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Jan Kučera

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SOLUTION IN LARGE OF CONTROL PROBLEM $\dot{x} = (A(1 - u) + Bu)\,x$

Jan Kučera, Praha

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Some notations used in this paper. The Euclidean n-dimensional space is denoted by E_n, its elements are written as matrices of n rows and one column. $e_i \in E_n$, $i = 1, 2, \ldots, n$, is that point which has the i-coordinate equal to one and the others equal to zero. We denote by δ_{ij} the Kronecker symbol equal to one if $i = j$ and to zero if $i \neq j$. If $x \in E_n$, then we use the norm $\|x\| = \sum |x_i|$ which induces the norm of n-by-n matrix $A = (a_{ij})$ equal to $\|A\| = \max\limits_{j} \sum\limits_{i=1}^{n} |a_{ij}|$.

The dimension of a (finite dimensional) vector space V one writes dim V. The symbol $\{x_1, x_2, \ldots, x_r\}$ denotes the linear hull of the elements $x_1, x_2, \ldots, x_r$ of a given linear space. If $\mathfrak{N}$ is a linear space, then $\mathfrak{M} \subset\subset \mathfrak{N}$ means that $\mathfrak{M}$ is a linear subspace of $\mathfrak{N}$.

The set of all elements $p \in P$, which have a property $P(p)$, is denoted by $E(p \in P; P(p))$. The symbol $O(t)$, $t \to 0$, represents a quantity, depending on t, which can be majorised by $c|t|$, where $c > 0$ is a sufficiently large constant, if t tends to zero.

If M is a given set, then $\overline{M}$ is its adherence. Empty set one writes $\emptyset$. If φ is an one-to-one mapping, then φ^{-1} is an inverse mapping. We use the sings $\cap$, $\cup$ to represent respectively the intersection and the union of sets. All measures and integrals, which occur later on, are meant in the Lebesgue sense.

The space of all n-by-n matrices one writes $\mathfrak{E}_n$. If $A, B \in \mathfrak{E}_n$, and $A_1, A_2, \ldots, A_r \in \mathfrak{E}_n$, we denote $[A, B] = BA - AB$, $[A_1, A_2, \ldots, A_r] = [A_1, [A_2, \ldots, [A_{r-1}, A_r] \ldots]]$.

We shall often meet the matrix $[A_1, A_2, \ldots, A_r]$, where $A_1 = A_2 = \ldots = A_{r-1}$. If there will be no danger to be mistaken then we shall write it briefly $[A_1, A_2, \ldots, A_r] = [A_1^{r-1} A_r]$.

The zero matrix is denoted by 0 and the unit matrix by E. A nonsingular matrix A possesses an inverse A^{-1}. The "bracket operation" $[A, B]$ possesses the following properties: $[A, B] + [B, A] = 0$, $[A_1 + A_2, B] = [A_1, B] + [A_2, B]$, $[A, B, C] + [B, C, A] + [C, A, B] = 0$, $[A, B, C, D] + [B, C, D, A] + [C, D, A, B] + [D, A, B, C] = [[A, C], [B, D]]$.

Definition 1. A connected set $S \subset E_n$ is called an r-dimensional manifold if for each $x \in S$ there exists an open non-empty set $G \subset E_r$ and an one-to-one mapping φ of G into S such that the following properties are satisfied:

1) $x \in \varphi(G)$,

2) $\varphi(G)$ is open in S,

3) the functional matrix $\partial\varphi/\partial\xi$ exists for all $\xi \in G$, is continuous and has rank r on G.

We often also say that the manifold S is given in an environ of the point x by the mapping φ.

If we have already an r-dimensional manifold $S \subset E_n$, then the set $\overline{S}$ is called r-dimensional closed manifold. And finally, the set $S \subset E_n$ that contains only one element, one calls O-dimensional closed manifold.

Let us have an r-dimensional manifold S, given by a mapping φ, then every vector

$$\left.\frac{\partial\varphi}{\partial\xi}\right|_{\xi=\varphi^{-1}(x)} \cdot \eta ,$$

where $\eta \in E_r$, $x \in S$, is called the tangent vector to the manifold S at the point x. The set of all tangent vectors to S at x is an r-dimensional vector space which we denote by $T(x)$ and call tangent space to S at x.

It is obvious that the present definition of the dimension of a manifold and of a tangent space is independent on the choise of the mapping φ.

The set $S \subset E_n$ cannot be the adherence of two manifolds S_p, S_q with different dimensions p, q. If it would be so, then $\overline{S}_p = \overline{S}_q = S$. Let us suppose $S_p \cap S_q = \emptyset \Rightarrow S - S_p \supset S_q \Rightarrow \overline{S - S_p} \supset \overline{S}_q = S$. This is impossible as S_p is open in S. So let be $x \in S_p \cap S_q$, then there exist homeomorphisms φ, ψ of $G_p \subset E_p$, $G_q \subset E_q$ onto $\varphi(G_p) \subset S$, $\psi(G_q) \subset S$, respectively. Let $\varepsilon > 0$ be so small that the open sphere $K \subset E_n$, with centre at x and radius ε, has empty intersection with $(S - S_p) \cap (S - S_q)$ and further that $(K \cap S) \subset (\varphi(G_p) \cap \psi(G_q))$. Then the open set $\varphi^{-1}(K \cap S) \subset E_p$ is homeomorphic with the open set $\psi^{-1}(K \cap S) \subset E_q$ and $p = q$.

Definition 2. $\mathfrak{A} \subset\subset \mathfrak{E}_n$. The mapping which assigns to every point $x \in E_n$ the vector space $\mathfrak{B}(x) = E(Ax;\, A \in \mathfrak{A})$ is called linear distribution created by $\mathfrak{A}$. As we will not investigate other distributions than linear, we will further omitt the adjective linear. Sometimes we write $\mathfrak{B}_{\mathfrak{A}}$ to underline that $\mathfrak{B}$ has been created by $\mathfrak{A}$. If it holds: $A, B \in \mathfrak{A} \Rightarrow [A, B] \in \mathfrak{A}$, then $\mathfrak{A}$ is called involutive. If for given distribution $\mathfrak{B}_{\mathfrak{A}}$ there exists an involutive space $\mathfrak{B} \subset\subset \mathfrak{E}_n$ such that $\mathfrak{B}_{\mathfrak{A}} = \mathfrak{B}_{\mathfrak{B}}$, then $\mathfrak{B}_{\mathfrak{A}}$ is also called involutive.

Example. One distribution can be created by different matrix-spaces. Even there are non-involutive matrix-spaces which create an involutive distribution. If we have two matrix-spaces $\mathfrak{A}_{1,2}$ which create the same distribution $\mathfrak{B}$, then the intersection $\mathfrak{A}_1 \cap \mathfrak{A}_2$ need not create $\mathfrak{B}$. To demonstrate it, let us put:

$$\mathfrak{A} = \left\{ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}; \quad \mathfrak{B} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\};$$

$$\mathfrak{C} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}; \quad \mathfrak{D} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}.$$

Then

$$\mathfrak{A} \cap \mathfrak{B} = \left\{ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}.$$

$\mathfrak{A}$, $\mathfrak{B}$ are involutive and $\mathfrak{C}$, $\mathfrak{D}$ are not. It holds $\mathfrak{B}_{\mathfrak{A}} = \mathfrak{B}_{\mathfrak{B}} = \mathfrak{B}_{\mathfrak{C}} = \mathfrak{B}_{\mathfrak{D}} \neq \mathfrak{B}_{\mathfrak{A} \cap \mathfrak{B}}$, while the spaces $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ are different each other.

Example. The set $E(k; \exists_{x \in E_n} \dim \mathfrak{B}(x) = k)$ may but need not contain all non-negative integers $k \leqq n$. Let us put

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 2 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 3 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & (n-1) \\ 0 & 0 & 0 & 0 & \ldots & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix},$$

and write $B = B_1$, $[A, B_i] = B_{i+1}$, $i = 1, 2, \ldots$, $\mathfrak{A} = \{A, B_1, B_2, \ldots, B_{n-1}\}$.

Then $\mathfrak{A}$ is involutive and create a distribution $\mathfrak{B}$ for which holds: if $(x, e_i) \neq 0$, $(x, e_j) = 0$, $j = i + 1$, $i + 2, \ldots, n$, then $\mathfrak{B}(x) = \{e_1, e_2, \ldots, e_i\}$, $i = 1, 2, \ldots, n$. Here the symbol (x, e_i) means the scalar product.

1. SOLUTION OF EQUATION $\dot{x} \in \mathfrak{B}(x)$

Let $\mathfrak{B}$ be a given distribution in E_n and $\omega \in E_n$. In this paragraph we will find a necessary and sufficient condition for the existence of a solution of an equation

$$(1.1) \qquad \dot{x} \in \mathfrak{B}(x), \quad x(0) = \omega,$$

and present the explicit form of that solution. What is to be understood as a solution of (1.1) is said in the following two definitions.

Definition 1.1. Each vector-function x, defined on an interval $\langle 0, T)$, $0 < T \leqq \infty$, local-absolutely continuous, which satisfies the conditions: 1) $x(0) = \omega$, 2) if there exists $dx(t)/dt$, then $dx(t)/dt \in \mathfrak{B}(x(t))$, is called a solution of (1.1).

Definition 1.2. Let $\mathfrak{B}$ be a distribution and $S \subset E_n$ be a manifold. If for each $x \in S$ holds $T(x) = \mathfrak{B}(x)$, where $T(x)$ is the tangent space to S at x, then S is called an integral manifold of the distribution $\mathfrak{B}$.

Lemma 1.1. *Let $\mathfrak{A} \subset\subset \mathfrak{E}_n$ be involutive. Let us have a local-absolutely integrable matrix-function $A(t)$, $t \geqq 0$, and a matrix $B \in \mathfrak{A}$. Suppose that $A(t) \in \mathfrak{A}$ for all $t \geqq 0$. Let us denote by $X(t)$ that fundamental matrix of the equation $\dot{x} = A(t)\,x$, for which $X(0) = E$.*

Then it holds $X^{-1}(t)\,BX(t) \in \mathfrak{A}$, $t \geqq 0$.

We prove it in two steps: 1) Let $A \in \mathfrak{A}$ be a constant matrix, then $e^{-A}Be^{A} \in \mathfrak{A}$.

$$e^{-A}Be^{A} = \sum_{i,j=0}^{\infty} \frac{(-1)^i}{i!\,j!} A^iBA^j = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{i=0}^{k} (-1)^i \binom{k}{i} A^iBA^{k-i} =$$

$$= \sum_{k \geqq 0} \frac{1}{k!} [\underbrace{A, A, \ldots, A}_{k}, B] \in \mathfrak{A}\,.$$

2) Let $T > 0$. If $A(t)$ is piecewise constant on $\langle 0, T\rangle$, then our statement is already proved by successive use of the first step. Let us choose a sequence of piecewise constant matrices $A_1(t), A_2(t), \ldots, t \in \langle 0, T\rangle$, which have all their values in $\mathfrak{A}$ and which tend asymptotically to $A(t)$ on $\langle 0, T\rangle$. Let X_k be the fundamental matrix of the equation $\dot{x} = A_k x$, for which $X_k(0) = E$, $k = 1, 2, \ldots$ Then $X_k(t) - X(t) = = \int_0^t A_k(\tau)(X_k(\tau) - X(\tau))\,d\tau + \int_0^t (A_k(\tau) - A(\tau))\,X(\tau)\,d\tau$, $\|X(t)\| \leqq \|E\| \exp \int_0^t \|A(t)\|\,d\tau$. As $X(t)$ is bounded on $\langle 0, T\rangle$, it holds $\lim_{k\to\infty} \int_0^t (A_k(\tau) - A(\tau))\,X(\tau)\,d\tau = 0$ uniformly on $\langle 0, T\rangle$. The asymptotic convergence of $\|A_k\|$, $k = 1, 2, \ldots$ implies $\sup_k \exp \int_0^t \|A_k\|\,d\tau = K < \infty$. If we denote $b_k = \max_{t \in \langle 0, T\rangle} \|\int_0^t (A_k(\tau) - A(\tau))\,X(\tau)\,d\tau\|$, we get the approximation $\|X_k(t) - X(t)\| \leqq b_k \,.\, \exp. \int_0^t \|A_k(\tau)\|\,d\tau \leqq K \,.\, b_k$, $t \in \langle 0, T\rangle$, where $\lim_{k\to\infty} b_k = 0$.

