence of the extension theorem: the possibility of obtaining if from sequential assumptions is based on the countable character of the double limit condition. The extension theorem may be formulated in many different forms – some of the most interesting ones are collected in section five. Section six treats the corresponding questions with separate continuity replaced by continuity.

In this case the results are much more superficial; they are included, however, since they put into evidence the similarities and differences between the weak and uniform topology.

In a suitable interpretation, they contain the classical theorems of Arzelà, Ascoli and Dini.

1. DEFINITIONS AND NOTATION

In the whole paper, we shall be dealing with completely regular topological spaces only. If T is a completely regular topological space, we shall denote by $C_\beta(T)$ the linear space of all bounded continuous functions on T with the norm

$$|x| = \sup_{t\in T} |x(t)| .$$

Clearly $C_\beta(T) = C(\beta T)$. The space $C_\beta(T)'$ consists of all continuous linear functionals on the Banach space $C_\beta(T)$. The space $C_\beta(T)'$ will always be taken in the weak-star topology, the topology $\sigma(C_\beta(T)', C_\beta(T))$. It is a well-known and obvious fact that the mapping E which assigns to each $t \in T$ the corresponding evaluation functional $E(t) \in C_\beta(T)'$

$$\langle x, E(t)\rangle = x(t)$$

is a homeomorphism. Hence it is possible to identify T and $E(T)$; accordingly, we shall adopt the following convention: we shall consider T as a subset of $C_\beta(T)'$ and we shall not distinguish between a point $t \in T$ and the evaluation functional it generates.

Let T be a completely regular topological space and F a class of continous functions on T. We shall denote by T/F the quotient space of T with respect to the equivalence E on T defined by the postulate $t_1 E t_2$ iff $f(t_1) = f(t_2)$ for all $f \in F$.

Let K be a compact space and H a dense subset of K. There is an obvious imbedding of $C(K)$ in a cartesian product P of real lines, one for each point $h \in H$. The topology induced on $C(K)$ by this imbedding will be called the point topology corresponding to H.

Further, consider a bounded function g on a cartesian product $S \times T$; we say that g satisfies the double limit condition on $S \times T$ if it is impossible to find two sequences $s_i \in S$ and $t_j \in T$ such that both $\lim_i \lim_j g(s_i, t_j)$ and $\lim_j \lim_i g(s_i, t_j)$ exist and are different from each other.

A subset A of a locally convex topological linear space is said to fulfill the double limit condition, if the scalar product $\langle x, x'\rangle$ satisfies the double limit condition on $A \times U^0$ for every neighbourhood of zero U.

We shall frequently use the following abbreviation: if R is a set of real numbers and α a real number, the symbol $R \leqq \alpha$ means $r \leqq \alpha$ for each $r \in R$. Similarly, we write $|R| \leqq \alpha$ for the system of inequalities $|r| \leqq \alpha$, $r \in R$.

The proof of the main theorem is based on the combinatorial lemma on convex means [8]. The reader is referred to [9] for all information and notation connected with this lemma and its application to problems concerning weak compactness.

2. THE EXTENSION THEOREM

The following lemma will be used in the proof of the main theorem.

(2,1) *Let X and Y be two completely regular topological spaces and $B(x, y)$ a separately continuous function on $X \times Y$. Suppose that B is bounded on $X \times Y$. Let us define a mapping h of X into $C_\beta(Y)$ and a mapping k of Y into $C_\beta(X)$ by the relation*

$$\langle h(x), y\rangle = \langle x, k(y)\rangle = B(x, y)\,.$$

Suppose further that B satisfies the double limit condition on $X \times Y$. Let $R \subset X \subset$ $\subset C_\beta(X)'$ and suppose that $r_0 \in C_\beta(X)'$ belongs to the closure of R. Let $\varepsilon > 0$. Then there exists a convex mean $\sum_{r\in R} \lambda(r)\, r$ such that

$$|\langle \sum \lambda(r)\, r - r_0\,,\ k(Y)\rangle| \leqq \varepsilon\,.$$

Proof. Let W be the subset of $R \times Y$ where

$$|\langle r - r_0,\ k(y)\rangle| \geqq \tfrac{1}{4}\varepsilon$$

and let M be the subset of $R \times Y$ where

$$|\langle r - r_0,\ k(y)\rangle| < \tfrac{1}{8}\varepsilon\,.$$

Let W be the system of all sets $W(y)$ with $y \in Y$. (The symbols that we use here and in the sequel are defined in [9] pp. 438 and 440.) Let β be such that $|B(x, y)| \leqq \beta$ on $X \times Y$. Suppose that $M(R, W, \varepsilon/(8\beta))$ is empty; it follows from Theorem (3.1) of [9] that there exists two sequences $r_n \in R$ and $y_n \in Y$ such that

$$r_n \in M(y_1) \cap \ldots \cap M(y_{n-1}) \cap W(y_n) \cap W(y_{n+1}) \cap \ldots$$

so that the double limit condition is violated on $R \times Y$. There exists, accordingly, a $\lambda \in M(R, W, \varepsilon/(8\beta))$. We have, for $y \in Y$,

$$|\langle \sum_{r\in R} \lambda(r)\, r - r_0,\ k(y)\rangle| \leqq \sum_{r\in R} \lambda(r)\, |\langle r - r_0,\ k(y)\rangle| =$$

$$= \sum_{r\in W(y)} + \sum_{r\in R - W(y)} \leqq \frac{\varepsilon}{8\beta}\, 2\beta + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}$$

and the proof is complete.

