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SEMIGROUPS CERTAIN OF WHOSE SUBSEMIGROUPS HAVE IDENTITIES

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KOLIBIAROVÁ [4; 5; 6] and VOROB'EV [10; 11] have established certain characterizations of semigroups S having one of the following properties: (α) every subsemigroup of S has an [left] identity, (β) every [principal] left ideal of S has an [left] identity. We characterize the semigroups S having one of the following properties: (γ) every [cyclic, finitely generated] subsemigroup of S has an [left, right] identity, (δ) every [principal, finitely generated] left ideal has an [left, right] identity, and in addition impose the uniqueness condition on one-sided identities.

Section I is a preliminary one. In it we establish certain connections between the maximal semilattice homomorphic image Y of a semigroup S which is a union of groups and the set E of idempotents of S. Specifically, we obtain the conditions on such a semigroup, in terms of the elements of E, in order that S be, e.g., a semilattice of right groups, or that Y be a linearly ordered set. Section II deals with the problems stated in (γ) above. We begin with the weakest of such conditions and then systematically impose more stringent ones. This keeps the proofs of single theorems relatively short and makes the dependence of all the conditions involved more transparent. In section III we perform a similar analysis, this time considering the problems stated in (δ) above. In it we also obtain some characterizations of inverse semigroups in terms of identities of principal one-sided ideals.

The analysis performed cannot be applied to semigroups satisfying analogous conditions for (two-sided) ideals. For, as BRUCK ([1], Theorem 8.3, p. 48) has shown, every semigroup S can be embedded in a simple semigroup S^* with identity. In S^* every ideal trivially has an identity (actually there is only one ideal, namely S). It is obvious that the structure of S^* may be very complicated. All the semigroups of section II and some of section III (precisely those all of whose [principal, finitely generated] one-sided ideals have a one-sided identity on the same side) are unions of groups whose idempotents form a semigroup satisfying certain conditions (those of section II are periodic). The structure of such semigroups was described by FANTHAM [3] and for a special case (idempotents commute) by CLIFFORD (see [2], Theorem 4.11, p. 128). The remaining cases of semigroups in section III are all regular semigroups whose idempotents form a semigroup satisfying certain conditions. The structure of these semigroups is little known except in the case of inverse semigroups. In the whole discussion, two factors are of fundamental importance: Clifford's theorem (stated below) and the notion of the maximal semilattice decomposition of a semigroup (discussed in detail in [8]).

Our results contain all the principal results in the papers mentioned above; we obtain most of them as corollaries to stronger statements. There is one exception here, namely, our results do not agree with certain statements in [5]. In fact, Theorems 2, 3, and 4 of [5] are incorrect as stated (consider the bicyclic semigroup); as a consequence the statements (concerning A): I, II, and IV (p. 13–14) are also incorrect. Furthermore, the word "unique" is omitted in the introduction of the paper and in both the Russian and the German summaries. In the papers cited, the authors have established certain additional properties of the semigroups in question. We have refrained from giving any extra properties of these semigroups except in a few instances when these properties followed directly from the statements established.

We follow the notation and terminology of Clifford and Preston [2]. Throughout the whole paper, S denotes an arbitrary semigroup, E the set of its idempotents, Y the maximal semilattice homomorphic image of S [8], unless expressly stated otherwise. The elements of Y are denoted by lower case Greek letters, and to $\alpha \in Y$ the corresponding N-class of S (that is, the complete inverse image of α) is denoted by N_α. The term "finitely generated" is abbreviated to f.g. and "linearly [well] ordered set" by l.o.s. [w.o.s.]. The statements that may be obtained by the "left-right" duality are omitted. Square brackets are used for alternative readings and for the reference to the bibliography.

We say that a partially ordered set T is *downward well-ordered* if every non-empty subset of T has a greatest element ([2], exerc. 6, p. 129). If the maximal semilattice homomorphic image Y of S, as a partially ordered set, is linearly ordered, we say that S is a linearly ordered set of N-classes of S; these may belong to some special classes of semigroups, e.g., right groups, so we say that S is a l.o.s. of right groups. If in such a case every N-class of S consists of a single element, we say that S (itself) is linearly ordered. The meaning of phrases "S is a downward w.o.s. of right groups" etc. is clear. If e is an idempotent of S, H_e denotes the maximal subgroup of S having e as its identity; hence if S is the union of groups, $S = \bigcup_{e \in E} H_e$. Note that H_e is contained in a single N-class of S.