Now it holds $X_k^{-1}(t)\,B\,X_k(t) \in \mathfrak{A}$, $k = 1, 2, \ldots, t \in \langle 0, T\rangle$, $\lim_{k\to\infty} X_k^{-1}(t)\,B\,X_k(t) = = X^{-1}(t)\,B\,X(t) \in \mathfrak{A}$.

Lemma 1.2. *Let $\mathfrak{A} \subset\subset \mathfrak{E}_n$ be involutive and create the distribution $\mathfrak{B}$. Let $x \in E_n$, $A_i \in \mathfrak{A}$, $i = 1, 2, \ldots, k$, $y = e^{A_1t_1}e^{A_2t_2} \ldots e^{A_kt_k}x$, where t_i, $i = 1, 2, \ldots, k$, are real numbers.*

Then $\dim \mathfrak{B}(y) = \dim \mathfrak{B}(x)$.

Proof. Let $Q \in \mathfrak{A}$ and $\mathfrak{B}(x) = \{P_1x, P_2x, \ldots, P_rx\}$, where $P_i \in \mathfrak{A}$, $i = 1, 2, \ldots, r$. For brevity we denote $\Phi = e^{A_1t_1}e^{A_2t_2} \ldots e^{A_kt_k}$. It holds $Qy = Q\Phi x = \Phi(\Phi^{-1}Q\Phi)\,x$. According to lemma 1.1 $\Phi^{-1}Q\Phi \in \mathfrak{A}$ and $\Phi^{-1}Q\Phi x \in \{P_1x, P_2x, \ldots, P_rx\} \Rightarrow Qy \in \in \{\Phi P_1x, \Phi P_2x, \ldots, \Phi P_rx\}$. This implies $\dim \mathfrak{B}(y) \leqq \dim \mathfrak{B}(x)$.

If we change the signs of all numbers t_i; $i = 1, 2, \ldots, k$, we get the inverse inequality.

Lemma 1.3. *Let $\mathfrak{A} \subset\subset \mathfrak{E}_n$ be involutive and create the distribution $\mathfrak{B}$. Let $x \in E_n$,* $\dim \mathfrak{B}(x) = r$.

Then x is contained in an r-dimensional integral manifold S of the distribution $\mathfrak{B}$.

Moreover, if $\mathfrak{B}(x) = \{P_1x, P_2x, \ldots, P_rx\}$, where $P_i \in \mathfrak{A}$, $i = 1, 2, \ldots, r$, then S can be given in an environ of x by the mapping

$$(1.2) \qquad \varphi(t_1, t_2, \ldots, t_r) = e^{P_1t_1}e^{P_2t_2} \ldots e^{P_rt_r}x\,, \quad t \in G\,,$$

where $G \subset E_r$ is an open set which contains the origin.

Proof. If $\dim \mathfrak{B}(x) = 0$, then the statement is trivial, so let be $\dim \mathfrak{B}(x) = r > 0$. Obviously $x = \varphi(0)$. Let us show that there is $\varepsilon > 0$ such that the mapping φ is one-to-one on the set $G_\varepsilon = E(t \in E_r;\ \|t\| < \varepsilon)$.

If it is not so, there would exist two sequences $t^1, t^2, \ldots, \tau^1, \tau^2, \ldots$, such that $t^k \to 0$, $\tau^k \to 0$, $t^k \neq \tau^k$, $\varphi(t^k) = \varphi(\tau^k)$, $k = 1, 2, \ldots$ Let us put $t^k - \tau^k = \Delta^k$, $k = 1, 2, \ldots$, then it holds $0 = \varphi(t^k) - \varphi(\tau^k) = (P_1\Delta_1^k + P_2\Delta_2^k + \ldots + P_r\Delta_r^k)x + O(\|\Delta^k\|) \,.\, O(\|t^k\| + \|\tau^k\|)$,

$$\lim_{k\to\infty} \frac{1}{\|\Delta^k\|} \|(P_1\Delta_1^k + P_2\Delta_2^k + \ldots + P_r\Delta_r^k)\, x\| = 0\,.$$

The vectors P_ix, $i = 1, 2, \ldots, r$, are linearly independent so that it holds

$$\min_{\Delta\neq 0} \frac{1}{\|\Delta\|} \|(P_1\Delta_1 + P_2\Delta_2 + \ldots P_r\Delta_r)\, x\| > 0\,.$$

This contrary proves the univalency of φ on G_ε.

The functional matrix $\partial\varphi/\partial t$ is continuous on E_n and at $t = 0$ has the rank equal to r. So we can choose $\varepsilon > 0$ so small that the matrix $\partial\varphi/\partial t$ has its rank equal to r for all $t \in G_\varepsilon$.

It remains to show that $\varphi(G_\varepsilon)$ is an integral manifold of $\mathfrak{B}$. Obviously $P_i\,\varphi(t) \in \mathfrak{B}(\varphi(t))$, $i = 1, 2, \ldots, r$, $t \in E_r$. According to lemma 1.1

$$\frac{\partial\varphi}{\partial t_i} = e^{P_1t_1}e^{P_2t_2} \ldots e^{P_{i-1}t_{i-1}}P_ie^{P_it_i} \ldots e^{P_rt_r}x =$$

$$= e^{P_1t_1} \ldots e^{P_{i-1}t_{i-1}}P_ie^{-P_{i-1}t_{i-1}} \ldots e^{-P_1t_1}\varphi(t) \in \mathfrak{B}(\varphi(t))\,; \quad i = 1, 2, \ldots, r,\ t \in E_r\,.$$

So we have got $\{\partial\varphi/\partial t_1, \partial\varphi/\partial t_2, \ldots, \partial\varphi/\partial t_r\} \subset \mathfrak{B}(\varphi(t))$, $t \in E_r$. For $t \in G_\varepsilon$ there must hold the equality $\{\partial\varphi/\partial t_1, \partial\varphi/\partial t_2, \ldots, \partial\varphi/\partial t_r\} = \mathfrak{B}(\varphi(t))$, because we have chosen the set G_ε so that on G_ε the rank of the matrix $\partial\varphi/\partial t$ is equal to r and according to lemma 1.2 $\dim \mathfrak{B}(\varphi(t)) = r$ for $t \in E_r$. This proves the equality.

Lemma 1.4. *Let $\mathfrak{B}$ be an involutive distribution. Let the point $x \in E_n$ is contained in two integral manifolds $S_{1,2}$ of $\mathfrak{B}$.*

Then there exists an integral manifold S *of* $\mathfrak{B}$ *which is contained in the intersection* $S_1 \cap S_2$ *and contains the point* x.

Proof. If $\dim \mathfrak{B}(x) = 0$, then the statement is trivial. So we assume $\dim \mathfrak{B}(x) = r > 0$. Let $\mathfrak{B}$ be created by an involutive space $\mathfrak{A} \subset\subset \mathfrak{E}_n$; $\mathfrak{B}(x) = \{P_1 x, P_2 x, \ldots, P_r x\}$, $P_i \in \mathfrak{A}$, $i = 1, 2, \ldots, r$. Let the manifold S_3 be given by the formula (1.2). It will do to show that S_1 and S_3 have a common r-dimensional submanifold which contains the point x.

Let us choose $\Delta > 0$ so small that if $\|y - x\| < \Delta$, then the vectors $P_1 y, P_2 y, \ldots, P_r y$ are linearly independent. Let the neighbourhood of x in S_1 be given by a mapping $\psi(\tau)$, $\|\tau\| < \delta$, $\psi(0) = x$. Let $\delta > 0$ is already chosen so small that the inequality $\|\tau\| < \delta$ implies $\|\psi(\tau) - x\| < \Delta$.

If $S_{1,3}$ have not a common submanifold, then there exists a sequence $\tau_1, \tau_2, \ldots$, such that $\|\tau_k\| < \delta$, $\psi(\tau_k) \notin S_3$, $k = 1, 2, \ldots$, $\tau_k \to 0$. The sequence $\tau_1, \tau_2, \ldots$ has a partial sequence (let us denote it again by $\tau_1, \tau_2, \ldots$) such that there exists $\lim_{k\to\infty} \tau_k / \|\tau_k\| = \Omega$. Then there exists a diferentiable rectificable curve $\tau = \tau(\vartheta)$, $\vartheta \in \langle 0, \vartheta_0 \rangle$, such that $\tau(0) = 0$, $\|\tau(\vartheta)\| < \delta$ for $\vartheta \in \langle 0, \vartheta_0 \rangle$, $\mathrm{d}\tau/\mathrm{d}\vartheta|_{v=0} = \Omega$, and such that there is a sequence $\vartheta_0 > \vartheta_1 > \vartheta_2 > \ldots > 0$ which satisfies $\tau(\vartheta_k) = \tau_k$, $k = 1, 2, \ldots$

For the curve $\Gamma_1(\vartheta) = \psi(\tau(\vartheta))$, $\vartheta \in \langle 0, \vartheta_0 \rangle$, it holds:

$$\frac{\mathrm{d}\Gamma_1(\vartheta)}{\mathrm{d}\vartheta} = \frac{\partial\psi}{\mathrm{d}\vartheta} \cdot \frac{\mathrm{d}\tau}{\mathrm{d}\vartheta} = \sum_{k=1}^{r} p_k(\vartheta)\, P_k \Gamma_1(\vartheta) \qquad \vartheta \in \langle 0, \vartheta_0 \rangle . \tag{1.3}$$

As the vectors $P_k \Gamma_1(\vartheta)$, $k = 1, 2, \ldots, r$, $\vartheta \in \langle 0, \vartheta_0 \rangle$, are continuous and linearly independent and as the vector-function $\partial\psi/\partial\tau \,.\, \mathrm{d}\tau/\mathrm{d}\vartheta$ is continuous on $\langle 0, \vartheta_0 \rangle$, too, the coefficients $p_k(\vartheta)$, $k = 1, 2, \ldots, r$ are also continuous functions on $\langle 0, \vartheta_0 \rangle$.