(2,2) The Extension Theorem. *Let S, T be two completely regular topological spaces and let $B(s, t)$ be a separately continuous function on $S \times T$. Suppose that B is bounded and that it satisfies the double limit condition on $S \times T$. Then B may be extended to a separately continuous bilinear form on $C_\beta(S)' \times C_\beta(T)'$.*

Proof. I. We define first a mapping h of S into $C_\beta(T)$ and a mapping k of T into $C_\beta(S)$ by the relation

$$(1) \qquad \langle h(s), t\rangle = \langle s, k(t)\rangle = B(s, t) .$$

If $p \in C_\beta(S)'$, define a function $k'(p)$ on T by the relation

$$(2) \qquad \langle k'(p), t\rangle = \langle p, k(t)\rangle .$$

Let us show that $k'(p)$ is continuous on T. Indeed, suppose that $M \subset T$ and $t_0 \in T$ belongs to the closure of M and that $|\langle k'(p), m - t_0\rangle| \geqq \varepsilon$ for all $m \in M$ and some $\varepsilon > 0$. Divide the set M into two parts $M^{(+)}$ and $M^{(-)}$ according to the sign of $\langle k'(p), m - t_0\rangle$. Since t_0 has to belong to the closure of one of them, we may clearly assume that t_0 is in the closure of $M^{(+)}$. According to (2,1) there exists a convex mean $\sum\limits_{m \in M(+)} \lambda(m)\, m$ such that

$$|\langle h(S), \textstyle\sum \lambda(m)\, m - t_0\rangle| \leqq \varepsilon/(2|p|)$$

whence $|\langle S, \sum \lambda(m)\, k(m) - k(t_0)\rangle| \leqq \varepsilon/(2|p|)$. It follows that $|\langle p, \sum \lambda(m)\, k(m) - - k(t_0)\rangle| \leqq \frac{1}{2}\varepsilon$. This is a contradiction, since $\langle k'(p), m - t_0\rangle \geqq \varepsilon$ for each $m \in M^{(+)}$, whence

$$\begin{aligned}\langle p, \textstyle\sum \lambda(m)\, k(m) - k(t_0)\rangle &= \langle k'(p), \textstyle\sum \lambda(m)\,(m - t_0)\rangle = \\ &= \textstyle\sum \lambda(m)\, \langle k'(p), m - t_0\rangle \geqq \varepsilon .\end{aligned}$$

It follows that k' is a mapping of $C_\beta(S)'$ into $C_\beta(T)$. By (2) and (1), we have

$$\langle k'(s), t\rangle = \langle s, k(t)\rangle = \langle h(s), t\rangle$$

so that k' is an extension of h.

II. In the same manner we obtain a mapping h' of $C_\beta(T)'$ into $C_\beta(S)$ defined by

$$(3) \qquad \langle s, h'(q)\rangle = \langle h(s), q\rangle .$$

III. Now let $p \in C_\beta(S)'$, $q \in C_\beta(T)'$. Since $h'(q) \in C_\beta(S)$, the expression $\langle p, h'(q)\rangle$ has a meaning; similarly, $\langle k'(p), q\rangle$ also may be defined. If we show that

$$(4) \qquad \langle k'(p), q\rangle = \langle p, h'(q)\rangle$$

it will be sufficient to put $B^*(p, q) = \langle k'(p), q\rangle$ to have the desired extension. Indeed, $B^*(s, t) = \langle k'(s), t\rangle = \langle s, k(t)\rangle = B(s, t)$ by (2) and (1). If p is fixed, we have $k'(p) \in C_\beta(T)$ so that $k'(p)$ is continuous on $C_\beta(T)'$. If q is fixed, we have $h'(q) \in C_\beta(S)$ so that $h'(q)$ is continuous on $C_\beta(S)'$.

IV. To prove (4), suppose that $|p| \leqq 1$, $|q| \leqq 1$ and let $\varepsilon > 0$ be given. Let V be the set of all linear combinations $\sum \omega_i s_i$ with $s_i \in S$ and $\sum |\omega_i| \leqq 1$ so that V is dense in the unit ball of $C_\beta(S)'$. Let R be the set of those $v \in V$ for which

$$|\langle v - p, h'(q) \rangle| \leqq \varepsilon \tag{5}$$

so that p belongs to the closure of R. Let us show now that it is sufficient to find a $v \in R$ such that

$$|\langle v - p, k(T) \rangle| \leqq \varepsilon . \tag{6}$$

Indeed, we have by (2) and (6)

$$|\langle k'(p), t \rangle - \langle k'(v), t \rangle| = |\langle p - v, k(t) \rangle| \leqq \varepsilon$$

for all $t \in T$ whence

$$|\langle k'(p), q \rangle - \langle k'(v), q \rangle| \leqq \varepsilon . \tag{7}$$

Since $v \in V$ and k' is an extension of h, we have further

$$\langle k'(v), q \rangle = \langle h(v), q \rangle = \langle v, h'(q) \rangle$$

which, together with (7), yields

$$|\langle k'(p), q \rangle - \langle v, h'(q) \rangle| \leqq \varepsilon . \tag{8}$$

On the other hand, $v \in R$ so that, by (5),

$$|\langle v, h'(q) \rangle - \langle p, h'(q) \rangle| \leqq \varepsilon$$

and this, combined with (8) gives

$$|\langle k'(p), q \rangle - \langle p, h'(q) \rangle| \leqq 2\varepsilon .$$

V. The proof will be complete if we show that there exists a $v \in R$ such that

$$|\langle v - p, k(T) \rangle| \leqq \varepsilon .$$

Since p belongs to the closure of R, it follows from (2,1) that there exists a convex mean $\sum_{r \in R} \lambda(r)\, r$ with $|\langle \sum \lambda(r)\, r - p, k(T) \rangle| \leqq \varepsilon$ or there exist two sequences r_i, t_j with

$$r_n \in M(t_1) \cap \ldots \cap M(t_{n-1}) \cap W(t_n) \cap W(t_{n+1}) \cap \ldots \tag{9}$$

where M and W are the subsets of $R \times T$ where $|\langle r - p, k(t) \rangle|$ is respectively $< \frac{1}{2}\varepsilon$ and $\geqq \varepsilon$.

If we show that (9) is impossible it will be sufficient to take $v = \sum \lambda(r)\, r$.

Now let t_j^* be a subsequence of t_j and $t_0 \in C_\beta(T)'$ an accumulation point of the sequence t_j^* such that

$$\lim_j \langle h(r_i), t_j^* \rangle = \langle h(r_i), t_0 \rangle \quad \text{for each} \quad i \tag{10}$$

and

$$|\langle k'(p), t_j^* - t_0 \rangle| \leqq \tfrac{1}{8}\varepsilon \quad \text{for each} \quad j. \tag{11}$$

By (2,1) there exists a convex mean $\sum \lambda_j t_j^*$ such that

$$|\langle h(S), \textstyle\sum \lambda_j t_j^* - t_0 \rangle| \leqq \tfrac{1}{8}\varepsilon. \tag{12}$$

Let i be given. It follows from (9) that, for large j,

$$|\langle r_i - p, k(t_j) \rangle| \geqq \varepsilon$$

or, which is the same,

$$|\langle h(r_i), t_j \rangle - \langle k'(p), t_j \rangle| \geqq \varepsilon;$$

this, together with (10) and (11), yields

$$|\langle h(r_i), t_0 \rangle - \langle k'(p), t_0 \rangle| \geqq \tfrac{7}{8}\varepsilon. \tag{13}$$