Throughout the whole paper, the words "finitely generated" (f.g.) may be replaced by "with at most two generators". The proofs of statements containing "f.g." are arranged so as to contain the proof of this fact. For example, in the proof of Theorem 5, we implicitly prove the following implications: b) implies c) and c) implies a), where

a) every f.g. subsemigroup of S has a left identity;

b) every subsemigroup of S having at most two generators has a left identity;

c) S is a l.o.s. of periodic right groups.

Note that a) implies b) trivially.

I

In our investigations the following result is fundamental (see [2], Theorem 4.6, p. 126).

Theorem (Clifford). *S is a semilattice of completely simple semigroups if and only if S is a union of groups.*

This theorem will be referred to as "Clifford's theorem". By 4.2 of [8], the semilattice in question is the maximal semilattice decomposition Y of S, that is, the completely simple semigroups mentioned coincide with N-classes of S; we will use this fact without explicit mention. The following lemma will be quite useful.

Lemma 1. *Let S be a union of groups. If, for every N-class N of S, $E \cap N$ is a subsemigroup of S, then E itself is a subsemigroup.*

Proof. By Clifford's theorem and the remark above, every N-class N of S is completely simple. By the hypothesis, the set of idempotents $E \cap N$ of N is a subsemigroup, and thus $E \cap N$ is a rectangular band by Lemma 1 of [7].

Now let $e, f \in E$; then $ef \in H_g, fe \in H_h$ for some $g, h \in E$. Further, $e \in N_\alpha, f \in N_\beta$ for some $\alpha, \beta \in Y$ and thus $ef, fe \in N_{\alpha\beta}$. Consequently $g, h \in E \cap N_{\alpha\beta}$ which implies $g = ghg$ since $E \cap N_{\alpha\beta}$ is a rectangular band. Since H_g and H_h are groups, $ef = efg$, $fe = feh$ and there exist $x \in H_g$, $y \in H_h$ such that $efx = g$, $fey = h$. Consequently

$$\begin{aligned} ef &= efg = ef(efx) = (efe)(efx) = efeg = e(feh)g = efe(fey)g = \\ &= efef(fey)g = efefhg = ef(efg)hg = efef(ghg) = efefg = \\ &= efef = (ef)^2. \end{aligned}$$

Definition. S is said to be a *rectangular group* if it is isomorphic to the Cartesian (direct) product of a rectangular band F and a group G, $S \cong F \times G$. In such a case, clearly E is a semigroup and $E \cong F$.

Theorem 1. *S is a semilattice of rectangular groups if and only if S is a union of groups and E is a subsemigroup of S.*

Proof. Since a rectangular group is a completely simple semigroup whose idempotents form a semigroup, necessity follows by Clifford's theorem and Lemma 1. Sufficiency follows from Clifford's theorem and Lemma 1 of [7].

Theorem 2. *S is a semilattice of right groups if and only if S is a union of groups and for every $e, f \in E$, $efe = fe$.*

Proof. Necessity. Theorem 1 implies that S is a union of groups and that E is a subsemigroup. If $e, f \in E$, then both ef and fe are contained in the same N-class N

of S. It follows from the hypothesis that N is a right group. Consequently $E \cap N$ is a right zero semigroup which implies $(ef)(fe) = fe$, that is, $efe = fe$.

Sufficiency. The hypothesis implies that E is a semigroup and thus S is a semilattice of rectangular groups by Theorem 1. It follows that for any $e, f \in E \cap N$, where N is an N-class of S, $efe = e$ and by the hypothesis, $efe = fe$. Consequently $fe = e$, that is to say, $E \cap N$ is a right zero semigroup and thus N is a right group.

Corollary 1 ([2], exerc. 8(b), p. 129). *A band T is a semilattice of right zero semigroups if and only if for all $e, f \in T$, $efe = fe$.*

Corollary 2 (cf. § 4.2 of [2]). *S is a semilattice of groups if and only if S is a union of groups and the elements of E commute. In such a case, Y and E are isomorphic semilattices.*

Remark. Under the hypothesis of Theorem 2, every right ideal of S is two-sided. In fact, every N-class of S is right simple and hence, by 4.4 of [8], every right ideal of S is a union of N-classes and is thus a two-sided ideal. Consequently every left or right ideal of S satisfying the hypothesis of Corollary 2 above is two-sided.

