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Vlastimil Dlab

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A REMARK TO A PAPER OF GH. PIC

VLASTIMIL DLAB, Canberra

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Throughout the paper, G denotes a group and, for a natural n, $\{G_n\}$ – the (fully invariant) subgroup of G generated by all $g \in G$ with $g^n = 1$. The symbol $[n_1, n_2, \ldots$ $\ldots, n_k]$ is used to denote the least common multiple of the natural numbers n_i $(1 \leqq$ $\leqq i \leqq k)$.

In his paper [3], GH. PIC has attempted to "dualize" the results of [1] on the powers $\{G^n\}$ of a group G to the case of the subgroups $\{G_n\}$. The following simple theorem improves the results of [3].

Theorem. *If, for* $1 \leqq i \leqq k$, *the subgroups* $\{G_{n_i}\} \subseteq G$ *possess a property* $\mathscr{P}$ *of type* $(*)$, *then* $\{G_{[n_1,n_2,\ldots,n_k]}\}$ *possesses the property* $\mathscr{P}$, *as well. Here, a property* $\mathscr{P}$ *is said to be of type* $(*)$ *if any group possessing* $\mathscr{P}$ *is locally nilpotent and if, for any finite set* π *of primes* p, *the direct product* $\prod_{p\in\pi} G_p$ *of* p*-groups* G_p *possesses* $\mathscr{P}$ *if and only if all* G_p *possess* $\mathscr{P}$.

Proof. In view of the relation $[n_1, n_2, \ldots, n_{k-1}, n_k] = [[n_1, n_2, \ldots, n_{k-1}], n_k]$, we can restrict ourselves to $k = 2$. Thus, $\{G_{n_1}\}$ and $\{G_{n_2}\}$ are locally nilpotent normal subgroups of G; therefore, $\{G_{[n_1,n_2]}\} = \{G_{n_1}\}\{G_{n_2}\}$ is locally nilpotent[1]) (see e.g. K. HIRSCH [2]). Being generated by elements of finite order, the subgroups $\{G_{n_1}\}$, $\{G_{n_2}\}$ and $\{G_{[n_1,n_2]}\}$ are the direct products of their respective Sylow subgroups, finite in number: $\{G_{[n_1,n_2]}\} = \prod_{p\in\pi} G_p$ with $G_p \neq 1$ and $\{G_{n_i}\} = \prod_{p\in\pi} G_p^{(i)}$ for $i = 1, 2$. Moreover, if p^{t_i} is the highest power of p dividing n_i, then all elements $g \in G$ such that $g^{p^{t_i}} = 1$ generate $G_p^{(i)}$ $(i = 1, 2)$. Therefore, since $[n_1, n_2] = p^{\max(t_1,t_2)}n'$ with n' relatively prime to p, $G_p = G_p^{(i)}$ for one of $i = 1, 2$. Thus, all G_p possess $\mathscr{P}$, and, consequently, $\{G_{[n_1,n_2]}\}$ possesses $\mathscr{P}$, as required.

Since the properties of being nilpotent, nilpotent regular, abelian, direct product of cyclic groups or cyclic are, evidently, of type $(*)$, we get immediately the validity of statements "dual" to those of [1]; this, in particular, shows that the counter-example of § 3 in [3] is defective.

[1]) In the same manner, $\{G^{(n_1,n_2,\ldots,n_k)}\}$ is (locally) nilpotent provided that all $\{G^{n_i}\}$ are (locally) nilpotent; this answers the third question of F. SZÁSZ [4]).

Also, an immediate consequence of Theorem reads that if every cyclic subgroup of group G has the from $\{G_n\}$ for a suitable n, then G is torsion and locally cyclic, i.e. G is a subgroup of the (multiplicative) group of all complex roots of unity.

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Author's address: Department of Mathematics, Institute of Advanced Studies, Australian National University, Canberra, A.C.T.