Now let i be greater than any of the indices of the t_s which occur in the expression $\sum \lambda_j t_j^*$. It follows from (9) that, for $i > s$,

$$|\langle r_i - p, k(t_s) \rangle| < \tfrac{1}{2}\varepsilon$$

or

$$|\langle h(r_i), t_s \rangle - \langle k'(p), t_s \rangle| \leqq \tfrac{1}{2}\varepsilon;$$

together with (11), we have

$$|\langle h(r_i), t_s \rangle - \langle k'(p), t_0 \rangle| \leqq \tfrac{5}{8}\varepsilon$$

so that it follows from (12)

$$|\langle h(r_i), t_0 \rangle - \langle k'(p), t_0 \rangle| \leqq \tfrac{6}{8}\varepsilon$$

which is a contradiction with (13). The proof is complete.

As we have seen, the main point of the proof is a sort of Fubini theorem. What we have actually proved is, roughly speaking, the following: if $f(s, t)$ is a bounded separately continuous function on $S \times T$ for which the order of the two countable limits $\lim_i \lim_j f(s_i, t_j)$ is invertible then the order of two integrations may be inverted as well. In other words, if μ is a Radon measure on βS and ν a Radon measure on βT then

(1) the function $p(s) = \int f(s, t) \, d\nu$ is continuous on S

(2) the function $q(t) = \int f(s, t) \, d\mu$ is continuous on T

$$\int p(s) \, d\mu = \int q(t) \, d\nu. \tag{3}$$

3. THE THEOREM OF KREIN

To show how the Extension Theorem may be applied let us give a proof of Krein's theorem based on it. We shall see that the extension theorem yields an immediate proof of a result which includes both the Eberlein theorem and the Krein theorem. We only formulate it for Banach spaces, the extension to complete convex spaces being obvious.

Since the proof of the main theorem involves some unpleasant technicalities, we prove first a slightly weaker result in which the application of the extension theorem is immediate.

(3,1) *Let E be a Banach space, S a bounded subset of E which satisfies the double limit condition. Then S^{00} is weakly compact.*

Proof. Take for T the unit ball U^0 of E' in the weak star topology and consider the scalar product $\langle s, t\rangle$ on $S \times T$. By the extension theorem, the scalar product may be extended to a separately continuous bilinear form $B(p, q)$ on $C_\beta(S)' \times C_\beta(T)'$. Take an $s^\wedge \in C_\beta(S)'$. It is easy to see that $B(s^\wedge, t)$ is a continuous linear form on T and may, accordingly, be identified with an element $P(s^\wedge)$ of E. We have thus obtained a mapping P of $C_\beta(S)'$ into E. Since $B(s^\wedge, t)$ is continuous as a function of $s^\wedge$, the mapping P is clearly weakly continuous. Let V be the unit ball of $C_\beta(S)'$. The mapping P being weakly continuous, $P(V)$ is a weakly compact absolutely convex set which contains S. The proof is complete.

(3,2) *Let E be a Banach space, S a bounded subset of E. Suppose that the double limit condition is satisfied on $S \times T$, where T is some norm-generating subset of E'. Then S^{00} is weakly compact.*

Proof. Denote by ω the natural imbedding of E in $C_\beta(T)$. Since T is norm-generating, ω is an open injection so that ω' maps $C_\beta(T)'$ onto E'. Define now a separately continuous function B on $S \times T$ by the relation $B(s, t) = \langle s, t\rangle$ and extend it as a separately continuous bilinear form to $C_\beta(S)' \times C_\beta(T)'$. If $s^\wedge \in C_\beta(S)'$, denote by $P(s^\wedge)$ the linear form on E' defined by the relation

$$\langle P(s^\wedge), y'\rangle = B(s^\wedge, \omega'^{-1}(y'))\,.$$

Of course, it remains to show that the value $B(s^\wedge, t^\wedge)$ does not depend on the choice of $t^\wedge$ in $\omega'^{-1}(y')$. This can be easily seen as follows. First, we have $B(s, t^\wedge) = \langle s, \omega'(t^\wedge)\rangle$ for $s \in S$ and $t^\wedge \in C_\beta(T)'$ by continuity. Suppose now that $\omega'(t_1^\wedge) = \omega'(t_2^\wedge)$. It follows that $B(s, t_1^\wedge) = B(s, t_2^\wedge)$ for each $s \in S$ whence $B(s^\wedge, t_1^\wedge) = B(s^\wedge, t_2^\wedge)$ as well. To see that the linear form $P(s^\wedge)$ is continuous on E', it is sufficient to observe that ω' is open in the weak topologies. This is a consequence of the fact that $\omega(E)$ is closed in $C_\beta(T)$. The proof is complete.

It is not difficult to see that the assumptions of the preceding theorems are satisfied if S is, e.g., weakly pseudocompact. We shall see that in section five, lemma (5,1).

4. DOUBLE SEQUENCES

Some of the consequences of the Extension Theorem may be conveniently formulated in terms of properties of double sequences.

(4,1) Definition. *Let a_{pq} be a double sequence and suppose that $\lim_q a_{pq} = a_{p0}$ exists for each p. We shall say that the convergence is almost uniform with respect to p if, for each $\varepsilon > 0$ and each infinite set R of indices q, there exists a finite $K \subset R$ such that*

$$\min_{k\in K} |a_{pk} - a_{p0}| < \varepsilon \quad \text{for each} \quad p .$$

(4,2) Definition. *Let a_{pq} be a double sequence and suppose that $\lim a_{pq} = a_{p0}$ exists for each p.*

We shall say that the convergence is uniform in the mean with respect to p if, for each $\varepsilon > 0$ and each infinite set R of indices q, there exists a finite $K \subset R$ and nonnegative λ_k such that $\sum_{k\in K} \lambda_k = 1$ and

$$\Big|\sum_{k\in K} \lambda_k(a_{pk} - a_{p0})\Big| < \varepsilon \text{ for each } p.$$

(4,3) Definition. *A double sequence a_{pq} of real numbers is said to be convergent if $\lim_q a_{pq} = a_{p0}$ exists for each p and $\lim_p a_{pq} = a_{0q}$ exists for each q.*