Theorem 3. *S is a l.o.s. of completely simple semigroups if and only if S is a union of groups and for any $e, f \in E$, either $e \in efS$ or $f \in feS$.*

Proof. Necessity. First note that every N-class is completely simple. Let $e, f \in E$; then $e \in N_\alpha$, $f \in N_\beta$. Suppose that $\alpha \leqq \beta$ (ordering of the semilattice Y); the case $\beta < \alpha$ is treated analogously. Then $efe \in N_\alpha$, and we have $e \in H_{i\lambda}$, $efe \in H_{j\mu}$ for some $\mathscr{H}$-classes $H_{i\lambda}$, $H_{j\mu}$ of N_α. Complete simplicity of N_α yields $efe = e(efe)\, e \in \in H_{i\lambda}H_{j\mu}H_{i\lambda} \subseteq H_{i\lambda}$. Letting u be the inverse of efe in $H_{i\lambda}$, which is a group, we obtain $efeu = e$. But $u \in H_{i\lambda}$ implies $eu = u$ whence $efu = e$.

Sufficiency. By Clifford's theorem it suffices to show that Y is linearly ordered. For any N-classes N_α and N_β of S, let $e \in N_\alpha$, $f \in N_\beta$ be idempotents. Then $e \in efS$ implies $\alpha \leqq \beta$ and $f \in feS$ implies $\beta \leqq \alpha$.

The following propositions follow easily from the results already proved.

Proposition 1. *S is a l.o.s. of rectangular groups if and only if S is a union of groups and for any $e, f \in E$, either $efe = e$ or $fef = f$.*

Proposition 2. *A band T is a l.o.s. of rectangular bands if and only if for any $e, f \in T$, either $efe = e$ or $fef = f$.*

Proposition 3. *S is a l.o.s. of right groups if and only if S is a union of groups and for any $e, f \in E$, either $ef = f$ or $fe = e$.*

Proposition 4. *A band T is a l.o.s. of right zero semigroups if and only if for any $e, f \in T$, either $ef = f$ or $fe = e$.*

Proposition 5. *S is a l.o.s. of groups if and only if S is a union of groups and E is linearly ordered.*

II

We now turn to the study of semigroups all of whose [cyclic, f.g.] subsemigroups have an [left, unique left] identity.

Theorem 4. *Every cyclic subsemigroup of S has an an identity if and only if S is a semilattice of periodic completely simple semigroups.*

Proof. Ncessity. Let $x \in S$; then $\langle x \rangle$ has an identity e. Hence $e = x^m$ for some natural number m, and thus $x^{m+1} = xe = x$. Theorem 7 of [9] implies that $\langle x \rangle$ is a group, and $\langle x \rangle$ consists of n elements $x, x^2, \ldots, x^{n-1}$ if n is the smallest natural number for which $x^n = e$. By Clifford's theorem, S is a semilattice of periodic completely simple semigroups.

Sufficiency. It follows that S is a union of periodic groups and hence a union of finite cyclic groups. Consequently every cyclic subsemigroup of S is contained in a finite cyclic group and is thus itself a finite cyclic group.

Theorem 5. *Every f.g. subsemigroup of S has a left identity if and only if S is a l.o.s. of periodic right groups.*

Proof. Necessity. By Theorem 4, S is a semilattice of periodic completely simple semigroups N_α, $\alpha \in Y$. If Y is not linearly ordered, then there exist $\alpha, \beta \in Y$ such that $\alpha\beta \neq \alpha$, $\alpha\beta \neq \beta$. Let $a \in N_\alpha$, $b \in N_\beta$ be arbitrary, and let $T = \langle a, b \rangle$. Let e be a left identity of T. Then $e \in N_\gamma$ for some $\gamma \in Y$; the equations $ea = a$, $eb = b$ imply $\alpha \leqq \gamma$, $\beta \leqq \gamma$, respectively. Since $T \subseteq N_\alpha \cup N_\beta \cup N_{\alpha\beta}$ and by the hypothesis $\alpha\beta < \alpha$, $\alpha\beta < \beta$, we must have either $\gamma = \alpha$ or $\gamma = \beta$ since $e \in T$. But if, e.g., $\gamma = \alpha$, then $\beta \leqq \alpha$, a contradiction since α and β were assumed to be incomparable. Consequently Y must be linearly ordered.