Now let us take a curve $\Gamma_2(\vartheta) = \varphi(t(\vartheta))$, $\vartheta \in \langle 0, \vartheta_0 \rangle$, where the function $t(\vartheta)$ has a continuous derivative on $\langle 0, \vartheta_0 \rangle$ and $t(0) = 0$, otherwise let it be in the meanwhile arbitrary.

$$\frac{\mathrm{d}\Gamma_2(\vartheta)}{\mathrm{d}\vartheta} = \frac{\partial\varphi}{\partial t} \cdot \frac{\mathrm{d}t}{\mathrm{d}\vartheta} = \sum_{s=1}^{r} \left(e^{P_1 t_1} \ldots e^{P_{s-1} t_{s-1}} P_s e^{-P_{s-1} t_{s-1}} \ldots e^{-P_1 t_1}\right) \Gamma_2(\vartheta) \cdot \frac{\mathrm{d}t_s}{\mathrm{d}\vartheta} =$$

$$= \sum_{s=1}^{r} \left(\sum_{k=1}^{r} a_{ks}(t)\, P_k \right) \Gamma_2(\vartheta) \frac{\mathrm{d}t_s}{\mathrm{d}\vartheta} = \sum_{k=1}^{r} \left(\sum_{s=1}^{r} a_{ks}(t) \frac{\mathrm{d}t_s}{\mathrm{d}\vartheta} \right) P_k \Gamma_2(\vartheta).$$

The functions a_{ks}; $k, s = 1, 2, \ldots, r$, are entire functions of the argument t and it holds $a_{ks}(0) = \delta_{ks}$; $k, s = 1, 2, \ldots, r$. If we now put

$$p_k(\vartheta) = \sum_{s=1}^{r} a_{ks}(t) \cdot \frac{\mathrm{d}t_s}{\mathrm{d}\vartheta}, \quad k = 1, 2, \ldots, r\,,$$

we get a system of equations from which we can calculate the derivatives $\mathrm{d}t_s/\mathrm{d}\vartheta$ $s = 1, 2, \ldots, r$, in an environ of the origin in E_r, and we get another system which has obviously a unique solution satisfying the initial condition $t(0) = 0$. So there exists $\tilde{\vartheta} > 0$ such that on $\langle 0, \tilde{\vartheta} \rangle$ the vector-functions $\Gamma_{1,2}$ solve the linear system of equations (1.3) and satisfy the initial condition $\Gamma_1(0) = \Gamma_2(0) = x$. Hence the both functions are on $\langle 0, \tilde{\vartheta} \rangle$ identical, what is the sought contrary.

We have proved that there exists a neighbourhood $G \subset E_r$ of the origin in E_r such that $\psi(G) \subset S_3$. Let G be so small that the ranks of the matrices $\partial\psi/\partial\tau$, $\partial\varphi/\partial\tau$ are both equal to r for all $\tau \in G$. Then there exists a neighbourhood $G_0 \subset G$ of the origin in E_r such that the equation $\psi(\tau) - \varphi(t) = 0$ has the unique solution $\tau = \tau(t)$ for $t \in G_0$. And the set $\psi(\tau(G_0)) = \varphi(G_0)$ is the sought common r-dimensional submanifold of the manifolds S_1, S_3.

Lemma 1.5. *Let $\mathfrak{B}$ be a distribution in E_n and let each point $x \in E_n$ be contained in an integral manifold S_x of $\mathfrak{B}$. Then $\mathfrak{B}$ is involutive.*

Proof. Let us put $\mathfrak{A} = E(A \in \mathfrak{E}_n;\ Ax \in \mathfrak{B}(x)$ for all $x \in E_n)$, choose $x \in E_n$ and $A, B \in \mathfrak{A}$. The point x is contained in a manifold S_x. According to lemma 1.4 it holds $e^{-Bt}e^{-At}e^{Bt}e^{At}x \in S_x$ for sufficiently small real t. This implies

$$\lim_{t \to 0} t^{-2}(e^{-Bt}e^{-At}e^{Bt}e^{At} - E)\,x = [A, B]\,x \in T(x)$$

and $[A, B]\,x \in \mathfrak{B}(x)$. As it holds for each $x \in E_n$, it is $[A, B] \in \mathfrak{A}$.

Lemma 1.6. *Let $\mathfrak{A} \subset\subset \mathfrak{E}_n$ be involutive and create the distribution $\mathfrak{B}$. Let for matrices $B, C \in \mathfrak{E}_n$ be $Bx \in \mathfrak{B}(x)$, $Cx \in \mathfrak{B}(x)$ for all $x \in E_n$.*

Then $[B, C]\,x \in \mathfrak{B}(x)$ for all $x \in E_n$.

Proof follows immediately from the proof of lemma 1.5.

Supplement to definition 2. Let $\mathfrak{B}$ be an involutive distribution. Let us form the space

$$(1.4) \qquad \mathfrak{A} = E(A \in \mathfrak{E}_n;\ Ax \in \mathfrak{B}(x) \text{ for all } x \in E_n)$$

Then according to lemma 1.6 the space $\mathfrak{A}$ is involutive. So by (1.4) we can uniquely to each involutive distribution $\mathfrak{B}$ determine the involutive space $\mathfrak{A} \subset\subset \mathfrak{F}_n$ such that $\mathfrak{B} = \mathfrak{B}_{\mathfrak{A}}$.

The definition of being involutive for a given distribution might be now given as follows: A given distribution $\mathfrak{B}$ is called involutive iff the space (1.4) is involutive.

Theorem 1.1. *Let $\mathfrak{B}$ be a distribution in E_n. Let us denote $Z_r = E(x \in E_n; \dim \mathfrak{B}(x) = r)$, $r = 0, 1, \ldots, n$, and take one connected component Z of a set Z_r.*

Then each $x \in Z$ is contained in a unique r-dimensional integral manifold S_x, of the distribution $\mathfrak{B}$, which is maximal in the sense of inclusion of sets, if and only if $\mathfrak{B}$ is involutive.

Moreover, if $\mathfrak{B}$ *is created by an involutive space* $\mathfrak{A} \subset\subset \mathfrak{E}_n$, $\mathfrak{B}(x) = \{P_1x, P_2x, \ldots, \ldots, P_rx\}$, $P_i \in \mathfrak{A}$, $i = 1, 2, \ldots, r$, *then the integral manifold* S_x *is given by the mapping* (1.2).

Proof. 1) The necessity follows from lemma 1.5. 2) Let $\mathfrak{B}$ be involutive. Then according to lemma 1.3 each point $x \in Z$ is contained in an integral manifold of $\mathfrak{B}$. Let us choose a new topology in Z like in [2]. That topology consists of all sets which can be represented as unions of integral manifolds of $\mathfrak{B}$. Then the sought manifold S_x is that component of Z which contains the point x.

The supplement is an immediate consequence of lemma 1.3.

Theorem 1.2. *Let* $\mathfrak{B}$ *be an involutive distribution in* E_n *and* $\omega \in E_n$. *Then the integral manifold* S_ω *is the set of all points* $x \in E_n$ *for which there exists a solution* $x(t)$ *of the equation* (1.1) *and* $T > 0$ *such that* $x = x(T)$.

Proof. 1) From the proof of lemma 1.4 it follows that all points lying on a solution of (1.1) are contained in S_ω.

2) Let $x \in S_\omega$, then x can be linked up with ω by a finite chain of integral manifolds S_i, $i = 1, 2, \ldots, k$, given by the formula (1.2), $\omega \in S_1$, $S_i \cap S_{i+1} \neq \emptyset$, $i = 1, 2, \ldots, \ldots, k - 1$, $x \in S_k$. And obviously if we have two arbitrary points $x_{1,2} \in S_1$, then there exists a solution $x(t)$ of the equation $\dot{x} \in \mathfrak{B}(x)$, $x(0) = x_1$, and a number $T > 0$ such that $x(T) = x_2$. This completes the proof.

2. CONTROL PROBLEM

In this paragraph we will investigate the equation

$$\frac{d}{dt} x(t) = (A(1 - u(t)) + B\,u(t))\,x(t)\,, \quad x(0) = \omega\,, \quad t \geqq 0\,, \tag{2.1}$$

where $A, B \in \mathfrak{E}_n$, $\omega \in E_n$ and u is a measurable function on $\langle 0, \infty)$, values of which lie in $\langle 0, 1\rangle$ for all $t \geqq 0$.

The matrices $A, B \in \mathfrak{E}_n$ will be fixed. As the matrix $B - A$ will appear very frequently we shall consistently denote it by C.

The set of all functions measurable on $\langle 0, \infty)$, values of which lie in $\langle a, b\rangle$, $a < b$, we denote by $M(a, b)$. The function $u \in M(0, 1)$ one call the control. The solution of (2.1), which corresponds to a given control $u \in M(0, 1)$, we denote by $x(t, u)$. And at last we denote by $X(t)$ the fundamental matrix of (2.1) for which $X(0) = E$. Here we do not indicate explicitely the dependence of $X(t)$ on the control u, because it will be still clear what control will be dealt with.

Definition 2.1. The smallest linear involutive space of n-by-n matrices, which contains the matrices A, B, we denote by $\mathfrak{A}$ (or by $\mathfrak{A}(A, B)$) and the distribution created by $\mathfrak{A}$ we denote by V.

Definition 2.2. By $\mathfrak{B}$ (or by $\mathfrak{B}(A, B)$) we denote the smallest linear space, of n-by-n matrices, which contains the matrix C and with each $P \in \mathfrak{B}$ contains also both matrices $[A, P]$ and $[B, P]$.

The distribution created by $\mathfrak{B}$ we denote by $\mathscr{V}$.

Lemma 2.1. $\mathfrak{B}(A, B)$ *is involutive.*

Proof. Let us call the matrix $P \in \mathfrak{E}_n$ elementary of grade p, if there exists a sequence of matrices $P_1, P_2, \ldots, P_{p-1}$, where $P_i = A$ or $P_i = B$, $i = 1, 2, \ldots, p - 1$ such that $P = [P_1, P_2, \ldots, P_{p-1}, C]$. We do not care that an elementary matrix could have different grades.

Evidently $\mathfrak{B}$ is the linear hull of all elementary matrices. To prove the involutivity of $\mathfrak{B}$, it is sufficient to show that if there are given elementary matrices P, Q, with grades p, q, respectively, then the matrix $[P, Q]$ is a linear combination of elementary matrices of grades less or equal to $p + q$.

If $p + q = 2$, then $[P, Q] = 0$, and the statement is obvious. Let us assume that our statement holds for all elementary matrices which have the grades p, q, where $p + q < r$. Now, let matrices P, Q have the grades p, q, respectively, and $p + q = r$. If $p = 1$, then obviously $[P, Q]$ is elementary. If $p > 1$, then, without loss of generality, let be $P = [A, R]$, where R is elementary of grade $p - 1$.

$$[P, Q] = [[A, R], Q] = [A, [R, Q]] + [R, [Q, A]] .$$

The matrix $[R, Q]$ is a linear combination of elementary matrices of the grades at most $p + q - 1$. As $[A, Q]$ is elementary and R has its grade less then p we get by means of the mathematical induction with respect to p that also $[R, [Q, A]]$ is a linear combination of elementary matrices of grades less or equal to r. This completes the induction with respect to r.

Lemma 2.2. $A, B \in \mathfrak{E}_n$; $\gamma_1, \gamma_2 \in E_1$; $\gamma_1 \neq \gamma_2$. *Let* $\mathfrak{M} \subset\subset \mathfrak{E}_n$ *be the smallest linear space with the properties*:

1) $C \in \mathfrak{M}$

2)' $P \in \mathfrak{M} \Rightarrow [A + \gamma_i C, P] \in \mathfrak{M}$, $i = 1, 2$.

Then $\mathfrak{M} = \mathfrak{B}(A, B)$.

Proof. $P \in \mathfrak{M} \Rightarrow [A, P] = (1/(\gamma_2 - \gamma_1)(\gamma_2[A + \gamma_1 C, P] - \gamma_1[A + \gamma_2 C, P]) \in \mathfrak{M}$, $[B, P] = (1/(\gamma_2 - \gamma_1)((\gamma_2 - 1)[A + \gamma_1 C, P] - (\gamma_1 - 1)[A + \gamma_2 C, P]) \in \mathfrak{M}$.