Let us now recall the following result

(4,4) *Let a_{pq} be a bounded convergent double sequence. Then the following statements are equivalent:*

1° *the convergence $\lim_q a_{pq} = a_{p0}$ is almost uniform with respect to p*

2° *the convergence $\lim_p a_{pq} = a_{0q}$ is almost uniform with respect to q*

3° *both limits $\lim_p a_{p0}$ and $\lim_q a_{0q}$ exist and are equal to each other,*

4° *the convergence $\lim_q a_{pq} = a_{p0}$ is uniform in the mean with respect to p,*

5° *the convergence $\lim_q a_{pq} = a_{0q}$ is uniform in the mean with respect to q.*

The proof of the equivalence of the first three statements is quite elementary; the rest is less so but may easily be obtained from the combinatorial lemma on convex means. This can be found in [9], theorem (4,3). With view to the equivalence (4,4) it will be convenient to introduce the following abbreviation: Let a_{pq} be a bounded convergent double sequence. The convergence is said to be almost uniform if the sequence satisfies one of the conditions of (4,4).

We use this opportunity to state the corresponding result for uniform convergence. It will not be used until section six.

(4,5) *Let* a_{pq} *be a bounded convergent double sequence. The following are equivalent*:

1° *the convergence* $\lim_q a_{pq} = a_{p0}$ *is uniform with respect to* p,

2° *the convergence* $\lim_p a_{pq} = a_{0q}$ *is uniform with respect to* q.

In both cases both $\lim_p a_{p0}$ *and* $\lim_q a_{0q}$ *exist and are equal to each other.*

The proof is elementary. If one of the above conditions is satisfied, we shall say that the convergence is uniform.

5. THE INDUCTIVE TOPOLOGY

In this section we intend to collect some immediate corollaries of the theorem. Let us begin with a simple lemma.

(5,1) *Let* S *and* T *be two completely regular topological spaces and* f *a separately continuous function on* $S \times T$. *Let* S *be pseudocompact and* T *countably compact. Then* f *satisfies the double limit condition on* $S \times T$.

Proof. Let $s_i \in S$ and $t_j \in T$ be two sequences such that

$$\alpha_i = \lim_j f(s_i, t_j)\,,$$

$$\beta_j = \lim_i f(s_i, t_j)$$

exist for each $i = 1, 2, \ldots$ and $j = 1, 2, \ldots$. Suppose further that $\alpha = \lim \alpha_i$ and $\beta = \lim \beta_j$ both exist. Let $t_0 \in T$ be an accumulation point of the sequence t_j so that $f(s_i, t_0) = \alpha_i$ for $i = 1, 2, \ldots$. The space S being pseudocompact, there exists a point $s_0 \in S$ such that

$$\lim_i f(s_i, t_j) = f(s_0, t_j)$$

for $j = 0, 1, \ldots$. It follows that

$$f(s_0, t_j) = \beta_j \quad \text{for} \quad j = 1, 2, \ldots$$

and

$$f(s_0, t_0) = \lim_i f(s_i, t_0) = \lim \alpha_i = \alpha\,.$$

On the other hand, t_0 being an accumulation point of the sequence t_j, we have

$$f(s_0, t_0) = \lim_j f(s_0, t_j) = \lim_j \beta_j = \beta\,.$$

It follows that $\alpha = \beta$ and the proof is complete.

(5,2) Theorem. *Let S and T be two completely regular topological spaces and f a bounded separately continuous function on $S \times T$. Define a mapping h of S into $C_\beta(T)$ and a mapping k of T into $C_\beta(S)$ by the relation*

$$\langle h(s), t\rangle = \langle s, k(t)\rangle = f(s, t)\,.$$

Now let $p \in C_\beta(S)'$; since $k(t) \in C_\beta(S)$, we may form the scalar product $\langle p, k(t)\rangle$. We define a function $k'(p)$ on T by the relation

$$\langle k'(p), t\rangle = \langle p, k(t)\rangle$$

so that k' is an extension of h. Similarly an extension h' of k is defined for $q \in C_\beta(T)'$ by the relation

$$\langle s, h'(q)\rangle = \langle h(s), q\rangle\,.$$

Then the following statements are equivalent:

1° *f has a separately continuous extension to $C_\beta(S)' \times C_\beta(T)'$,*

2° *f has a separately continuous extension to $\beta S \times \beta T$,*

3° *h' is a weakly continuous mapping of $C_\beta(T)'$ into $C_\beta(S)$,*

4° *h' is a weakly continuous mapping of βT into $C_\beta(S)$,*

5° *h' is a continuous mapping of $C_\beta(T)'$ into $C_\beta(S)$ equipped with the point topology corresponding to βS,*

6° *h' is a continuous mapping of βT into $C_\beta(S)$ equipped with the point topology corresponding to βS,*

7° *$k(T)$ is weakly relatively compact in $C_\beta(S)$,*

8° *$k(T)$ is relatively compact in $C_\beta(S)$ equipped with the point topology corresponding to βS.*

9° *f satisfies the double limit condition on $S \times T$,*

10° *if s_i and t_j are two sequences such that $f(s_i, t_j)$ is a convergent double sequence then the convergence is almost uniform,*

11° *further six conditions obtained from conditions 3°–8° by interchanging S and T.*

If S is pseudocompact, then the above conditions are also equivalent with each of the following ones.

12° *h' is a mapping of $C_\beta(T)'$ into $C_\beta(S)$,*

13° *h' is a mapping of βT into $C_\beta(S)$,*

14° *h is weakly continuous as a mapping from S into $C_\beta(T)$,*

15° *h is continuous as a mapping from S into $C_\beta(T)$ equipped with the point topology corresponding to βT,* 16° $k(T)$ *is relatively compact in* $C_\beta(S)$ *equipped with the point topology corresponding to* S.

Remark. In particular, each of the conditions above is equivalent to the corresponding condition with S and T interchanged; thus, e.g., $k(T)$ is weakly relatively compact in $C_\beta(S)$ if and only if $h(S)$ is weakly relatively compact in $C_\beta(T)$. Some of these equivalences are not without interest.