It remains to show that each N-class of S is a right group. It thus suffices to show that a completely simple semigroup T all of whose subsemigroups generated by at most two elements have left identities, is a right group. This follows immediately from a simple calculation in T if T is represented in the form $\mathcal{M}(G; I, \Lambda; P)$ (see [2], chapter 3).

Sufficiency. Let $T = \langle a_1, a_2, \ldots, a_n \rangle$ with $a_i \in S$, $i = 1, 2, \ldots, n$. Then $a_i \in N_{\alpha_i}$ for some $\alpha_i \in Y$, and we may suppose (and do) that $\alpha_1 \geqq \alpha_i$, $i = 1, 2, \ldots, n$. Further $a_i \in H_{e_i}$ for some $e_i \in E$, $i = 1, 2, \ldots, n$. Each N_{α_i} is a right group and hence the idempotents of N_{α_i} form a right zero semigroup. If $\alpha_i = \alpha_1$ then $e_1 e_i = e_i$. By Proposition 3, either $e_1 e_i = e_i$ or $e_i e_1 = e_1$. Hence if $\alpha_i < \alpha_1$, then $e_i e_1 \in N_{\alpha_i}$, $e_1 \in N_{\alpha_1}$, and $N_{\alpha_i} \neq N_{\alpha_1}$, which implies that $e_i e_1 \neq e_1$. Thus $e_1 e_i = e_i$, and we obtain for any $1 \leqq i \leqq n$

$$e_1 a_i = e_1(e_i a_i) = (e_1 e_i)\, a_i = e_i a_i = a_i\,.$$

Since H_{e_1} is periodic, $a_1^m = e_1$ for some m and thus $e_1 \in T$. Consequently e_1 is a left identity of T.

Remark. From the proof of sufficiency, it follows that any e_i such that $a_i \in H_{e_i}$, $\alpha_i = \alpha_1$, is a left identity of T. Conversely, these are all the left identities of T. For let e be a left identity of T. Then in particular $ee_1 = e_1$ and thus $e \in N_{\alpha_1}$. Further, $e \in T \cap N_{\alpha_1}$ implies $e = a_{\beta_1}^{k_1} a_{\beta_2}^{k_2} \dots a_{\beta_m}^{k_m}$ for some $a_{\beta_i} \in \{a_1, a_2, \dots, a_n\} \cap N_{\alpha_1}$ and some k_i, $i = 1, 2, \dots, m$, with $1 \leqq m \leqq n$. Since $ee_{\beta_m} = e_{\beta_m}$ and $a_{\beta_m} e_{\beta_m} = a_{\beta_m}$, we have $e = a_{\beta_1}^{k_1} a_{\beta_2}^{k_2} \dots a_{\beta_m}^{k_m} = e_{\beta_m}$, where $e_{\beta_m} \in N_{\alpha_1}$, which proves the assertion. Consequently the set of all different e_i with $a_i \in H_{e_i}$, $\alpha_i = \alpha_1$, constitutes the set of all left identities of T. Hence their number is at most n.

Corollary. *The following conditions on S are equivalent*:

a) *every f.g. subsemigroup of S has an identity*;
b) *every f.g. subsemigroup of S has a unique left [right] identity*;
c) *S is a l.o.s. of periodic groups.*

Proof. The equivalence of a) and c) follows from Theorem 5 and its dual. Item a) obviously implies b). Suppose that every subsemigroup of S with at most two generators has a unique left identity; the case of "right" is dual. By Theorem 5, S is a l.o.s. of periodic right groups N_α, $\alpha \in Y$. If $a \in H_e$, $b \in H_f$ where $e, f \in N_\alpha$, then the semigroup $\langle a, b \rangle$ has both e and f as its left identities by the remark following Theorem 5. By the hypothesis, $e = f$ and consequently $H_e = N_\alpha$, i.e., N_α is a group and c) holds.

Theorem 6. *Every subsemigroup of S has a left identity if and only if S is a downward w.o.s. of periodic right groups.*

Proof. Necessity. By Theorem 5, S is a l.o.s. of periodic right groups N_α, $\alpha \in Y$. Let A be a non-empty subset of Y. Let $T = \bigcup_{\alpha \in A} N_\alpha$; then T is a semigroup since Y is linearly ordered. Let e be a left identity of T. Then $e \in N_\beta$ for some $\beta \in A$. If $x \in T$, where $x \in N_\gamma$, then $ex = x$ implies $\beta \geqq \gamma$. Consequently β is a maximal element of A and Y is well-ordered downward.