Lemma 2.3. $\mathfrak{A}(A, B) = \{A, \mathfrak{B}(A, B)\}$, $V(x) = \{Ax, \mathscr{V}(x)\}$ *for all* $x \in E_n$.

Proof follows immediately from the proof of lemma 2.1.

Definition 2.3. $\omega \in E_n$, $T \geqq 0$. Then the set of all points $x \in E_n$ for which there exists such $u \in M(0, 1)$ that $x = x(T, u)$ ($x(t, u)$ is a solution of (2.1)), we denote by $\mathscr{S}_\omega(T)$ and we write $\bigcup_{t \in \langle 0, T \rangle} \mathscr{S}_\omega(t) = S_\omega(T)$.

Lemma 2.4. *The sets $\mathscr{S}_\omega(T)$, $S_\omega(T)$ are compact and connected for all $\omega \in E_n$, $T \geqq 0$.*

Proof. 1) $S_\omega(T)$ is bounded. Denote $\alpha = \max_{u \in \langle 0,1 \rangle} \|A + uC\|$, then for every solution of (2.1) it holds $\|x(t, u)\| \leqq \|\omega\| + \alpha \int_0^t \|x(\tau, u)\| \, d\tau \Rightarrow \|x(t, u)\| \leqq \|\omega\| \, . \, e^{\alpha t}$.

2) $S_\omega(T)$ is compact. Let a sequence $x_i \in S_\omega(T)$, $i = 1, 2, \ldots$ be given. Then there exist $t_i \in \langle 0, T \rangle$ and $u_i \in M(0, 1)$, $i = 1, 2, \ldots$ such that $x_i = x(t_i, u_i)$, $i = 1, 2, \ldots$ There is a convergent subsequence of $t_1, t_2, \ldots$ Let its limit be t_0 and let us suppose, without loss of generality, that already $t_1, t_2, \ldots$ is convergent to t_0. The sequence $x(t, u_i)$, $t \in \langle 0, t_0 \rangle$, is uniformly bounded due to boundeness of $S_\omega(T)$ and uniformly continuous. So there exists a subsequence (let it be again the original sequence) that converges uniformly to a function $x(t)$, $t \in \langle 0, t_0 \rangle$.

The sequence $u_1, u_2 \ldots$ is bounded in the space $L_2(0, t_0)$ of all square-integrable functions on $\langle 0, t_0 \rangle$ and we can choose from it a subsequence (let it be again $u_1, u_2, \ldots$) weakly convergent (in $L_2(0, t_0)$) to a function $u \in L_2(0, t_0)$. Let us show that $u \in M(0, 1)$. If $\varepsilon > 0$, denote χ the characteristic function of the set $Q = E(t \in \langle 0, t_0 \rangle; u(t) > 1 + \varepsilon)$. Then $\mu(Q) \geqq \int_0^{t_0} u_i(\tau) \chi(\tau) \, d\tau$, $i = 1, 2, \ldots$, $\mu(Q) \geqq \int_0^{t_0} u(\tau) \chi(\tau) \, d\tau \geqq (1 + \varepsilon) \mu(Q) \Rightarrow \mu(Q) = 0$. In the same way we can prove that $u(t) \geqq 0$ almost everywhere on $\langle 0, t_0 \rangle$.

It holds: $\left|\int_0^{t_0} (u_i x_i - ux) \, d\tau\right| \leqq \left|\int_0^{t_0} u_i(x_i - x) \, d\tau\right| + \left|\int_0^{t_0} (u_i - u) \, x \, d\tau\right| \to 0$ $\int_0^{t_0} (A + u_i C) \, x_i \, d\tau \to \int_0^{t_0} (A + uC) \, x \, d\tau = x(t_0, u) - \omega$. We have got $x(t) = x(t, u)$ for $t \in \langle 0, t_0 \rangle$, hence $x(t_0) = x(t_0, u) \in S_\omega(T)$. The proof of the compacteness of $\mathscr{S}_\omega(T)$ is similar.

3) $S_\omega(T)$ is obviously connected. We prove that also $\mathscr{S}_\omega(T)$ is connected.

Let $x(T, u)$, $x(T, v) \in \mathscr{S}_\omega(T)$, then the set $E(x(T, u(1 - \lambda) + v\lambda); \lambda \in \langle 0, 1 \rangle)$ is connected, included in $\mathscr{S}_\omega(T)$ and includes both points $x(T, u)$, $x(T, v)$.

Let us fix $\delta \in (0, \frac{1}{2})$, $u \in M(\delta, 1 - \delta)$ and put in (2.1) instead of the control u the control $u + \varepsilon v$, where $v \in M(-1, 1)$, $\varepsilon \in \langle 0, \delta \rangle$. Obviously $u + \varepsilon v \in M(0, 1)$ and we get

$$\dot{x} = (A + Cu) \, x + \varepsilon C v x \,, \quad x(0) = \omega \,. \tag{2.2}$$

The solution of (2.2) is an analytic function of the parameter ε.

$$x(t, u + \varepsilon v) = x_0(t, v) + \varepsilon x_1(t, v) + \varepsilon^2 x_2(t, v) + \ldots \tag{2.3}$$

$$\dot{x}_0 = (A + Cu) \, x_0 \,, \quad x_0(0, v) = \omega \,, \tag{2.4}$$

$$\dot{x}_k = (A + Cu) \, x_k + C v x_{k-1} \,, \quad x_k(0, v) = 0 \,, \quad k = 1, 2, \ldots \tag{2.5}$$

Let us put $\alpha = \max_{u \in \langle 0,1 \rangle} \|A + Cu\|$ and estimate $\|x_k(t, v)\|$, $k = 1, 2, \ldots$

$$\|x_0(t, v)\| \leqq \|\omega\| \, . \, e^{\alpha t}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\|x_k\| \leqq \alpha\|x_k\| + \|\mathrm{C}\| \,.\, \|x_{k-1}\| \Rightarrow \|x_k(t, v)\| \leqq \|\mathrm{C}\| \int_0^t e^{\alpha(t-\tau)} \|x_{k-1}\| \,\mathrm{d}\tau$$

$$\|x_k(t, v)\| \leqq \|\omega\| \,.\, \|\mathrm{C}\|^k \frac{t^k}{k!} e^{\alpha t}\,, \quad k = 0, 1, 2, \ldots$$

$$\|x(t, u + \varepsilon v)\| \leqq \sum_{k \geqq 0} \varepsilon^k \|x_k(t, v)\| \leqq \|\omega\| \,.\, e^{(\alpha + \varepsilon\|c\|)t}\,.$$

Thus the serie (2.3) is locally uniformly absolutely convergent. We can write

$$(2.6) \qquad x_1(T, v) = \mathrm{X}(T) \int_0^T \mathrm{X}^{-1}(t)\, \mathrm{C}x_0(t, v)\, v(t)\, \mathrm{d}t = \\ = \mathrm{X}(T) \int_0^T \mathrm{X}^{-1}(t)\, \mathrm{C}\mathrm{X}(t)\, v(t)\, \mathrm{d}t\, \omega\,.$$

Lemma 2.5. *The set $K_u(T)$, $T > 0$, of all vectors (2.6), where $v \in M(-1, 1)$, is convex and symetric with the centre at origin.*

Proof follows immediately from (2.6).

Lemma 2.6. *The linear hull of $K_u(T)$ is identical with the linear hull of the vectors*

$$(2.7) \qquad \mathrm{X}(T)\, \mathrm{X}^{-1}(t)\, \mathrm{C}\mathrm{X}(t)\, \omega\,, \quad t \in \langle 0, T\rangle\,.$$

Proof. By integration of (2.7) we cannot leave the linear hull. On the contrary let us fix $t \in \langle 0, T\rangle$ and put

$$v_\alpha(\tau) = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \quad \text{for} \quad \begin{Bmatrix} \tau \in \langle t - \alpha,\ t + \alpha\rangle \cap \langle 0, T\rangle \\ \tau \in \langle 0, T\rangle - \langle t - \alpha,\ t + \alpha\rangle \end{Bmatrix}, \quad \alpha > 0$$

Then

$$\lim_{\alpha \to 0+} (2\alpha)^{-1}\, \mathrm{X}(T) \int_0^T \mathrm{X}^{-1}(\tau)\, \mathrm{C}\mathrm{X}(\tau)\, v_\alpha(\tau)\, \mathrm{d}\tau\, \omega = \\ = \varphi(t)\, \mathrm{X}(T)\, \mathrm{X}^{-1}(t)\, \mathrm{C}\mathrm{X}(t)\, \omega\,,$$

where

$$\varphi(t) = \begin{Bmatrix} 1 \\ \frac{1}{2} \end{Bmatrix} \quad \text{for} \quad \begin{Bmatrix} t \neq 0,\ t \neq T \\ t = 0 \text{ or } t = T \end{Bmatrix}.$$

Lemma 2.7. *Let $T > 0$, $u_\lambda \in M(0, 1)$ for all $\lambda \in \langle 0, \lambda_0\rangle$, $\lambda_0 > 0$. If $u_\lambda \to u_0$ asymptotically on $\langle 0, T\rangle$, $\lambda \to 0$, then $\mathrm{X}(t, u_\lambda) \to \mathrm{X}(t, u_0)$ uniformly on $\langle 0, T\rangle$.*

Here $\mathrm{X}(t, u_\lambda)$ is the fundamental matrix-function of equation (2.1), where u is replaced by u_λ, and $\mathrm{X}(0, u_\lambda) = \mathrm{E}$.

Proof is contained in the second part of the proof of lemma 1.2.

Lemma 2.8. $T > 0$, $\delta \in (0, \frac{1}{2})$, $u \in M(\delta, 1 - \delta)$. *Let the function u be not constant on $\langle 0, T\rangle$ (not equivalent with a constant function), then*

$$\mathscr{V}(x(T, u)) \subset \bigcup_{r=1}^{\infty} r \,.\, K_u(T)\,. \tag{2.8}$$

Proof. According to lemma 2.6 the vectors (2.7) are contained in $\bigcup_{r=1}^{\infty} r \,.\, K_u(T)$. For almost all $t \in \langle 0, T\rangle$ has the function $\int_0^t u(\tau)\,\mathrm{d}\tau$ the derivative equal to $u(t)$. Let $G \subset \langle 0, T\rangle$ be the set where it is not true. Let us take $t_1 \in \langle 0, T\rangle - G$, then we can write $\mathrm{X}(T)\,\mathrm{X}^{-1}(t)\,\mathrm{C}\mathrm{X}(t)\,\omega = \mathrm{X}(T)\,\mathrm{X}^{-1}(t_1)\,(\mathrm{X}(t_1)\,\mathrm{X}^{-1}(t)\,\mathrm{C}\mathrm{X}(t)\,\mathrm{X}^{-1}(t_1))\,\mathrm{X}(t_1)\,.$ $.\,\mathrm{X}^{-1}(T)\,x(T, u)$. Let us, for brevity, denote $\mathrm{X}(t)\,\mathrm{X}^{-1}(t_1) = \mathrm{Y}(t)$ and divide the proof into two parts. In the first, resp. second, part we show that the linear hull L, resp. L_1, of matrices $\mathrm{Y}^{-1}(t)\,\mathrm{C}\mathrm{Y}(t)$, resp. $\mathrm{X}(T)\,\mathrm{X}^{-1}(t)\,\mathrm{C}\mathrm{X}(t)\,\mathrm{X}^{-1}(T)$, where t ranges through entire $\langle 0, T\rangle$, contains the space $\mathfrak{B}(\mathrm{A}, \mathrm{B})$.