Proof. We observe first that conditions 9° and 10° are equivalent by (4,4). Suppose now we have already established the equivalence of the first ten conditions. Then they must all stay equivalent if S and T are interchanged since condition 1° is invariant with respect to this change. Further we obtain at once the following three purely formal chains of implications

$$1° \to 3° \to 5° \to 6° \to 8°$$

$$1° \to 3° \to 4° \to 7° \to 8°$$

$$1° \to 2° \to 6° \to 8°$$

Now it follows from the Extension Theorem that 9° implies 1°. The proof will thus be complete if we show that 8° implies 9°. Let us denote by N the closure in $C_\beta(S)$ of $k(T)$ taken in the point topology corresponding to βS so that N is compact. If $s \in \beta S$ and $n \in N$, put $g(s, n) = n(s)$ so that g is separately continuous on $\beta S \times N$. It follows from lemma (5,1) that g satisfies the double limit condition on $\beta S \times N$. If $s \in S$ and $t \in T$, we have $f(s, t) = g(s, k(t))$ and it follows that f satisfies the double limit condition on $S \times T$. The proof of the first part is complete.

To establish the second part of the theorem, we observe first that conditions 13° and 15° are equivalent since they both express the fact that $\langle s, h'(t^*)\rangle = \langle h(s), t^*\rangle$ is continuous on S for each $t^* \in \beta T$. Further, the implications 12° $\to$ 13° and 14° $\to$ 15° are immediate. Let us show that 13° implies 16°. Assume 13° and consider the set $h'(\beta T)$. We have clearly $k(T) \subset h'(\beta T)$. It is sufficient to show that $h'(\beta T)$ is compact in the point topology corresponding to S; this, however, follows from the fact that h' is always continuous in the point topology corresponding to S. We have thus the chains of implications 12° $\to$ 13° $\to$ 16° and 14° $\to$ 15° $\to$ 13° $\to$ 16°. The proof will be complete if we show that, for S pseudocompact, 16° implies 9°. Indeed, using the equivalence established in the first part of the theorem, we see immediately that 9° implies both 12° and 14°. Assume S to be pseudocompact and suppose that 16° is satisfied.

Denote by M the closure in $C_\beta(S)$ of $k(T)$ taken in the point topology corresponding to S so that M is compact. If $s \in S$ and $m \in M$, put $h(s, m) = m(s)$ so that h is separately continuous on $S \times M$. It follows from lemma (5,1) that h satisfies the double limit condition on $S \times M$. If $s \in S$ and $t \in T$, we have $f(s, t) = h(s, k(t))$ so that f satisfies the double limit condition on $S \times T$. The proof is complete.

In this section we propose to treat the analogous questions for a function f which is continuous on $S \times T$, not only separately continuous. Roughly speaking, we obtain similar results with uniform topology for the mappings h and k instead of the weak topology in the separately continuous case. It is to be added that the results in the continuous case are much less deep but it is our opinion that they should be included: It is then possible to compare the weak and uniform topologies.

The following simple lemma is the counterpart of (5,1).

(6,1) *Let S and T be two completely regular topological spaces and f a continuous function on $S \times T$. Let S be pseudocompact and T compact. Then each convergent double sequence $f(s_i, t_j)$ converges uniformly.*

Proof. Let $\varepsilon > 0$ and let s be fixed. For each $t \in T$ there exists a neighbourhood $U(s, t) \times V(s, t)$ of $[s, t]$ on which the oscillation of f is less than ε. Since T is compact, there exists a finite subset $K(s)$ of T such that $\bigcup_{t \in K(s)} V(s, t) = T$. Put $U(s) = \bigcap_{t \in K(s)} U(s, t)$. It follows that $k(T)$ is uniformly continuous on S with respect to T. Indeed, if $s \in U(s_0)$ and $t \in T$ arbitrary, we have $|f(s, t) - f(s_0, t)| < \varepsilon$. In other words, the mapping h is continuous from S into $C_\beta(T)$. Since S is pseudocompact, $h(S)$ is compact.

Further, consider a convergent double sequence $f(s_i, t_j)$ and write $\beta_j = \lim_i f(s_i, t_j)$. Suppose there exists a positive σ and an infinite set of natural numbers R such that for each $r \in R$ there exists a $j(r)$ with

$$|f(s_r, t_{j(r)}) - \beta_{j(r)}| \geqq \sigma .$$

The set $h(S)$ being relatively compact in $C_\beta(T)$, there exists an accumulation point $z \in C_\beta(T)$ of the sequence $h(s_r)$, $r \in R$. It follows that, for each j, the number $z(t_j)$ is an accumulation point of $f(s_r, t_j)$ whence $z(t_j) = \beta_j$. Since z is an accumulation point of $h(s_r)$, there exists an $r \in R$ such that

$$|h(s_r) - z| < \tfrac{1}{2}\sigma .$$

It follows that

$$|h(s_r)(t_{j(r)}) - z(t_{j(r)})| < \tfrac{1}{2}\sigma$$

or, which is the same,

$$|f(s_r, t_{j(r)}) - \beta_{j(r)}| < \tfrac{1}{2}\sigma$$

and this is a contradiction.

(6,2) *Let S and T be two completely regular topological spaces and f a bounded continuous function on $S \times T$. Define the mappings h, k, h', k' in the same manner as in (5,2). Then the following statements are equivalent:*

1° *f has a continuous extension $f^\wedge$ to $\beta S \times \beta T$,*

2° *for each $\varepsilon > 0$ there exists a finite open covering $U_1, \ldots, U_p$ of S such that if s' and s'' belong to the same U_i, then $|f(s', t) - f(s'', t)| < \varepsilon$ for any $t \in T$,*

3° *the mapping h of S into $C_\beta(T)$ is continuous and the space S is praecompact in the pseudometric*

$$\varrho_S(s', s'') = |h(s') - h(s'')| \,,$$

4° *$S/k(T)$ and $T/h(S)$ are metrizable in such a manner that they become praecompact and f is uniformly continuous on their product,*

5° *$h(S)$ is relatively compact in $C_\beta(T)$,*

6° *let $s_i \in S$ and $t_j \in T$ be such that the limit $\lim_i f(s_i, t_j) = \beta_j$ exists for each j; then the convergence is uniform with respect to j.*

7° *Let $s_i \in S$ and $t_j \in T$ be two sequences such that the double sequence $f(s_i, t_j)$ is convergent; then the convergence is uniform.*

8° *h' is a continuous mapping from bounded subsets of $C_\beta(T)'$ into $C_\beta(S)$.*