Sufficiency. Let T be a subsemigroup of S, and let $A = \{\alpha \in Y \mid N_\alpha \cap T \neq \square\}$. Then A is a non-empty subset of Y and hence contains a maximal element β. If $x \in N_\beta \cap T$, then $x \in H_e$ for some $e \in N_\beta$. Since N_β is periodic, so is H_e and hence $e = x^n \in T$ for some n. Let $y \in T$; then $y \in N_\gamma$ for some $\gamma \in A$. If $\gamma = \beta$, then $ey = y$

since N_γ is a right group. Suppose that $\gamma < \beta$. We have $y \in H_f$ for some $f \in N_\gamma$. By Proposition 3, either $ef = f$ or $fe = e$. Since $fe \in N_\gamma$, $e \in N_\beta$ and $\gamma < \beta$, the second alternative is impossible. Hence $y = fy = efy = ey$ and e is a left identity of T.

Corollary. *The following conditions on S are equivalent:*

a) *every subsemigroup of S has an identity;*
b) *every subsemigroup of S has a unique left [right] identity;*
c) *S is a downward w.o.s. of periodic groups.*

III

In this section we perform an analysis similar to the one in the previous section by considering semigroups all of whose [principal, f.g.] left ideals have [one-sided, unique or not] identities. We start with some auxiliary statements.

Lemma 2. *The following conditions on an element a of S are equivalent:*

a) *a is regular;*
b) *$R(a)$ $[L(a)]$ has an idempotent generator;*
c) *$R(a)$ $[L(a)]$ has a left [right] identity.*

Proof. Items a) and b) are equivalent by Lemma 1.13, p. 27, [2]. If a) holds, then $a = axa$ for some $x \in S$, and ax $[xa]$ is a left [right] identity of $R(a)$ $[L(a)]$, and c) holds. If c) holds and e is a left [right] identity of $R(a)$ $[L(a)]$, then $e = xa = (ex)a$ $[e = ax = a(xe)]$ for some $x \in S^1$. Thus $a = a(ex)a$ $[a = a(xe)a]$ where $ex \in S$ and a) holds.

Corollary 1. *Every principal right [left] ideal of S has a left [right] identity if and only if S is regular.*

It is easy to see that if a is regular, then e is an idempotent generator of $R(a)$ $[L(a)]$ if and only if e is a left [right] identity of $R(a)$ $[L(a)]$ (and is of the form ax $[xa]$ for some $x \in S$).

Corollary 2. *The following conditions on an element a of S are equivalent:*

a) *a is regular and $a = axa = aya$ implies $ax = ay$;*
b) *$R(a)$ has a unique idempotent generator;*
c) *$R(a)$ has a unique left identity.*

Corollary 3. *The following conditions on an element a of S are equivalent:*

a) *a is regular and $a = axa = aya$ implies $xax = yay$;*
b) *$R(a)$ and $L(a)$ both have unique idempotent generators;*
c) *$R(a)$ has a unique left identity and $L(a)$ has a unique right identity;*
c) *a is regular and has a unique inverse.*

Proof. Item a) is equivalent to: a is regular and $a = axa = aya$ implies $ax = ay$, $xa = ya$. Hence the equivalence of a), b), and c) follows from Corollary 2 and its dual. Item a) implies d) directly. Suppose that d) holds and that $a = axa = aya$. By Lemma 1.14, p. 27, [2], both xax and yay are inverses of a so that $xax = yay$.

Corollary 4. *Every principal right ideal of S has a unique left identity and every principal left ideal of S has a unique right identity if and only if S is an inverse semigroup (cf. Theorem 1. 17, p. 28, [2]).*

Theorem 7. *Let S be a regular semigroup. Then E is a band which is a semilattice of left zero semigroups if and only if for all $a, x, y \in S$, $a = axa = aya$ implies $ax = ay$.*

Proof. Necessity. By the dual of Corollary 1 to Theorem 2, for any $e, f \in E$, $efe = ef$. Hence if $a = axa = aya$, then $ax, ay \in E$ and thus $(ax)(ay)(ax) = (ax)(ay)$. Consequently $ax = ay$.