1) For $t = t_1$ we get $\mathrm{C} \in L$. Let all elementary matrices from $\mathfrak{B}(\mathrm{A}, \mathrm{B})$ of the grades equal at most to $(k - 1)$ (the definition of the elementary matrix and its grade is given in the proof of lemma 2.1) belong to L. The matrix-function Y satisfies the equation $\dot{\mathrm{Y}} = (\mathrm{A} + u\mathrm{C})\,\mathrm{Y}$, $\mathrm{Y}(t_1) = \mathrm{E}$ and for $t \in (t_1, T\rangle$ we can write

$$\mathrm{Y}(t) = \mathrm{E} + \int_{t_1 < \tau_1 < t} (\mathrm{A} + u(\tau_1)\,\mathrm{C})\,\mathrm{d}\tau_1 + \ldots +$$

$$+ \int_{t_1 < \tau_k < \ldots < \tau_1 < t} (\mathrm{A} + u(\tau_1)\,\mathrm{C}) \ldots (\mathrm{A} + u(\tau_k)\,\mathrm{C})\,\mathrm{d}\tau_k \ldots \mathrm{d}\tau_1 +$$

$$+ \int_{t_1 < \tau_{k+1} < \ldots < \tau_1 < t} (\mathrm{A} + u(\tau_1)\,\mathrm{C}) \ldots (\mathrm{A} + u(\tau_{k+1})\,\mathrm{C})\,\mathrm{Y}(\tau_{k+1})\,\mathrm{d}\tau_{k+1} \ldots \mathrm{d}\tau_1$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Y}^{-1}(t) = -\mathrm{Y}^{-1}(t)\left(\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{Y}(t)\right)\mathrm{Y}^{-1}(t) = -\mathrm{Y}^{-1}(t)\,(\mathrm{A} + u(t)\,\mathrm{C})\,,$$

$\mathrm{Y}^{-1}(t_1) = \mathrm{E}$,

$$\mathrm{Y}^{-1}(t) = \mathrm{E} - \int_{t_1 < \tau_1 < t} (\mathrm{A} + u(\tau_1)\,\mathrm{C})\,\mathrm{d}\tau_1 + \ldots +$$

$$+ (-1)^k \int_{t_1 < \tau_k < \ldots < \tau_1 < t} (\mathrm{A} + u(\tau_k)\,\mathrm{C}) \ldots (\mathrm{A} + u(\tau_1)\,\mathrm{C})\,\mathrm{d}\tau_k \ldots \mathrm{d}\tau_1 +$$

$$+ (-1)^{k+1} \int_{t_1 < \tau_{k+1} < \ldots < \tau_1 < t} \mathrm{Y}^{-1}(\tau_{k+1})\,(\mathrm{A} + u(\tau_{k+1})\,\mathrm{C}) \ldots$$

$$\ldots (\mathrm{A} + u(\tau_1)\,\mathrm{C})\,\mathrm{d}\tau_{k+1} \ldots \mathrm{d}\tau_1$$

$$\mathrm{Y}^{-1}(t)\,\mathrm{C}\mathrm{Y}(t) = \mathrm{C} + \int_{t_1 < \tau_1 < t} [\mathrm{A} + u(\tau_1)\,\mathrm{C}, \mathrm{C}]\,\mathrm{d}\tau_1 +$$

$$+ \int_{t_1 < \tau_2 < \tau_1 < t} [A + u(\tau_2) C, A + u(\tau_1) C, C] \, d\tau_2 \, d\tau_1 + \ldots +$$

$$+ \int_{t_1 < \tau_k < \ldots < \tau_1 < t} [A + u(\tau_k) C, \ldots, A + u(\tau_1) C, C] \, d\tau_k \, d\tau_{k-1} \ldots d\tau_1 +$$

$$+ O((t - t_1)^{k+1}) \in L.$$

If we subtract from the matrix $Y^{-1}(t) \, CY(t)$ the first k addend, then according to the induction assumption we do not leave L and we get

$$\int_{t_1 < \tau_k < \ldots < \tau_1 < t} [A + u(\tau_k) C, \ldots, A + u(\tau_1) C, C] \, d\tau_k \ldots d\tau_1 + O((t - t_1)^{k+1}) \in L.$$

Let us multiply the left side by $(t - t_1)^{-k}$ and tend $t \to t_1$, then we get

$$[(A + u(t_1) \, C)^k \, C]/k! \in L.$$

We can take another point $t_2 \in \langle 0, T \rangle - G$ for which $u(t_2) \neq u(t_1)$ and by the same procedure we get $[(A + u(t_2) \, C)^k \, C] \in L$, $k = 0, 1, 2, \ldots$ Thus according to lemma 2.1 it holds $\mathfrak{B}(A, B) \subset L$.

2) For $t = T$ we get $X(T) \, X^{-1}(T) \, CX(a) \, X^{-1}(a) = C \in L_1$. Let us, for brevity, denote $X(t_1) \, X^{-1}(a) = Z$, then we can write $X(T) \, X^{-1}(t) \, CX(t) \, X^{-1}(T) = Z^{-1} Y^{-1}(t) \, CY(t) \, Z$, $t \in \langle 0, T \rangle$.

Let us choose $t_0 \in \langle 0, T \rangle$, then $[A, Z^{-1} Y^{-1}(t_0) \, CY(t_0) \, Z] = [Z^{-1}(ZAZ^{-1}) \, Z, Z^{-1} Y^{-1}(t_0) \, CY(t_0) \, Z] = Z^{-1}[ZAZ^{-1}, Y^{-1}(t_0) \, CY(t_0)] \, Z$. According to lemma 1.1 it holds $ZAZ^{-1} \in \mathfrak{A}(A, B)$ and thus in accordance with the first part of this proof we get $[ZAZ^{-1}, Y^{-1}(t_0) \, CY(t_0)] \in L$, $[A, Z^{-1} Y^{-1}(t_0) \, CY(t_0) \, Z] \in L_1$. In the same way we get $[B, Z^{-1} Y^{-1}(t_0) \, CY(t_0) \, Z] \in L_1$.

This concludes the proof.

Example. The assumption in lemma 2.8 that the control u is not constant is necessary. Let us put

$$A = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}, \quad u(t) \equiv \tfrac{1}{2}.$$

We shall show that then (2.8) will not hold for all $T > 0$.

It holds

$$[A, C] = [A + \tfrac{1}{2}C, C] = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$[\mathrm{A}, \mathrm{A}, \mathrm{C}] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}, \quad [\mathrm{A} + \tfrac{1}{2}\mathrm{C}, \mathrm{A} + \tfrac{1}{2}\mathrm{C}, \mathrm{C}] = [\mathrm{A}, \mathrm{C}],$$

$$\mathscr{V}(\omega) = \left\{ \begin{pmatrix} \omega_1 \\ \omega_1 \\ \omega_2 \end{pmatrix}, \begin{pmatrix} 0 \\ \omega_1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \omega_1 \end{pmatrix} \right\},$$

$$e^{(\mathrm{A} + \frac{1}{2}\mathrm{C})T} = \begin{pmatrix} e^T & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathscr{V}(e^{(\mathrm{A} + \frac{1}{2}\mathrm{C})T}\omega) = \left\{ \begin{pmatrix} e^T\omega_1 \\ e^T\omega_1 \\ \omega_2 \end{pmatrix}, \begin{pmatrix} 0 \\ e^T\omega_1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ e^T\omega_1 \end{pmatrix} \right\}.$$

If $\omega_1 \neq 0$, then for all $T \geqq 0$ it holds $\dim \mathscr{V}(e^{(\mathrm{A}+\frac{1}{2}\mathrm{C})T}\omega) = 3$. For $t \in \langle 0, T \rangle$ we get $\mathrm{X}(T)\,\mathrm{X}^{-1}(t)\,\mathrm{C}\mathrm{X}(t)\,\mathrm{X}^{-1}(T) = e^{-(\mathrm{A}+\frac{1}{2}\mathrm{C})(t-T)}\mathrm{C}e^{(\mathrm{A}+\frac{1}{2}\mathrm{C})(t-T)} = \mathrm{C} + \sum_{r=1}^{\infty} ((t - T)^r/r!)$. $[(\mathrm{A} + \frac{1}{2}\mathrm{C})^r\, \mathrm{C}] = \mathrm{C} + (e^{t-T} - 1)\,[\mathrm{A}, \mathrm{C}]$.

Thus it holds

$$\sum_{r=1}^{\infty} rK_u(T) = \{\mathrm{C}_\omega, [\mathrm{A}, \mathrm{C}]\,\omega\}, \quad \dim \sum_{r=1}^{\infty} rK_u(T) = 2.$$

Lemma 2.9. $T > 0$, $\delta \in (0, \frac{1}{2})$, $u \in M(\delta, 1 - \delta)$. *Then*

$$K_u(T) \subset \mathscr{V}(x(T, u)). \tag{2.9}$$

Proof. According to lemma 2.6 the set $K_u(T)$ is contained in the linear hull L of the vectors (2.7) and from lemma 1.1 it follows $L \subset \mathscr{V}(x(T, u))$.

Lemma 2.10. $T > 0$, $\delta \in (0, \frac{1}{2})$, $u \in M(\delta, 1 - \delta)$. *Let us denote by* $\mathscr{T}(x(T, u))$ *the set of all possible limits (if they exist),*

$$\lim_{k \to \infty} a \frac{x_k - x(T, u)}{\|x_k - x(T, u)\|},$$

where $x_k \in \mathscr{S}_\omega(T)$, $x_k \neq x(T, u)$, $k = 1, 2, \ldots$, $a \in \mathrm{E}_1$, $x_k \to x(T, u)$.

Then

$$K_u(T) \subset \mathscr{T}(x(T, u)). \tag{2.10}$$

Proof is evident from (2.3).

Lemma 2.11. *Let* $x(t, u)$ *be a solution of* (2.1), *then* $\dim \mathscr{V}(x(t, u)) = \dim \mathscr{V}(\omega)$, $t \geqq 0$.

Proof. According to theorem 1.1 it holds $\dim \mathrm{V}(x(t, u)) = \dim \mathrm{V}(\omega)$, $t \geqq 0$. So it is sufficient to prove the equivalence

$$\mathrm{A}\omega \in \mathscr{V}(\omega) \Leftrightarrow \mathrm{A}x(t, u) \in \mathscr{V}(x(t, u))$$

Let us fix $t \geqq 0$, then it holds $A\omega \in \mathscr{V}(\omega) \Rightarrow X^{-1}(t)\, AX(t)\, \omega \in \mathscr{V}(\omega) \Rightarrow$ such $A_i \in \mathfrak{B}(A, B)$ and $a_i \in E_1$, $i = 1, 2, \ldots, r$, exist that $X^{-1}(t)\, AX(t)\, \omega = \sum_{i=1}^{r} a_i A_i \omega \Rightarrow$

$\Rightarrow Ax(t, u) = AX(t)\, \omega = X(t)\, X^{-1}(t)\, AX(t)\, \omega = X(t) \sum_{i=1}^{r} a_i A_i \omega = (\sum_{i=1}^{r} a_i X(t)\, .$

$.\, A_i X^{-1}(t))\, x(t, u) \Rightarrow Ax(t, u) \in \mathscr{V}(x(t, u))$.

The inverse implication we get if we take the matrices $-A$, $-B$ instead of the matrices A, B.