9° *h' is a continuous mapping of βT into $C_\beta(S)$.*

10° *Any two points of $\beta(S \times T)$ which may be separated by f may already be separated either by a function $f(s, t_0)$ or a function $f(s_0, t)$. More precisely: If $t_0 \in T$ we denote by $p(t_0)$ the extension to $\beta(S \times T)$ of the function $p(s, t) = f(s, t_0)$. Let P be the family consisting of all $p(t_0)$, $t_0 \in T$. Similarly, let Q be the family of all $q(s_0)$, $s_0 \in S$ where $q(s_0)$ is the extension to $\beta(S \times T)$ of the function $q(s, t) = f(s_0, t)$. Any two points of $\beta(S \times T)$ which may be separated by f may already be separated by a function from P or Q.*

11° *for each $\varepsilon > 0$ there exist functions $g_1(s), \ldots, g_m(s)$ continuous on S and functions $h_1(t), \ldots, h_m(t)$ continuous on T such that $|f(s, t) - \sum g_i(s)\, h_i(t)| < \varepsilon$ for all $s \in S$ and $t \in T$,*

12° *further six conditions obtained from 2°, 3°, 5°, 6°, 8°, 9° by interchanging S and T.*

Suppose that S is pseudocompact. Then the preceding conditions are also equivalent with the following.

13° *f is uniformly continuous on S with respect to T (for each $\varepsilon > 0$ and each $s_0 \in S$ there exists a neighbourhood U of s_0 such that $|f(s, t) - f(s_0, t)| < \varepsilon$ whenever $s \in U$ and $t \in T$),*

14° *the mapping h is continuous from S into $C_\beta(T)$,*

15° *the set $k(T)$ is equicontinuous on S.*

Proof. 1° → 2°. Let $f^\wedge$ be the continuous extension of f to $\beta S \times \beta T$. For each $x \in \beta S \times \beta T$ there exists an open neighbourhood $P(x) \times Q(x)$ of x such that the oscillation of $f^\wedge$ on $P(x) \times Q(x)$ is less than ε. There is a finite subcollection $P_i \times Q_i$, $i \in F$ which covers $\beta S \times \beta T$. For each $s \in \beta S$ let $F(s)$ be the set of those indices $i \in F$ for which $s \in P_i$. The classes of the equivalence sEs' iff $F(s) = F(s')$ form a finite open covering of S which we shall denote by $U_1, \ldots, U_p$. Suppose now that both s and s'

belong to U_i and that $t \in T$ is given. It follows that $F(s) = F(s')$. The point $[s, t]$ belongs to some $P_i \times Q_i$. Since $F(s) = F(s')$ it follows that $[s', t] \in P_i \times Q_i$ as well whence $|f(s, t) - f(s', t)| < \varepsilon$.

$2° \to 3°$. Immediate.

$3° \to 5°$. The set $h(S)$ is a continuous map of the praecompact pseudometric space (S, ϱ_S).

$5° \to 6°$. Suppose there exists a positive σ and an infinite set of natural numbers R such that for each $r \in R$ there exists a $j(r)$ with

$$|f(s_r, t_{j(r)}) - \beta_{j(r)}| \geqq \sigma .$$

The set $h(S)$ being relatively compact in $C_\beta(T)$, there exists an accumulation point $z \in C_\beta(T)$ of the sequence $h(s_r)$, $r \in R$. It follows that, for each j, the number $z(t_j)$ is an accumulation point of $f(s_r, t_j)$ whence $z(t_j) = \beta_j$. Since z is an accumulation point of $h(s_r)$ there exists an $r \in R$ such that $|h(s_r) - z| < \frac{1}{2}\sigma$. It follows that $|h(s_r)(t_{j(r)}) - z(t_{j(r)})| < \frac{1}{2}\sigma$ or, which is the same, $|f(s_r, t_{j(r)}) - \beta_{j(r)}| < \frac{1}{2}\sigma$ and this is a contradiction.

$6° \to 7°$. Immediate.

$7° \to 8°$. Assume 7° and let us show first that both $h(S)$ and $k(T)$ are relatively compact. With view to the symmetry of condition 7° it will suffice to prove one of these statements only, 5° say. To this end we observe first that 7° implies condition 6°°, the counterpart of 6° and complete the proof by showing that 6°° implies 5°.

$6°° \to 5°$.

Suppose that 6°° is satisfied and let us prove that $h(S)$ is relatively compact in $C_\beta(T)$. Since f is bounded, $h(S)$ is bounded as well so that it is sufficient to prove that the functions $h(s)$ are equicontinuous on βT. Let $t^* \in \beta T$ be a point where equicontinuity is violated: there exists an $\alpha > 0$ such that the intersection of all sets $U(s) = \{t \in \beta T;\ |f(s, t) - f(s, t^*)| < \alpha\}$ is not a neighbourhood of t^*. Choose $s_0 \in S$. There exists a $t_1 \in T$ and $s_1 \in S$ such that

$$|f(s_0, t_1) - f(s_0, t^*)| < 1 ,$$

$$|f(s_1, t_1) - f(s_1, t^*)| \geqq \alpha .$$

Suppose we have already defined $s_1, \ldots, s_n \in S$ and $t_1, \ldots, t_n \in T$ so that

$$|f(s_i, t_j) - f(s_i, t^*)| < \frac{1}{j}, \quad 0 \leqq i < j \leqq n ,$$

$$|f(s_j, t_j) - f(s_j, t^*)| \geqq \alpha .$$

The set

$$\left\{t \in \beta T;\ |f(s_i, t) - f(s_i, t^*)| < \frac{1}{n + 1}, \quad 0 \leqq i \leqq n\right\}$$

being a neighbourhood of t^*, there exists an $s_{n+1} \in S$ and a $t_{n+1} \in T$ such that

$$|f(s_i, t_{n+1}) - f(s_i, t^*)| < \frac{1}{n+1}, \quad 0 \leqq i \leqq n$$

$$|f(s_{n+1}, t_{n+1}) - f(s_{n+1}, t^*)| \geqq \alpha ;$$

the induction is thus complete. If i is fixed, we have $\lim\limits_j f(s_i, t_j) = f(s_i, t^*)$. This convergence is, however, not uniform with respect to i since

$$|f(s_n, t_n) - f(s_n, t^*)| \geqq \alpha .$$

It follows that both $k(T)$ and $h(S)$ are relatively compact. Since $k(T)$ is relatively compact in $C_\beta(S)$, it follows that $k(T)$ is uniformly continuous on S.