Sufficiency. We first show that E is a semigroup. Let $e, f \in E$ and let a be an inverse of ef. Then

$$ef = (ef)(ae)(ef) = (ef)\,a(ef)$$

which implies $(ef)(ae) = (ef)a$ by the hypothesis. Hence

$$a = a(ef)a = a(efa) = a(efae) = (aefa)e = ae\,, \tag{1}$$

and thus $a = aefa = afa$. It follows in particular that $fa \in E$ and hence

$$fa = (fa)(fa)(fa) = (fa)(faf)(fa)$$

which, again by the hypothesis, implies $(fa)(fa) = (fa)(faf)$, that is, $fa = faf$. Consequently

$$a = afa = a(faf) = af$$

whence, using (1),

$$ef = (ef)\,a(ef) = (ef)(af)(ef) = (ef)\,a(fef) = (ef)(ae)(fef) = (efaef)(ef) = efef = (ef)^2\,,$$

that is to say, E is a semigroup. Consequently

$$ef = (ef)(ef)(ef) = (ef)\,e(ef)$$

which implies $(ef)(ef) = efe$, and thus $efe = ef$. The dual of Corollary 1 to Theorem 2 now implies that E is a semilattice of left zero semigroups.

Corollary 1. *Every principal right ideal of S has a unique left identity if and only if S is regular and E is a band which is a semilattice of left zero semigroups.*

Corollary 2. *Let S be a regular semigroup. Then E is a semilattice if and only if for all $a, x, y \in S$, $a = axa = aya$ implies $xax = yay$.*

This Corollary and Corollary 3 to Lemma 2 together imply the well-known fact that a semigroup is an inverse semigroup if and only if it is regular and its idempotents commute.

Theorem 8. *The following conditions on S are equivalent:*

a) *every principal left ideal of S has an identity;*
b) *every principal left ideal of S has a left identity;*
c) *S is a semilattice of right groups.*

Proof. Item a) trivially implies b). Suppose that b) holds. Let $a \in S$ and let e be a left identity of $L(a)$. Then $e = xa$ for some $x \in S^1$ and consequently

$$(xa)(ya) = ya \tag{1}$$

for all $y \in S^1$. If $x \notin S$, then $e = xa = a$ and thus $a = a^2$. Suppose that $x \in S$. Applying to x the argument by which formula (1) was obtained, we find $u \in S^1$ such that $(ux)(yx) = yx$ for all $y \in S^1$. From (1) we have $(xa)\,a = a$ and thus $a = xa^2$; analogously $x = ux^2$. Hence

$$a = xa^2 = (ux^2)\,a^2 = (ux)(xa^2) = uxa = u(xa)(xa) = (ux)(ax)\,a = axa\,.$$

Thus in any case, we have $a \in Sa^2 \cap aSa$, and Theorem 4.3 (D), p. 122, [2] implies that S is a union of groups. For $e \in E$ there is $v \in S^1$ such that for all $y \in S^1$, $(ve)(ye) = ye$ (cf. (1)). Hence $(ve)\,e = e$, that is, $e = ve$, which implies $eye = ye$. In particular for any $f \in E$, $efe = fe$ and thus by Theorem 2, S is a semilattice of right groups, that is to say, c) holds.

Suppose now that c) holds. By Theorem 2, S is a union of groups and for any $e, f \in E$, $efe = fe$. Let $a \in S$ with $a \in H_e$; we wish to show that e is an identity of $L(a)$. We have $a = ea = ae$ and thus a is a right identity of $L(a)$; it remains to show that for any $x \in S$, $xa = e(xa)$. Hence let $x \in S$ with $x \in H_f$. If $a \in N_\alpha$ and $x \in N_\beta$, then $ef, xe \in N_{\alpha\beta}$. Since $N_{\alpha\beta}$ is a right group and ef is idempotent, ef is a left identity of $N_{\alpha\beta}$. Consequently

$$exe = e(fx)\,e = (ef)(xe) = xe\,,$$

and thus

$$exa = e(fx)\,a = (efef)(xa) = (efe)(fx)\,a = (fe)\,xa = (fex)(ea) = f(exe)\,a = \\ = f(xe)\,a = (fx)(ea) = xa\,.$$

Remark. It follows from a) that the left identity in b) is unique and from c) that every principal right ideal has a left identity (S being regular; see Corollary 1 to Lemma 2).

