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COMPACT SEMITOPOLOGICAL SEMIGROUPS AND AFFINE SEMIGROUPS

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A *semitopological semigroup* is a semigroup and a Hausdorff topological space such that multiplication is continuous in each variable separately. The purpose of this note is to state a few theorems about compact semitopological semigroups.

Compact semitopological semigroups arise naturally in the study of weakly almost periodic functions by way of the weakly almost periodic compactification of a semitopological semigroup (in particular, of a topological semigroup or, even, of a topological group). GLICKSBERG and DE LEEUW [3] show that the C^*-algebra $W(S)$ of all weakly almost periodic functions on a semitopological semigroup S is isomorphic to the C^*-algebra of all continuous functions on the weakly almost periodic compactification ΩS of S. They also show that the invariant mean theory of $W(S)$ is reflected in the ideal theory of ΩS (see corollary 2.2 below).

At the present stage of development, no coherent structure theory for compact semitopological semigroups exists. In working with compact semitopological semigroups, the observation is quickly made that most of the techniques of the theory of compact topological semigroups are not applicable. Furthermore, the one-point compactification of the additive group of real numbers with the point at infinity as a zero already exhibits several pathologies (in terms of compact topological semigroups: for example, the group of units is not closed and the Čech cohomology is not carried by the minimal ideal). The main tool used to obtain the results listed herein was the theorem of Ellis on locally compact groups of transformations [4], which gives us joint continuity of multiplication for certain parts of the semigroup.

An *affine semigroup* is a semigroup S in a real vector space V such that S is convex and the functions $s \to st$ and $s \to ts$ are affine for every $t \in S$. Compact semitopological affine semigroups also arise naturally in the study of weakly almost periodic functions as the closed convex hull $\overline{\text{co}}\,(\Omega S)$ of the weakly almost periodic compactification ΩS of a semitopological semigroup S. (ΩS is a semigroup of operators on $W(S)$.) Theorem 2 makes a connection between the ideal structure of a compact semitopological affine semigroup and the ideal structure of a generating subsemigroup. For any semigroup S, let $E(S)$ denote the set of idempotents in S. An outstanding feature of a compact semitopological affine semigroup S is that $M(S) = E(M(S))$; that is, the minimal ideal of S consists of idempotents. In the case of convex semigroups of operators, this fact with Theorem 2 yields certain ergodic-type statements (see corollaries 2.1 and 2.2). Theorem 2 also combines with a theorem derived from the RYLL-NARDZEWSKI fixed point theorem [1] to yield results analogous to some of COHEN and COLLINS [2].

The first fundamental theorem of compact topological semigroups ([5], p. 57) states, among other things, that a compact topological semigroup S contains a unique minimal ideal $M(S)$ which is a compact topological paragroup. (A *paragroup* is a semigroup $[X, G, Y]_\sigma$ consisting of the set $X \times G \times Y$ together with the multiplication

$$(x, g, y)(x', g', y') = (x, g[y, x']\, g', y'),$$

where X and Y are sets, G is a group, and $\sigma : Y \times X \to G$ is a sandwich function with $\sigma(y, x) = [y, x]$.) Theorem 1 is the analogous theorem for compact semitopological semigroups.

The writer wishes to note his obligation to Professor KARL HEINRICH HOFMANN for his help and advice in the development of this material.

Theorem 1. *Let S be a compact semitopological semigroup. Then S contains a minimal ideal $M(S)$ which is a paragroup. Specifically, if e is an idempotent in $M(S)$ and if $G = eSe$, $X = E(Se)$, $Y = E(eS)$, and $\sigma : Y \times X \to G$ is defined by*

$$\sigma(y, x) = [y, x] = yx,$$

then $M(S)$ is algebraically isomorphic to $[X, G, Y]_\sigma$. Moreover, if $\eta_e : S \to [X, G, Y]_\sigma$ with

$$\eta_e(s) = (\varepsilon(se), ese, \varepsilon(es)),$$

where $\varepsilon : M(S) \to E(M(S))$ is the function which assigns to each $t \in M(S)$ the identity of the group containing t, and if $\vartheta_e : [X, G, Y]_\sigma \to M(S)$ with

$$\vartheta_e(x, g, y) = xgy,$$

then the following statements hold:

(i) *$[X, G, Y]_\sigma$ is a compact semitopological semigroup relative to the product topology.*

(ii) *$\varepsilon \mid X$ and $\varepsilon \mid Y$ are continuous.*

(iii) (1) *η_e is a continuous function.*

(2) *$\eta_e \mid \overline{M(S)}$ is a morphism of compact semitopological semigroups.*

(3) *$\eta_e \mid M(S)$ is bijective.*

(iv) *$\overline{M(S)}^2 = M(S)$; specifically, if $s, t \in \overline{M(S)}$, then $st = \vartheta_e \eta_e(st)$.*

(v) *The following statements are equivalent*:

(a) $\eta_e \mid M(S)$ *is an isomorphism of semitopological semigroups.*

(b) ϑ_e *is continuous.*

(c) $\vartheta_e \eta_e : S \to M(S)$ *is a continuous retraction.*

(*d*) $\overline{M(S)} = M(S)$.

(Examples show that $M(S)$ need not be closed.)