$5° \to 8°$. Let us show now that this implies that h' is a mapping into $C_\beta(S)$. Let $M \subset S$ and suppose that s_0 belongs to the closure of M. If $\varepsilon > 0$ is given, it follows from the equicontinuity of $k(T)$ that there exists an $s \in M$ such that

$$|\langle s - s_0, k(T)\rangle| \leqq \varepsilon \quad \text{or} \quad |\langle h(s) - h(s_0), T\rangle| \leqq \varepsilon .$$

It follows that

$$|\langle s - s_0, h'(q)\rangle| = |\langle h(s) - h(s_0), q\rangle| \leqq \varepsilon$$

for each $q \in C_\beta(T)'$ of norm $\leqq 1$. It follows that $h'(q)$ is continuous on S. If $s \in S$ and $q \in C_\beta(T)'$ we have

$$\langle s, h'(q)\rangle = \langle h(s), q\rangle$$

whence

$$|h'(q_1) - h'(q_2)| = \sup |\langle S, h'(q_1 - q_2)\rangle| = \sup |\langle h(S), q_1 - q_2\rangle| .$$

Since $h(S)$ is relatively compact in $C_\beta(T)$ it generates the weak topology on bounded subsets of $C_\beta(T)'$.

$8° \to 9°$. Immediate.

$9° \to 1°$. If $\sigma \in \beta S$ and $\tau \in \beta T$, put $f^\wedge(\sigma, \tau) = \langle \sigma, h'(\tau)\rangle$. Clearly $f^\wedge$ is an extension of f. Let us show that $f^\wedge$ is continuous. Let σ_0, τ_0 and $\varepsilon > 0$ be given. According to $9°$, there is a neighbourhood V of τ_0 such that $|h'(\tau) - h'(\tau_0)| < \varepsilon$ for $\tau \in V$. Further, $h'(\tau_0)$ being continuous, there exists a neighbourhood U of σ_0 such that $|\langle \sigma - \sigma_0, h'(\tau_0)\rangle| < \varepsilon$ for $\sigma \in U$. If $[\sigma, \tau] \in U \times V$, we have

$$|f^\wedge(\sigma, \tau) - f^\wedge(\sigma_0, \tau_0)| = |\langle \sigma, h'(\tau) - h'(\tau_0)\rangle| + |\langle \sigma - \sigma_0, h'(\tau_0)\rangle| < 2\varepsilon$$

and the continuity is established.

In this manner, the equivalence of $1°, 2°, 3°, 5°, 6°, 7°, 8°, 9°$ is established.

To complete the proof, we intend to prove $1° \to 4° \to 7° \to 3° \to 11° \to 1°$ and $1° \to 10° \to 1°$.

$1° \to 4°$. We already know that $1°$ implies $3°$. Clearly it also implies the analogous statement $3^{\circ\circ}$ with S and T interchanged. Thus $S/k(T)$ and $T/h(S)$ are praecompact in the metrics

$$\varrho_S(s', s'') = |h(s') - h(s'')| \quad \text{and} \quad \varrho_T(t', t'') = |k(t') - k(t'')| .$$

We have

$$|f(s, t) - f(s_0, t_0)| \leqq |f(s, t) - f(s, t_0)| + |f(s, t_0) - f(s_0, t_0)| \leqq$$
$$\leqq \varrho_T(t, t_0) + \varrho_S(s, s_0)$$

so that f is uniformly continuous on $(S, \varrho_S) \times (T, \varrho_T)$.

$4° \to 7°$. On the other hand, let $4°$ be fulfilled. There exists a uniformly continuous function f^0 on the product $S/k(T) \times T/h(S)$ such that $f(s, t) = f^0(\alpha(s), \beta(t))$, where α is the canonical map of S onto $S/k(T)$ and β the canonical map of T onto $T/h(S)$. Since both $S/k(T)$ and $T/h(S)$ are praecompact and f^0 uniformly continuous on their product, f^0 possesses a (uniformly) continuous extension f^{00} to its completion which clearly coincides with the product $X \times Y$ of the completionsi of $S/k(T)$ and $T/h(S)$. Completions of praecompact spaces being compact, we easly obtain the following fact: if $x_i \in X$ and $y_j \in Y$ are such that $f^{00}(x_i, y_j)$ is a convergent double sequence, then the convergence is uniform. It follows that f satisfies $7°$.

$3° \to 11°$. Let $\varepsilon > 0$ be given. There exist points $s_1, \ldots, s_n \in S$ such that the open ε-neighbourhoods U_i of s_i cover S. Put $g_i(s) = (1/\varepsilon) \max (\varepsilon - \varrho(s, s_i), 0)$ so that g_i is zero outside U_i and $g(s) = \sum g_i(s)$ is positive for each $s \in S$. If $e_i(s) = g_i(s)/g(s)$ put $r(s, t) = \sum e_i(s) f(s_i, t)$ so that $f(s, t) - r(s, t) = \sum e_i(s) (f(s, t) - f(s_i, t))$. Let $s \in S$, $t \in T$ be given. If $e_i(s) \neq 0$, we have $s \in U_i$ whence $|f(s, t) - f(s_i, t)| < \varepsilon$. It follows that $|f(s, t) - r(s, t)| \leqq \sum e_i(s)\, \varepsilon = \varepsilon$.

$11° \to 1°$. Let r_n be a sequence of functions of the form $\sum g_i(s)\, h_i(t)$ such that

$$|f(s, t) - r_n(s, t)| < \frac{1}{n} \text{ on } \quad S \times T .$$

Each of the r_n has an obvious continuous extension r_n^* to $\beta S \times \beta T$. If $p < q$, we have $|r_p^* - r_q^*| < 2/p$ in $C(\beta S \times \beta T)$ so that the sequence r_n^* converges uniformly to an $r \in C(\beta S \times \beta T)$ which clearly is the desired extension of f.

$1° \to 10°$. Let us denote by ω_S the quotient mapping of $\beta(S \times T)$ generated by the equivalence $x_1 E x_2$ iff $r(x_1) = r(x_2)$ for each continuous function r on $\beta(S \times T)$ such that its restriction to $S \times T$ does not depend on t. We intend to show first that $\omega_S(\beta(S \times T))$ is homeomorphic to βS. Let us introduce first a notation. If b is a bounded continuous function on S, let $b^\square$ be the extension to $\beta(S \times T)$ of the function b_0 defined on $S \times T$ by the formula $b_0(s, t) = b(s)$.