Corollary. *The following conditions on S are equivalent*:

a) *every principal left ideal and every principal right ideal of S has an identity*;

b) *every principal left ideal of S has a left identity and every principal right ideal of S has a right identity*;

c) *S is a semilattice of groups.*

Theorem 9. *Every f.g. right ideal of S has a left identity if and only if S is regular and E is a band which is a l.o.s. of right zero semigroups.*

Proof. Necessity. By Corollary 1 to Lemma 2, S is regular. Let $e, f \in E$, let R be the right ideal of S generated by e and f, and let g be a left identity of R. Then $R = R(e) \cup R(f)$ and thus either $g \in R(e)$ or $g \in R(f)$, that is, either $g = ex$ or $g = fy$ for some $x, y \in S^1$. If $g = ex$, then $f = gf = exf$ which implies $ef = f$. If $g = fy$, then $e = ge = fye$ which implies $fe = e$. It follows that $efef = ff = f = ef$ if $ef = f$ and $efef = eef = ef$ if $fe = e$. Thus E is a semigroup and Proposition 4 implies that E is a l.o.s. of right zero semigroups.

Sufficiency. Let R be the right ideal generated by the elements $a_1, a_2, \ldots, a_n$ of S. Since S is regular, $a_i = a_i x_i a_i$ for some $x_1, x_2, \ldots, x_n \in S$, and thus $a_i x_i \in E$. Denote by E_{α_i} the N-class of E containing $a_i x_i$. We may suppose that $\alpha_1 \geqq \alpha_i$, $i = 2, 3, \ldots, n$. Let $e_i = a_i x_i$; then $e_1 e_i \in E_{\alpha_i}$ since $\alpha_1 \geqq \alpha_i$. If $\alpha_i = \alpha_1$, then $e_1 e_i = e_i$ since E_{α_i} is a right zero semigroup. Suppose that $\alpha_i < \alpha_1$. By Proposition 4, either $e_i e_1 = e_1$ or $e_1 e_i = e_i$. But $e_i e_1 \in E_{\alpha_i}$, $e_1 \in E_{\alpha_1}$, and $\alpha_i < \alpha_1$ imply that the first alternative is impossible. Thus in any case $e_1 e_i = e_i$, which implies

$$a_i = a_i x_i a_i = e_i a_i = e_1 e_i a_i = e_1 a_i .$$

Since also $e_1 = a_1 x_1 \in R$, e_1 is a left identity of R.

Remark. From the proof of sufficiency, it follows that if $a_i = a_i x_i a_i$, $a_i x_i \in E_{\alpha_i}$, $\alpha_i = \alpha_1$, then $a_i x_i$ is a left identity of R. Conversely, every left identity of R is of this form. For let e be a left identity of R. Since $R = \bigcup_{i=1}^{n} R(a_i)$, we have $e = a_i y$ for some $1 \leqq i \leqq n$ and some $y \in S^1$; since e is idempotent, we may suppose that $y \in S$. We thus have $a_i = ea_i = a_i y a_i$ which proves the assertion.

Corollary. *The following conditions on S are equivalent*:

a) *every f.g. right ideal of S has a left identity and every f.g. left ideal of S has a right identity*;

b) *every f.g. right [left] ideal of S has a unique left [right] identity*;

c) *S is regular and E is a linearly ordered semilattice.*

Proof. The equivalence of a) and c) follows from Theorem 9 and its dual. The equivalence of b) and c) follows easily from Theorem 9, its dual, and the remark preceding this corollary (cf. the proof of the corollary to Theorem 5).

Theorem 10. *Every f.g. left ideal of S has a left identity if and only if S is a l.o.s. of right groups.*

Proof. Necessity. By Theorem 8, S is a semilattice of right groups N_α, $\alpha \in Y$. The proof that Y is linearly ordered is analogous to the corresponding part of the proof of necessity of Theorem 5 and is omitted.

Sufficiency. Let L be the left ideal generated by the elements $a_1, a_2, \dots, a_n$ of S. With the notation of the proof of sufficiency of Theorem 5, we obtain, as there, $e_1 a_i = a_i$ for $i = 1, 2, \dots, n$. As in the very last part of the proof of Theorem 8, we conclude that for any $x \in S$, $e_1 x a_i = x a_i$. Let u be the inverse of a_1 in H_{e_1}. Then $e_1 = u a_1 \in L$ which then implies that e_1 is a left identity of L since every element of L is of the form $x a_i$ for some $x \in S^1$.

Remark. A statement concerning the left identities of L analogous to the remark following Theorem 5 is valid here.