Theorem 2. *Let T be a compact semitopological affine semigroup. Suppose $S \subseteq T$ is a compact subsemigroup such that T is the closed convex hull $\overline{co}\,(S)$ of S. If $M(S)$ is a minimal left (right) ideal of S, then $M(T)$ is a minimal left (right) ideal of T. In case T has an identity, the converse holds.*

Natural examples of affine semigroups are convex semigroups of linear operators of a locally convex topological vector space E. Theorem 2, together with some results of de Leeuw and Glicksberg ([3], p. 92 and pp. 78–79), yields the following two corollaries:

Corollary 2.1. *Let E be a Banach space. Suppose that S is a weakly almost periodic semigroup of operators on E (i.e., S contains the identity operator and Sx is relatively weakly compact for every $x \in E$). Let T be the convex hull* co (S) *of S. Let $\bar{S}$, respectively, $\bar{T}$, denote closures in the weak operator topology. Then the following statements are equivalent*:

(a) $M(\bar{S})$ *is a minimal right (left) ideal of* $\bar{S}$.

(b) $M(\bar{T})$ *is a minimal right (left) ideal of* $\bar{T}$.

(c) $E(M(\bar{S}))$ *is a right (left) zero semigroup.*

(d) $M(\bar{T})$ *is a right (left) zero semigroup.*

Moreover, statements (a)–(d) *with "right" read every time are equivalent to each of the following statements*:

(e) $\bar{T}x = \overline{co}\,(Sx)$ *contains a fixed point of T.*

(f) $\bar{T}x = \overline{co}\,(Sx)$ *contains a fixed point of S.*

And statements (a)–(d) *with "left" read every time are equivalent to each of the following statements*:

(e′) $\{x \in E : O \in \bar{S}x\}$ *is a closed S-invariant subspace of E.*

(f′) $\{x \in E : O \in \bar{T}x = \overline{co}\,(Sx)\}$ *is T-invariant.*

Applying corollary 2.1 to weakly almost periodic functions, we get the following result:

Corollary 2.2. *Let S be a semitopological semigroup with identity. Let ΩS denote the weakly almost periodic compactification of S, $W(S)$ the C^*-algebra of all weakly almost periodic functions on S, and for $f \in W(S)$, let $O(f)$ denote the orbit* *of f given by*

$$O(f) = \{{}^s f : s \in S\},$$

where ${}^s f(x) = f(xs)$, $x \in S$. *Then the following statements are equivalent*:

(a) $M(\Omega S)$ *is a minimal right ideal of* ΩS.
(b) $W(S)$ *has a left invariant mean.*
(c) $\{f \in W(S) : \text{if } g \in O(f)^-, \text{ then } f \in O(g)^-\}$, *where closure are taken in the weak topology of* $W(S)$, *is the space* $U(S)$ *of all uniform limits of linear combinations of coefficients of finite dimensional unitary representations of* S.
(d) *The closed convex hull* $\overline{co}\,(O(f))$ *of the orbit of f contains a constant function for every* $f \in W(S)$.

And the following statements are equivalent:

(a) $M(\Omega S)$ *is a minimal left ideal of* ΩS.
(b) $W(S)$ *has a right invariant mean.*
(c) $\{f \in W(S) : O \text{ is a weak cluster point of } O(f)\}$ *is a closed translation invariant subspace* $N(S)$ *of* $W(S)$.
(d) *If* $O \in \overline{co}\,(O(f))$ *and* $g \in \text{co}\,(O(f))$, *then* $O \in \overline{co}\,(O(g))$.

Finally, the following statements are equivalent:

(a) $M(\Omega S)$ *is a compact topological group.*
(b) $W(S)$ *has an invariant mean.*
(c) $W(S) = N(S) \oplus U(S)$.
(d) $W(S) = N'(S) \oplus \mathscr{C}$, *where* $N'(S)$ *is a closed translation invariant subspace of* $W(S)$ *consisting of all* $f \in W(S)$ *such that* $O \in \overline{co}\,(O(f))$.

The Ryll-Nardzewski fixed point theorem [1] yields the following theorem:

Theorem 3. *Suppose that* S *is a compact semitopological semigroup and that* $T \subseteq S$ *is a dense subsemigroup with identity. Further suppose that* T *is topologically left simple (every left ideal of* T *is dense in* T*). Then* $M(S)$ *is a compact topological group.*

Remark. In a compact topological semigroup, maximal groups are always closed; however, this does not remain true for compact semitopological semigroups. Therefore it is important to observe that, as a consequence of Theorem 3, the closure $\overline{H(e)}$ of a maximal group $H(e)$ in a compact semitopological semigroup has a compact topological group as minimal ideal.

Theorems 2 and 3 combine to yield the following corollaries, which are analogous to results of Cohen and Collins [2].

Corollary 3.1. *If* S *is a compact semitopological affine semigroup with identity, and if* $S = \overline{co}\,(T)$, *where* T *consists of left units of* S, *then* S *has a zero.*

Remark. In sharp contrast to the situation in compact topological semigroups, a compact semitopological semigroup may have left units which are not right units. (Consider the one-point compactification of the discrete bicyclic semigroup with the point at infinity as a zero.)

Definition. Let K be a convex subset of a vector space V. A point $p \in K$ is called a *border point* if there is a point $q \in S$, $q \neq p$, such that if L is the line generated by p and q, then the line segment $L \cap K$ has p as an endpoint.

Corollary 3.2. *Let S be a compact semitopological affine semigroup. Let*

$$H = \bigcup\{H(e) : e \in E(S) \setminus M(S)\},$$

where $H(e)$ is the maximal group containing e. Then H is contained in the border of S.

It is planned that a complete discussion will appear among other results in a joint publication with KARL HEINRICH HOFMANN (Compact semitopological semigroups and weakly almost periodic functions (to appear)).

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