Theorem 2.1. *$T > 0$, $\omega \in E_u$, $\dim \mathscr{V}(\omega) = q$. Then $\mathscr{S}_\omega(T)$ is a closed, q-dimensional integral manifold of the distribution $\mathscr{V}$.*

Proof. Let us choose $x \in \mathscr{S}_\omega(T)$ and $\varepsilon > 0$. Then there exist $\delta \in (0, \frac{1}{2})$ and a non-constant control $u \in M(\delta, 1 - \delta)$ such that it holds $\|x - x(T, u)\| < \varepsilon$.

According to lemma 2.11 $\dim \mathscr{V}(x(T, u)) = q$ and from lemma 2.8 it follows that there exist such functions $v_i \in M(-1, 1)$, $i = 1, 2, \ldots, q$, that $\mathscr{V}(x(T, u)) = \{x_1(T, v_1), x_1(T, v_2), \ldots, x_1(T, v_q)\}$, where $x_1(T, v_i)$, $i = 1, 2, \ldots, q$, are vectors (2.6).

The function

$$x(T, u + \sum_{i=1}^{q} \vartheta_i v_i)\,; \quad \vartheta \in G = E\left(\vartheta \in E_q;\ \|\vartheta\| < \frac{1}{q}\,\delta\right), \tag{2.11}$$

represents a mapping of the open set $G \subset E_q$ into $\mathscr{S}_\omega(T)$, has continuous partial derivatives of first order with respect to ϑ_i, $i = 1, 2, \ldots, q$, which are for $\vartheta = 0$ solutions of the equation

$$\frac{d}{dt}\frac{dx}{d\vartheta_i} = (A + uC)\frac{\partial x}{\partial \vartheta_i} + Cv_i\, x(t, u)\,; \quad \left.\frac{\partial x}{\partial \vartheta_i}\right|_{t=0} = 0\,; \quad i = 1, 2, \ldots, q\,.$$

Hence, the matrix $dx/\partial\vartheta|_{\vartheta=0}$ has as its columns the vectors $x_1(T, \vartheta_i)$, $i = 1, 2, \ldots, q$, and in an environ of the point $\vartheta = 0$ has the rank q.

Thus the set $\mathscr{S}_\omega(T)$ is the adherence of a union of a system of q-dimensional integral manifolds of the distribution $\mathscr{V}$. Let $S_{1,2}$ be two manifolds of this system. Let us choose points $x(T, u_i) \in S_i$, $i = 1, 2$, where the controls u_i, $i = 1, 2$ are so chosen that for every $\lambda \in \langle 0, 1 \rangle$ the function $u_1(1 - \lambda) + u_2\lambda$ is not constant. Then the curve

$$x(T, u_1(1 - \lambda) + u_2\lambda)\,, \quad \lambda \in \langle 0, 1 \rangle\,, \tag{2.12}$$

links both points $x(T, u_i)$, $i = 1, 2$, and each point of (2.12) is contained in a q-dimensional integral manifold of the distribution $\mathscr{V}$, that is contained in $\mathscr{S}_\omega(T)$.

According to theorem 1.1 the theorem is proved.

Theorem 2.2. *$T > 0$, $\omega \in \mathrm{E}_n$, $\dim \mathrm{V}(\omega) = r$. Then $\mathrm{S}_\omega(T)$ is a closed r-dimensional integral manifold of the distribution* V.

Proof. At first let us again show that $\mathrm{S}_\omega(T)$ is an adherence of a system of r-dimensional integral manifolds of the distribution V. We shall distinguish two cases:

1) $\dim \mathrm{V}(\omega) = \dim \mathscr{V}(\omega)$. Then the statement follows immediately from theorem 2.1.

2) $\dim \mathrm{V}(\omega) > \dim \mathscr{V}(\omega)$. Let us choose $x \in \mathrm{S}_\omega(T)$ and $\varepsilon > 0$, then there exist $\delta \in (0, \frac{1}{2})$, $t \in (0, T)$ and a non-constant continuous on $\langle 0, T \rangle$ control $u \in M(\delta, 1 - \delta)$ so that $\|x - x(t, u)\| < \varepsilon$. If we choose functions $v_i \in M(-1, 1)$, $i = 1, 2, \ldots, r - 1$, like in theorem 2.1, where we have t instead of T, then the function

$$x\left(\tau, u + \sum_{i=1}^{r-1} \vartheta_i v_i\right), \tag{2.13}$$

$$|\tau - t| < \Delta = \min (t, T - t), \quad \vartheta \in G = E\left[\vartheta \in \mathrm{E}_{r-1};\ \|\vartheta\| < \frac{1}{r - 1}\,\delta\right],$$

is a mapping of the open set $(t - \Delta, t + \Delta) \times G \subset \mathrm{E}_r$ into $\mathrm{S}_\omega(T)$. The function (2.13) has all properties as the function (2.11) has and moreover

$$\left.\frac{\mathrm{d}x(\tau, u)}{\mathrm{d}\tau}\right|_{\vartheta = 0} = (\mathrm{A} + u(\tau)\,\mathrm{C})\,x(\tau, u)\,.$$

Hence, the functional matrix of the mapping (2.13) has at the point $\tau = t$, $\vartheta = 0$ the rank r.

Now, let us take two integral manifolds $\mathrm{S}_{1,2}$ of the distribution V, which are contained in $\mathrm{S}_\omega(T)$. We choose points $x(t_i, u_i) \in \mathrm{S}_i$, $i = 1, 2$, so that $t_i \in (0, T)$, the control u_i is continuous on $\langle 0, T \rangle$, $i = 1, 2$, and the function $u_1(1 - \lambda) + u_2\lambda$ is not constant on $\langle 0, \min(t_1, t_2) \rangle$. Let for example be $t_1 < t_2$, then we link both points $x(t_i, u_i)$ with the curve, composed of the following arcs:

$$x(t_1, u_1(1 - \lambda) + u_2\lambda)\,, \quad \lambda \in \langle 0, 1 \rangle\,, \quad x(t, u_2)\,, \quad t \in \langle t_1, t_2 \rangle\,.$$

The theorem is proved.

3. CASE OF 1-DIMENSIONAL MANIFOLD $\mathscr{S}_\omega(T)$

In this paragraph we show that every point on a 1-dimensional manifold $\mathscr{S}_\omega(T)$ can be reached by a piecewise constant control $u \in M(0, 1)$ and we describe the points at which u is discontinuous. In the whole paragraph we shall have fixed matrices $\mathrm{A}, \mathrm{B} \in \mathfrak{E}_n$.

Lemma 3.1.

$$(3.1)\qquad \frac{\mathrm{d}}{\mathrm{d}u}(\mathrm{A} + u\mathrm{C})^k = \sum_{i=1}^{k} \binom{k}{i} (-1)^{i-1} [(\mathrm{A} + u\mathrm{C})^{i-1} \mathrm{C}] (\mathrm{A} + u\mathrm{C})^{k-i} ;$$

$$k = 0, 1, 2, \ldots$$

Proof by the mathematical induction.

Lemma 3.2.

$$(3.2)\qquad \frac{\mathrm{d}}{\mathrm{d}u} e^{\mathrm{A}+u\mathrm{C}} = \left(\sum_{i \geqq 0} \frac{(-1)^i}{(i+1)!} [(\mathrm{A} + u\mathrm{C})^i \mathrm{C}] \right) e^{\mathrm{A}+u\mathrm{C}} .$$

Proof.

$$\frac{\mathrm{d}}{\mathrm{d}u} e^{\mathrm{A}+u\mathrm{C}} = \sum_{k \geqq 0} \frac{1}{k!} \frac{\mathrm{d}}{\mathrm{d}u} (\mathrm{A} + u\mathrm{C})^k = \sum_{k \geqq 0} \sum_{i=1}^{k} \frac{1}{k!} \binom{k}{i} (-1)^{i-1} [(\mathrm{A} + u\mathrm{C})^{i-1} \mathrm{C}] .$$

$$. (\mathrm{A} + u\mathrm{C})^{k-i} = \sum_{i \geqq 1} \sum_{s \geqq 0} \frac{1}{i!\, s!} (-1)^{i-1} [(\mathrm{A} + u\mathrm{C})^{i-1} \mathrm{C}] (\mathrm{A} + u\mathrm{C})^s =$$

$$= \sum_{i \geqq 0} \frac{(-1)^i}{(i+1)!} [(\mathrm{A} + u\mathrm{C})^i \mathrm{C}] e^{\mathrm{A}+u\mathrm{C}}$$

we use the new index $s = k - i$.

Lemma 3.3. $T > 0$, $u \in M(0, 1)$, $\omega \in \mathrm{E}_n$, $\dim \mathscr{V}(\omega) > 0$. *Then the set*

$$(3.3)\qquad E(t \in \langle 0, T \rangle;\ \mathrm{C}x(t, u) = 0)$$

is finite.

Proof. Let there exist a sequence $t_1, t_2, t_2, \ldots$ $t_i \in \langle 0, T \rangle$, $t_i \to t_0$, $t_i \neq t_0$, $\mathrm{C}x(t_i, u) = 0$, $i = 1, 2, \ldots$ Let us denote $x(t_0, u) = x_0$. Evidently $\mathrm{C}x_0 = 0$. Let us suppose $\mathrm{CA}^r x_0 = 0$, $r = 0, 1, \ldots, (k-1)$. Without loss of generality let us suppose that there exists an infinite number of those terms t_i in the sequence $t_1, t_2, \ldots$ for which $t_i > t_0$. Crossing to a subsequence we can assume that $t_i > t_0$ holds for all $i = 1, 2, \ldots$ Then $t_0 < T$ and we can write for $t \in (t_0, T\rangle$

$$(3.4)\qquad x(t, u) = x_0 + \int_{t_0 < \tau_1 < t} (\mathrm{A} + u(\tau_1)\, \mathrm{C})\, x_0\, \mathrm{d}\tau_1 + \ldots +$$

$$+ \int_{t_0 < \tau_k < \ldots < \tau_1 < t} (\mathrm{A} + u(\tau_1)\, \mathrm{C}) \ldots (\mathrm{A} + u(\tau_k)\, \mathrm{C})\, x_0\, \mathrm{d}\tau_k \ldots \mathrm{d}\tau_1 +$$

$$+ \int_{t_0 < \tau_{k+1} < \ldots < \tau_1 < t} (\mathrm{A} + u(\tau_1)\, \mathrm{C}) \ldots (\mathrm{A} + u(\tau_{k+1})\, \mathrm{C})\, x(\tau_{k+1}, u)\, \mathrm{d}\tau_{k+j} \ldots \mathrm{d}\tau_1 .$$

If the number of those terms of the sequence $t_1, t_2, \ldots$, for which $t_i > t_0$ holds, is finite, then $t_0 > 0$ and for $t < t_0$ we would have to change the signs of the inequalities in (3.4) and to alternate the signs of the integrals.

If we put in (3.4) $t = t_i$, $i = 1, 2, \ldots$, and multiply it by the matrix

$$\left(\frac{1}{(t_i - t_0)^k} \cdot \mathrm{C}\right),$$

we get

$$0 = \frac{1}{(t_i - t_0)^k} \mathrm{C}x(t_i, u) = \frac{1}{(t_i - t_0)^k} \int_{t_0 < \tau_k < \ldots < \tau_1 < t_i} \mathrm{CA}^k x_0 \, \mathrm{d}\tau_k \ldots \mathrm{d}\tau_1 +$$

$$+ \frac{1}{(t_i - t_0)^k} \int_{t_0 < \tau_{k+1} < \ldots < \tau_1 < t_i} (\mathrm{A} + u(\tau_1)\,\mathrm{C}) \ldots (\mathrm{A} + u(\tau_{k+1})\,\mathrm{C})\, x(\tau_{k+1}, u)\, \mathrm{d}\tau_{k+1} \ldots$$

$$\ldots \mathrm{d}\tau_1 = \frac{1}{k!} \mathrm{CA}^k x_0 + O(t_i - t_0)\,.$$

Thus we have found out that it holds $\mathrm{CA}^k x_0 = 0$ for all integers k.