Theorem 11. *The following conditions on S are equivalent:*

a) *every f.g. left [right] ideal of S has an identity;*

b) *every f.g. left [right] ideal of S has a unique left [right] identity;*

c) *every f.g. left ideal of S has a left identity and every f.g. right ideal of S has a right identity;*

d) *S is a l.o.s. of groups.*

Proof. In a) and b), we consider the case of left ideals; the case of right ideals is similar. The equivalence of a) and d) follows from Theorem 10 and the dual of Theorem 9, and the equivalence of c) and d) from Theorem 10 and its dual. Item a) obviously implies b). From Theorem 10 and the remark following it, we conclude that b) implies d) (cf. the proof of the corollary to Theorem 5).

Theorem 12. *Every right ideal of S has a left identity if and only if S is regular and E is a band which is a downward w.o.s. of right zero semigroups.*

Proof. Necessity. By Theorem 9, S is regular and E is a band which is a l.o.s. of right zero semigroups (which are the N-classes of E). Suppose that there is a set $\{E_{\alpha_i}\}_{i=1}^{\infty}$ of N-classes of E with the property $\alpha_i < \alpha_{i+1}$ for $i = 1, 2, \dots$. Let $A = \bigcup_{i=1}^{\infty} E_{\alpha_i}$, let B be the right ideal of S generated by A, and let e be a left identity of B. Then $B = A \cup AS$ and hence $e = gx$ for some $g \in A$ and $x \in S^1$. Thus $g \in E_{\alpha_i}$ for some i and $ge = e$. Since also $eg = g$, it follows that $e \in E_{\alpha_i}$. But then for any $f \in E_{\alpha_{i+1}}$, $ef \in E_{\alpha_i}$ since $\alpha_i < \alpha_{i+1}$ and thus $ef \neq f$ contradicting the hypothesis that e is a left identity of B. Hence E must be a downward w.o.s. of right zero semigroups.

Sufficiency. Let R be a right ideal of S. If $a \in R$, then $a = axa$ for some $x \in S$ since S is regular. Hence $ax \in E \cap R$ and thus the set C of all α in the maximal semilattice decomposition of E such that $E_\alpha \cap R \neq \square$ is not empty. By the hypothesis, C has a maximal element γ. The element γ plays the role of α_1 in the proof of sufficiency of Theorem 9; the remainder of the proof is the same as there.

Remark. A statement concerning the left identities of R analogous to the remark following Theorem 9 is valid here.

Corollary. *The following conditions on S are equivalent*:

a) *every right ideal of S has a left identity and every left ideal of S has a right identity*;

b) *every right [left] ideal of S has a unique left [right] identity*;

c) *S is regular and E is a downward well-ordered semilattice.*

Proof. The proof is a straightforward modification of the proof of the corollary to Theorem 9.

Theorem 13. *Every left ideal of S has a left identity if and only if S is a downward w.o.s. of right groups.*

Proof. The proof is an easy adaptation of the proof of Theorem 10 (cf. also the proof of Theorem 12).

Remark. It is easy to see that under the hypothesis of Theorem 13, the set $\bigcup_{f \leqq e} H_f \, (e \in E)$ is a right ideal of S and that all right ideals of S are of this form for some $e \in E$.

Theorem 14. *The following conditions on S are equivalent*:

a) *every left [right] ideal of S has an identity*;

b) *every left [right] ideal of S has a unique left [right] identity*;

c) *every left ideal of S has a left identity and every right ideal of S has a right identity*;

d) *S is a downward w.o.s. of groups.*

Proof. The proof is similar to the proof of Theorem 11 (cf. also the proof of Theorem 12).

Remark. It is easy to see that under the hypothesis d) of Theorem 14, the set $\bigcup_{f \leqq e} H_f \, (e \in E)$ is an ideal of S and that every left or right ideal of S is of this form for some $e \in E$.

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Резюме

ПОЛУГРУППЫ, ДЛЯ КОТОРЫХ НЕКОТОРЫЕ КЛАССЫ ПОДПОЛУГРУПП ИМЕЮТ ЕДИНИЦЫ

МАРИО ПЕТРИЧ, (Mario Petrich), Пеннсылвания

В работе исследуется строение полугрупп, которые имеют некоторое из следующих свойств: 1) Всякая подполугруппа (циклическая подполугруппа, подполугруппа с конечным числом порождающих элементов) имеет односраннюю единицу. 2) Всякий односторонний идеал (главный односторонний идеал, идеал, порожденный конечным числом элементов) имеет левую (правую) единицу. Исследуются тоже условия, при которых существует в упомянутых подмножествах только одна одностранняя единица.