If $s \in S$, clearly the whole set $\{s\} \times T$ is contained in a single class of the equivalence E which we shall denote by $m(s)$. If s_1 and s_2 are two different points of S, there is a bounded continuous function g on S which separates them. If t_0 is a given point of T, clearly $g^\square$ separates $[s_1, t_0]$ and $[s_2, t_0]$ so that $m(s_1) \neq m(s_2)$. Clearly m is a homeomorphic imbedding of S in $\omega_S(\beta(S \times T))$. Now let a be a bounded continuous function on S. Since $a^\Box$ does not depend on t on $S \times T$ there exists a function a^+ on $\omega_s(\beta(S \times T))$ such that $a^\Box(x) = a^+(\omega_s(x))$ for all $x \in \beta(S \times T)$. If F is a closed subset of the real line, we have $a^+(\xi) \in F$ iff $\xi \in \omega_S(a^{\Box-1}(F))$ which is a compact set. Hence a^+ is continuous on $\omega_S(\beta(S \times T))$. The space $\omega_S(\beta(S \times T))$ is compact and contains as a dense subspace the set $\omega_S(S \times T) = m(S)$ which is isomorphic with S. For every bounded continuous function a on S, the function $a \circ m^{-1}$ possesses a continuous extension a^+ to $\omega_S(\beta(S \times T))$ defined by

$$a^+(\omega_S(x)) = a^\Box(x) .$$

It follows that $\omega_S(\beta(S \times T))$ is homeomorphic to βS. We shall identify $\omega_S(\beta(S \times T))$ with βS. In this manner, we have established the following correspondence between functions on βS and $\beta(S \times T)$: If a is bounded continuous on S, we have $a^\Box = a^* \circ \omega_S$.

A further abbreviation: if $t_0 \in T$, let $k^*(t_0)$ be the extension to βS of $k(t_0)$.

Let us show now that $\beta(S \times T)/P$ is homeomorphic to $\beta S/k^*(T)$. This is, however, an immediate consequence of the fact that $p(t_0) = k^*(t_0) \circ \omega_S$ for each t_0. Let Q_S be the quotient mapping of βS on $\beta S/k^*(T)$. We know already that there exists a uniformly continuous function f^0 on the product of the metric spaces $S/k(T)$ and $T/h(S)$ equipped with the metrics ϱ_S and ϱ_T such that

$$f(s, t) = f^0(Q_S(s),\ Q_T(t)) .$$

Now $\beta S/k^*(T)$ is compact in the metric ϱ_S and contains as a dense subspace $S/k(T)$. Let f^{00} be the extension of f^0 to $\beta S/k^*(T) \times \beta T/h^*(S)$. It is not difficult to see that $f^{00}(Q_S\omega_S(x), Q_T\omega_T(x))$ is continuous on $\beta(S \times T)$ and extends f so that it coincides with the continuous extension f^* of f. Suppose now that $\zeta_1, \zeta_2 \in \beta(S \times T)$ and $f^*(\zeta_1) \neq f^*(\zeta_2)$. Since $f^* = f^{00} \circ (Q_S \circ \omega_S \times Q_T \circ \omega_T)$ it follows that either $Q_S \circ \omega_S$ or $Q_T \circ \omega_T$ assumes different values on ζ_1 and ζ_2. Let this be the first coordinate, say. Since $Q_S \circ \omega_S\zeta_1 \neq Q_S \circ \omega_S\zeta_2$, there exists a $t_0 \in T$ such that $k^*(t_0)\, \omega_S\zeta_1 \neq k^*(t_0)\, \omega_S\zeta_2$. Since $p(t_0) = k^*(t_0) \circ \omega_S$, we have $p(t_0)\, \zeta_1 \neq p(t_0)\, \zeta_2$.

$10° \rightarrow 1°$. Suppose now that $10°$ is satisfied. We have then the following implication: if ζ_1 and ζ_2 are points of $\beta(S \times T)$ such that $\omega_S(\zeta_1) = \omega_S(\zeta_2)$ and $\omega_T(\zeta_1) = \omega_T(\zeta_2)$ then $f^*(\zeta_1) = f^*(\zeta_2)$. It follows that there exists a function $f^\wedge$ on $\beta S \times \beta T$ such that $f^\wedge(\omega(x)) = f^*(x)$ for $x \in \beta(S \times T)$ and $\omega = \omega_S \times \omega_T$. If F is a closed subset of the reals then $f^\wedge(\omega(x)) \in F$ iff $\omega(x) \in \omega(f^{*-1}(F))$ which is a compact set. It follows that $f^\wedge$ is continuous on $\beta S \times \beta T$ and extends f so that $1°$ is satisfied.

The equivalence of the first eleven conditions is thus established. The three remaining conditions are clearly equivalent without any assumptions on S. They are evidently implied by the corresponding conditions of the first part of the theorem. If S is pseudocompact, it follows from $14°$ that $h(S)$ is compact in $C_\beta(T)$ so that $5°$ is satisfied.

To conclude, let us point out some questions arising in connection with the present remark. A systematic study of the convex extension of a given topological space seems to be indicated. Also, it would be interesting to obtain more information about the inductive topology of the cartesian product $S \times T$, i.e. the topology which yields as continuous functions exactly the system of all separately continuous functions.

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Резюме

ОБ ОДНОЙ ТЕОРЕМЕ О РАСШИРЕНИИ ЧАСТИЧНО НЕПРЕРЫВНЫХ ФУНКЦИЙ И ЕЕ ПРИМЕНЕНИИ В ФУНКЦИОНАЛЬНОМ АНАЛИЗЕ

ВЛАСТИМИЛ ПТАК (Vlastimil Pták), Прага

Главным результатом работы является следующая теорема: *Пусть f – – частично непрерывная ограниченная функция на произведении $S \times T$ двух вполне регулярных топологических пространств. Тогда следующие утверждения эквивалентны друг другу:*

(1) *если $s_i \in S$ $t_j \in T$ – две последовательности такие, что существуют пределы $\lim_i \lim_j f(s_i, t_j)$ и $\lim_j \lim_i f(s_i, t_i)$, то эти пределы равны друг другу,*

(2) *функция $f(s, t)$ обладает расширением на частично непрерывную билинейную форму на произведении $C_\beta(S)' \times C_\beta(T)'$.*

Эта теорема находит применение в изучении слабой компактности в линейных пространствах. Она содержит в себе, напр., теорему Эберлейна и теорему Крейна, а также позволяет из предположений о счетном характере получить утверждение о непрерывности в топологии, не имеющей счетного характера.