Now, let us rewrite the equation (2.1) into the following form:

$$\dot{x} = (\mathrm{A} + u\mathrm{C})\, x = (\mathrm{B} + (u - 1)\,\mathrm{C})\, x\,, \quad x(0) = \omega\,.$$

By the same procedure we would find out that also $\mathrm{CB}^k x_0 = 0$, $k = 0, 1, 2, \ldots$ Hence, according to lemma 2.2 it holds $\dim \mathscr{V}(x_0) = 0$ and from lemma 2.11 follows $\dim \mathscr{V}(\omega) = 0$. We have got a contradiction.

Lemma 3.4. $T > 0$, $x \in \mathrm{E}_n$, $\dim \mathscr{V}(x) = 1$, $u_1 < u_2$. *Let us put* $g(\vartheta, t) = \mathrm{e}^{(\mathrm{A}+\vartheta\mathrm{C})t} x$, $\vartheta \in \langle u_1, u_2 \rangle$, $t \in (0, T\rangle$.

Then there exists an analytic function $\psi(\vartheta, t)$, *defined on the set*

$$G = E(\vartheta \in \langle u_1, u_2 \rangle, t \in (0, T\rangle;\ \mathrm{C}g(\vartheta, t) \neq 0)\,,$$

such that

$$\frac{\partial g(\vartheta, t)}{\partial \vartheta} = \psi(\vartheta, t)\, \mathrm{C}g(\vartheta, t)\,, \quad (\vartheta, t) \in G\,. \tag{3.5}$$

Moreover, if there exists such $\vartheta_0 \in \langle u_1, u_2 \rangle$ *that* $\mathrm{C}g(\vartheta_0, t) \neq 0$ *for all* $t \in (0, T\rangle$, *then* $\psi(\vartheta_0, t) > 0$ *for all* $t \in (0, T\rangle$ *and there exists* $\lim_{t \to 0+} t^{-1} \,.\, \psi(\vartheta_0, t) > 0$.

Proof. In accordance with lemma 3.2 it holds

$$\frac{\partial g(\vartheta, t)}{\partial \vartheta} = \sum_{k \geq 0} (-1)^k \frac{t^{k+1}}{(k+1)!} [(\mathrm{A} + \vartheta\mathrm{C})^k\, \mathrm{C}]\, g(\vartheta, t)\,.$$

According to lemma 2.11 it holds $\dim \mathscr{V}(g(\vartheta, t)) = 1$ and thus

$$[(\mathrm{A} + \vartheta \mathrm{C})^k \mathrm{C}]\, g(\vartheta, t) \in \{\mathrm{C}g(\vartheta, t)\}\,, \quad k = 0, 1, 2, \ldots, \; (\vartheta, t) \in G\,.$$

Let us denote $[(\mathrm{A} + \vartheta \mathrm{C})^k \mathrm{C}] = \mathrm{D}_k(\vartheta)$, $k = 0, 1, 2, \ldots$ As the sequence of norms $\|\mathrm{D}_k(\vartheta)\|$, $k = 0, 1, 2, \ldots$ can be majorised by a geometric sequence with the quotient. $2 \max\limits_{\vartheta \in \langle u_1, u_2 \rangle} \|\mathrm{A} + \vartheta \mathrm{C}\|$, the first part of the lemma is proved.

It holds:

$$e^{-(\mathrm{A}+\vartheta\mathrm{C})t} \frac{\partial g}{\partial \vartheta} = e^{-(\mathrm{A}+\vartheta\mathrm{C})t} \sum_{k \geq 0} (-1)^k \frac{t^{k+1}}{(k+1)!} \mathrm{D}_k(\vartheta)\, e^{(\mathrm{A}+\vartheta\mathrm{C})t} x =$$

$$= \sum_{k \geq 0} (-1)^k \frac{t^{k+1}}{(k+1)!} e^{-(\mathrm{A}+\vartheta\mathrm{C})t} \mathrm{D}_k(\vartheta)\, e^{(\mathrm{A}+\vartheta\mathrm{C})t} x =$$

$$= \sum_{k \geq 0} (-1)^k \frac{t^{k+1}}{(k+1)!} \left(\sum_{e \geq 0} \frac{t^e}{e!} \mathrm{D}_{k+e}(\vartheta) \right) x = \sum_{r \geq 0} \frac{t^{r+1}}{(r+1)!} \mathrm{D}_r(\vartheta)\, x\,,$$

$$e^{-(\mathrm{A}+\vartheta\mathrm{C})t} \frac{\partial g}{\partial \vartheta} = e^{-(\mathrm{A}+\vartheta\mathrm{C})t} \psi(\vartheta, t)\, \mathrm{C}g =$$

$$= \psi(\vartheta, t)\, e^{-(\mathrm{A}+\vartheta\mathrm{C})t} \mathrm{C} e^{(\mathrm{A}+\vartheta\mathrm{C})t} x = \psi(\vartheta, t) \sum_{r \geq 0} \frac{t^r}{r!} \mathrm{D}_r(\vartheta)\, x\,.$$

For $\vartheta = \vartheta_0$ and for sufficiently small $t > 0$ both sides of (3.5) are not zero. Let us denote by $\beta(t)$ one non-zero coordinate of the vector $e^{-(\mathrm{A}+\vartheta_0\mathrm{C})t} \,.\, (\partial g / \partial \vartheta)$, then we get the equation

$$\beta(t) = \psi(\vartheta_0, t) \,.\, \beta'(t)\,, \quad t \in (0, t_0)\,, \quad t_0 = \min\Big(T, \inf_{\substack{\beta(\tau) = 0 \\ \tau > 0}} \tau\Big)\,.$$

Let be $\mathrm{D}_i(\vartheta_0)\, x = 0$; $i = 0, 1, \ldots, r - 1$; $\mathrm{D}_r(\vartheta_0)\, x \neq 0$, then evidently $\lim\limits_{t \to 0+} t^{-1} \,.$ $.\, \psi(\vartheta_0, t) = 1/(r + 1) > 0$.

Let us take $t_1 \in (0, t_0)$ so small that $\psi(\vartheta_0, t_1) > 0$, then

$$\beta(t) = \beta(t_1) \exp \{\textstyle\int_{t_1}^t (1/\psi(\vartheta_0, t))\, \mathrm{d}t\}\,, \quad t \in \langle t_1, t_2 \rangle\,, \quad t_2 = \min(T, \inf\limits_{\beta'(\tau) = 0 \;\; \tau > t_1} \tau)\,.$$

It is not possible to be $t_0 < t_2$ as for $t \in \langle t_1, t_2 \rangle$ it holds $|\beta(t)| \geq |\beta(t_1)| > 0$.

If it holds $t_2 < t_0$, then there is $\lim\limits_{t \to t_2-} \psi(\vartheta_0, t) = +\infty$, what is again impossible, as $\psi(\vartheta_0, t)$ is bounded on $\langle t_1, T \rangle$. Thus it holds $t_0 = t_2$.

If it holds $t_2 < T$, then another coordinate $\tilde{\beta}$ of the vector $\exp\{-(\mathrm{A} + \vartheta_0 \mathrm{C})\, t\}$. $.\, (\partial g / \partial \vartheta)$ has its derivative at the point t_2 different from zero and again it holds

$$\tilde{\beta}(t) = \psi(\vartheta_0, t) \,.\, \tilde{\beta}'(t)\,, \quad t \in \langle t_2 - \Delta, t_3 \rangle\,, \quad t_3 = \min\Big(T, \inf_{\substack{\tilde{\beta}'(\tau) = 0 \\ \tau > t_2}} \tau\Big)\,,$$

where $\Delta > 0$ is so small that $\tilde{\beta}'(t) \neq 0$ on the interval $\langle t_2 - \Delta, t_2 \rangle$.

This completes the proof.

Lemma 3.5. $u_{1,2} \in \langle 0, 1 \rangle$, $t_{1,2} > 0$, $\omega \in E_n$, $\dim \mathscr{V}(\omega) = 1$. *Let us put*

$$x = e^{(A + u_2 C) t_2} e^{(A + u_1 C) t_1} \omega .$$

Then it exists a piecewise constant control

$$u \in M(\min(u_1, u_2), \max(u_1, u_2))$$

such that:

1) $x = x(t_1 + t_2, u)$,

2) *if t is a point of discontinuity of the control u, then* $Cx(t, u) = 0$.

Proof. The case $u_1 = u_2$ is trivial. Let be $u_1 < u_2$. Let us denote by u_0 the control given by the prescription:

$$\left.\begin{array}{l} u_0(t) = u_1 \\ u_0(t) = u_2 \end{array}\right\} \text{ for } \left\{\begin{array}{l} t \in \langle 0, t_1) \\ t \in \langle t_1, t_1 + t_2 \rangle . \end{array}\right.$$

If it is $t_1 \in F = E(t \in (0, t_1 + t_2\rangle; Cx(t, u_0) = 0)$, then there is nothing to be proved. So let be $t_1 \notin F$. Without loss of generality we can assume that $F = \emptyset$. Then on some open neighbourhood G of the set of all points $x(t, u_0)$, where $t \in (0, t_1 + t_2\rangle$, it holds $x \in G \Rightarrow Cx \neq 0$.

Let us choose $\Delta \in (0, t_2\rangle$ so small that for all $\Delta_{1,2} \geqq 0$, $\Delta_1 + \Delta_2 \leqq \Delta$, it holds

$$e^{(A + u_2 C) \Delta_2} e^{(A + u_1 C)(t_1 + \Delta_1)} \omega \in G .$$

Let us put

$$f(\tau, \zeta) = e^{(A + u_2 C)\tau} e^{(A + u_1 C)(t_1 + \zeta - \tau)} \omega ; \quad \tau, \zeta \in \langle 0, \Delta \rangle ,$$

Then it holds:

$$\frac{\partial f(\tau, \zeta)}{\partial \tau} = (u_2 - u_1)\, e^{(A + u_2 C)\tau} C e^{(A + u_1 C)(t_1 + \zeta - \tau)} \omega = \tag{3.6}$$

$$= (u_2 - u_1)\, e^{(A + u_2 C)\tau} C e^{-(A + u_2 C)\tau} f(\tau, \zeta) .$$

The right side of (3.6) is for $\tau = 0$ different from zero. We can choose Δ so small that $\partial f(\tau, \zeta)/\partial \tau \neq 0$ for $\tau, \zeta \in \langle 0, \Delta \rangle$.

Let us further put

$$g(\vartheta, t) = e^{(A + \vartheta C)t} \omega , \quad \vartheta \in \langle u_1, u_2 \rangle , \quad t \in \langle 0, t_1 + t_2 \rangle .$$

According to lemma 3.4 we can define functions φ, ψ as follows:

$$\frac{\partial f(\tau, \zeta)}{\partial \tau} = \varphi(\tau, \zeta)\, Cf(\tau, \zeta) , \quad \frac{\partial g(\vartheta, t)}{\partial \vartheta} = \psi(\vartheta, t)\, Cg(\vartheta, t) .$$

Then φ is analytic and different from zero on the set $\tau, \zeta \in \langle 0, \Delta \rangle$. As $\varphi(0, \zeta) = u_2 - u_1 > 0$, $\zeta \in \langle 0, \Delta \rangle$, the function φ is positive. The function ψ is according to lemma 3.4 analytic on the set of all (ϑ, t) for which $Cg(\vartheta, t) \neq 0$ and according to the supplement to lemma 3.4 it is $\psi(u_1, t) > 0$ for $t \in (0, t_1 + \Delta \rangle$.

Let us put $K_1 = \max \varphi(\tau, \zeta)$, where $\tau, \zeta \in \langle 0, \Delta \rangle$ and choose $\Delta_1 > 0$ so small that $\psi(\vartheta, t) > 0$ for $\vartheta \in \langle u_1, u_1 + \Delta_1 \rangle$, $t \in \langle t_1, t_1 + \Delta \rangle$. Let us put $K_2 = \min \psi(\vartheta, t)$, where $\vartheta \in \langle u_1, u_1 + \Delta_1 \rangle$, $t \in \langle t_1, t_1 + \Delta \rangle$, $\tau_0 = \min (\Delta, (K_2/K_1)\, \Delta_1, (K_2/K_1)(u_2 - u_1))$ and take the equation

$$\frac{\mathrm{d}\vartheta}{\mathrm{d}\tau} = \frac{\varphi(\tau, \zeta)}{\psi(\vartheta, t_1 + \zeta)}, \quad \vartheta(0, \zeta) = u_1, \quad \tau \in \langle 0, \tau_0 \rangle, \quad \zeta \in \langle 0, \Delta \rangle .$$

The function φ is defined on $\langle 0, \tau_0 \rangle$ and it holds

$$u_1 \leqq \vartheta(\tau, \zeta) \leqq u_1 + \int_0^\tau \frac{\varphi(\tau, \zeta)}{\psi(\vartheta, t_1 + \zeta)}\, \mathrm{d}\tau \leqq u_1 + \frac{K_1}{K_2}\tau \leqq \min (u_1 + \Delta_1, u_2) .$$

Hence, the solution ϑ exists on the interval $\langle 0, \tau_0 \rangle$ for $\zeta \in \langle 0, \Delta \rangle$.

Let us define the function $h(\tau) = g(\vartheta(\tau, \tau_0), t_1 + \tau_0)$, $\tau \in \langle 0, \tau_0 \rangle$. Then it holds

$$\frac{\mathrm{d}h(\tau)}{\mathrm{d}\tau} = \frac{\partial g(\vartheta(\tau, \tau_0), t_1 + \tau_0)}{\partial \vartheta} \cdot \frac{\partial \vartheta(\tau, \tau_0)}{\partial \tau} = \varphi(\tau, \tau_0)\, Cg(\vartheta(\tau, \tau_0), t_1 + \tau_0) =$$

$$= \varphi(\tau, \tau_0)\, Ch(\tau) \quad \text{for} \quad \tau \in \langle 0, \tau_0 \rangle .$$

$$h(0) = g(\vartheta(0, \tau_0), t_1 + \tau_0) = g(u_1, t_1 + \tau_0) = e^{(A + u_1 C)(t_1 + \tau_0)}\omega = f(0, \tau_0) .$$

Thus we have got $h(\tau) = f(\tau, \tau_0)$ for $\tau \in \langle 0, \tau_0 \rangle$. If we put $\tau = \tau_0$, we get $f(\tau_0, \tau_0) = g(\vartheta(\tau_0, \tau_0), t_1 + \tau_0)$ i.e.

$$e^{(A + u_2 C)\tau_0}\, e^{(A + u_1 C)t_1}\, \omega = e^{(A + \vartheta(\tau_0, \tau_0)C)(t_1 + \tau_0)}\, \omega .$$

If it is $\vartheta(\tau_0, \tau_0) = u_2$, then the proof is finished. If it is $\vartheta(\tau_0, \tau_0) < u_2$ and if the set

$$E(t \in (0, t_1 + \tau_0 \rangle;\ Ce^{(A + \vartheta(\tau_0, \tau_0)C)t}\, \omega = 0)$$

is empty, we get the original problem and we can repeat the whole procedure. So we get (finite or infinite) sequences (let us write only the case of infinite sequences):

$$u_1 < \vartheta_0 < \vartheta_1 < \vartheta_2 \ldots < u_2 , \quad \tau_0, \tau_1, \tau_2, \ldots ; \quad \tau_i > 0 , \quad i = 0, 1, 2, \ldots$$

such that either $\sum_{i \geq 0} \tau_i = t_2$, then $x = e^{(A+\tilde{\vartheta}C)(t_1+t_2)}\omega$, where $\tilde{\vartheta} = \lim_{i \to \infty} \vartheta_i$ or $\sum_{i \geq 0} \tau_i = \tilde{\tau} < t_2$. If in the second case the set

$$F_1 = E(t \in (0, t_1 + \tilde{\tau}\rangle;\ Ce^{(A+\tilde{\vartheta}C)t}\omega = 0)$$

is empty, we could use our procedure in the case

$$x = e^{(A+u_2C)(t_2-\tilde{\tau})}\, e^{(A+\tilde{\vartheta}C)(t_1+\tilde{\tau})}\, \omega\,.$$

If $F_1 \neq \emptyset$, then there exists $\bar{t} = \max_{t \in F_1} t$ (F_1 is according to lemma 3.3 finite). Then we define $u(t) = \tilde{\vartheta}$ for $t \in \langle 0, \bar{t}\rangle$ and we get the original problem:

$$x = e^{(A+u_2C)(t_2-\tilde{\tau})}\, e^{(A+\tilde{\vartheta}C)(t_1+\tilde{\tau}-\bar{t})}\, x(\bar{t}, u)\,.$$

According to lemma 3.3 we must reach the point x after finite number of such steps and the proof is finished.

Theorem 3.1. $\omega \in E_n$, $\dim \mathscr{V}(\omega) = 1$, $T > 0$, $x \in \mathscr{S}_\omega(T)$. *Then such piecewise constant control* $u \in M(0, 1)$ *exists that it holds*:

1) $x = x(T, u)$,

2) *u has a finite number of discontinuities. Moreover, if $t \in \langle 0, T\rangle$ is a point of discontinuity of u, then* $Cx(t, u) = 0$.

Proof. It exists such $v \in M(0, 1)$ that $x = x(T, v)$. Let $v_k \in M(0, 1)$, $k = 1, 2, \ldots$ be a sequence of piecewise constant on $\langle 0, T\rangle$ controls such that $v_k \to v$ asymptotically on $\langle 0, T\rangle$. According to lemma 3.5 for each integer k it exists a piecewise constant on $\langle 0, T\rangle$ control w_k that has only such discontinuity-points t at which $Cx(t, \omega_k) = 0$ and it holds $x(T, w_k) = x(T, v_k)$.

We can choose from the sequence w_k, $k = 1, 2, \ldots$, such subsequence (let it be the original sequence) that converges asymptotically to a control w. Then according to lemma 2.7 it holds

$$x(t, w_k) \to x(t, w) \quad \text{uniformly on} \quad \langle 0, T\rangle\,. \tag{3.7}$$

According to lemma 3.3 the set $E(t \in \langle 0, T\rangle;\ Cx(t, w) = 0)$ is finite. Let us denote its elements by t_i, $i = 1, 2, \ldots, r$, $t_1 < t_2 < t_3 \ldots < t_r$.

Let us choose $\varepsilon \in (0, \frac{1}{2}(t_2 - t_1))$ and put $\varrho = \min \|y - x(t, w)\|$, where $y \in E(x \in E_n;\ Cx = 0)$, $t \in \langle t_1 + \varepsilon, t_2 - \varepsilon\rangle$.

Evidently $\varrho > 0$. According to (3.7) it exists such integer k_0 that for all $k > k_0$ it holds $\|x(t, w_k) - x(t, w)\| < \varrho$, where $t \in \langle t_1 + \varepsilon, t_2 - \varepsilon\rangle$. Thus the control w_k, $k > k_0$, is on $\langle t_1 + \varepsilon, t_2 - \varepsilon\rangle$ constant. The limit w must be also constant on

$\langle t_1 + \varepsilon, t_2 - \varepsilon\rangle$ for all $\varepsilon \in (0, \frac{1}{2}(t_2 - t_1))$. Hence, w is constant on (t_1, t_2), what was to be proved.

Theorem 3.2. $\omega \in \mathrm{E}_n$, $\dim \mathscr{V}(\omega) = 1$, $T > 0$, $x \in \mathscr{S}_\omega(T)$. *Let the matrix* C *be regular. Then there exists such constant control* $u \in M(0, 1)$ *that* $x = x(T, u)$.

The proof follows immediately from theorem 3.1, as the solution of the equation $\mathrm{C}x = 0$ is just only $x = 0$.

Example. Let us take

$$\mathrm{A} = \begin{pmatrix} 1 & \frac{1}{2} & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathrm{C} = \begin{pmatrix} 2 & \gamma_2 & \gamma_3 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \omega \in \mathrm{E}_3, \ \omega_3 = 0$$

and for $t \in \langle 0, T\rangle$ put:

$$f(t) = e^{\mathrm{A}(T-t)}\, e^{(\mathrm{A}+\mathrm{C})t}\, \omega = e^T \begin{pmatrix} e^{2t}\left(\omega_1 + \dfrac{2\gamma_2 + 1}{4}\,\omega_2\right) + \dfrac{(T-t)\,2 - (2\gamma_2 + 1)}{4}\,\omega_2 \\ \omega_2 \\ 0 \end{pmatrix}$$

$$g(t) = e^{(\mathrm{A}+(t/T)\mathrm{C})T}\, \omega = e^T \begin{pmatrix} e^{2t}\omega_1 + (e^{2t} - 1)\dfrac{2t\gamma_2 + T}{4t}\,\omega_2 \\ \omega_2 \\ 0 \end{pmatrix}.$$

Evidently $f(0) = g(0)$, $f(T) = g(T)$.

If we take such ω that satisfies the inequalities:

$$2\omega_1 + \gamma_2\omega_2 < 0\,, \quad 4(2\omega_1 + \gamma_2\omega_2) + \omega_2 > 0\,,$$

then it holds $\dim \mathscr{V}(\omega) = 1$ and for $T > 1$ the first coordinate of the vector f, resp. g, at first decreases and then increases, resp. still increases, on $\langle 0, T\rangle$.

Hence, we cannot reach every point of $\mathscr{S}_\omega(T)$, where $T > 1$, by a constant control at the time T.

References

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Author's address: Praha 1, Žitná 25, ČSSR (Matematický ústav ČSAV).

Резюме

РЕШЕНИЕ В ЦЕЛОМ УРАВНЕНИЯ УПРАВЛЕНИЯ $\dot{x} = (\mathrm{A}(1 - u) + \mathrm{B}u)\, x$

ЯН КУЧЕРА (Jan Kučera), Прага

В данной работе исследуется множество $\mathscr{S}_\omega(T)$ или $\mathrm{S}_\omega(T)$ всех точек, в которые возможно попасть из данной начальной точки ω по решению уравнения (2.1) за данное время T или во время менше или равно T. Доказано в теоремах 2.1 и 2.2 что эти множества являются замыканием некоторых многообразий, которые локально заданы отображениями (2.11) и (2.13) или отображением (1.2). Размерность этих многообразий равняется размерности некоторого распределения, которое введено в [2], в начальной точке ω.

В третьем параграфе изучается более подробно случай когда $\mathscr{S}_\omega(T)$ кривая. Потом я каждую из точек $\mathscr{S}_\omega(T)$ можно попасть при помощи по частьях постоянного управления u, точки перерыва которого соответствуют пересечениям $x(t, u)$ с пространством решений уравнения $(\mathrm{A} - \mathrm{B})\, x = 0$.