Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 19 (1969), No. 3, 534–546

Persistent URL: http://dml.cz/dmlcz/100920

Czechoslovak Mathematical Journal, 19 (94) 1969, Praha

ON COUPLES OF LINE CONGRUENCES WITH THE PROJECTIVE CONNECTION HAVING ROZENFELD IMAGE WITH THE CHARACTER SMALLER THAN 5

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(Received January 6, 1969)

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Let a couple L_1, L_2 of the line congruences in a three dimensional projective space P_3 be given formed by the corresponding lines $p = p(u, v)$, $q = q(u, v)$ where $(u, v) \in \Omega \subset E_2$. To each couple of the corresponding lines we associate a frame consisting of linearly independent analytic points A_1, A_2, A_3, A_4 (vertices) such that

$$[A_1, A_2, A_3, A_4] = 1 \tag{1.1}$$

and $(A_1, A_2) = p$, $(A_3, A_4) = q$.

Fundamental equations of this moving frame are

$$\mathrm{d}A_i = \omega_i^k A_k\,, \quad i,\ k = 1, 2, 3, 4 \tag{1.2}$$

where ω_i^k are linear differential forms depending on two principal and fifteen secondary parameters and satisfying the structure equations of the projective space

$$\mathrm{d}\omega_i^k = \omega_i^j \wedge \omega_j^k\,, \quad j = 1, \ldots, 4\,. \tag{1.3}$$

The normalization (1.1) of vertices gives

$$\omega_1^1 + \omega_2^2 + \omega_3^3 + \omega_4^4 = 0\,, \quad \pi_1^1 + \pi_2^2 + \pi_3^3 + \pi_4^4 = 0 \tag{1.4}$$

where π_i^k is the usual notation for ω_i^k if the differentials of the principal parameters annul.

If we denote by $[i, k]$ the analytic line $[A_i, A_k]$, then

$$\mathrm{d}[i, k] = (\omega_i^i + \omega_k^k)\,[i, k] + \omega_i^j[j, k] + \omega_k^j[i, j]\,, \quad \text{for} \quad j \neq i, k\,. \tag{1.5}$$

From (1.5) for $i = 1$, $k = 2$ and for $i = 3$, $k = 4$ we obtain that the principal forms are

$$\omega_1^3, \omega_1^4, \omega_2^3, \omega_2^4\,, \quad \omega_3^1, \omega_3^2, \omega_4^1, \omega_4^2\,. \tag{1.6}$$

We shall specialize the frame in such a way that we place its vertices A_i into the foci of the congruences L_1, L_2. For π_i^k we have $\pi_i^k = 0$ $(i \neq k)$.

Now we assume the forms ω_1^3 and ω_3^1 to be independent (here we eliminate from our considerations the couple Θ of Popov, see [2]) and since the principal forms in (1.6) depend on two independent parameters, we have

$$\begin{aligned} \omega_1^4 &= a_1\omega_1^3 + a_2\omega_3^1\,, \quad \omega_2^3 = \alpha\omega_1^3 + \beta\omega_3^1\,, \\ \omega_2^4 &= b_1\omega_1^3 + b_2\omega_3^1\,, \quad \omega_3^2 = a_2^*\omega_1^3 + a_1^*\omega_3^1\,, \\ \omega_4^1 &= \beta^*\omega_1^3 + \alpha^*\omega_3^1\,, \quad \omega_4^2 = b_2^*\omega_1^3 + b_1^*\omega_3^1\,. \end{aligned} \tag{1.7}$$

Considering that the points of the frame $\{A_i\}_{i=1}^4$ are situated in the foci of the congruences L_1, L_2 we obtain

$$\begin{gathered} a_2 = a_2^* = 0\,, \\ \alpha b_2 - \beta b_1 = \alpha^* b_2^* - \beta^* b_1^* = 0\,, \\ (b_2 - \beta a_1)(b_2^* - \beta^* a_1^*) \neq 0\,. \end{gathered} \tag{1.8}$$

By the exterior differentiation of (1.7) and by Cartan's lemma we obtain for the remaining secondary parameters

$$\begin{gathered} \delta\alpha + \alpha(\pi_1^1 - \pi_2^2) = 0\,, \qquad \delta\alpha^* + \alpha^*(\pi_3^3 - \pi_4^4) = 0\,, \\ \delta\beta + \beta(2\pi_3^3 - \pi_1^1 - \pi_2^2) = 0\,, \quad \delta\beta^* + \beta^*(2\pi_1^1 - \pi_3^3 - \pi_4^4) = 0\,. \end{gathered} \tag{1.9}$$

Using these relations we make the following normalization of vertices

$$\begin{aligned} &1^\circ \qquad \alpha = 1\,, && \pi_1^1 - \pi_2^2 = 0\,, \\ &2^\circ \qquad \alpha^* = 1\,, && \pi_3^3 - \pi_4^4 = 0\,, \\ &3^\circ \quad \beta\beta^{*-1} = 1\,, && 3(\pi_1^1 - \pi_3^3) + \pi_2^2 - \pi_4^4 = 0\,. \end{aligned} \tag{1.10}$$

By this normalization we exclude the so called T – couple and therefore also the so called stratifiable couple of congruences (see [2], p. 71).

By the choice (1.10) the frame $\{A_i\}_{i=1}^4$ of the couple of congruences L_1, L_2 is completely canonized. If we denote with use of (1.8)

$$\begin{gathered} \frac{b_2}{\beta} = \frac{b_1}{\alpha} = \varphi\,, \quad a_1 = \psi\,, \\ \frac{b_2^*}{\beta^*} = \frac{b_1^*}{\alpha^*} = \varphi^*\,, \quad a_1^* = \psi^*\,, \\ (\varphi - \psi)(\varphi^* - \psi^*) \neq 0\,, \end{gathered} \tag{1.8'}$$

we may write the equations of the infinitesimal motion of our canonical frame $\{A_i\}$ as follows:

$$
\begin{aligned}
(1.11)\quad dA_1 &= (\alpha_1^1\omega_1^3 + \beta_1^1\omega_3^1)\,A_1 + (\sigma_1\omega_1^3 + \sigma_2\omega_3^1)\,A_2 + \omega_1^3 A_3 + \psi\omega_1^3 A_4\,,\\
dA_2 &= (\gamma_1\omega_1^3 + \gamma_2\omega_3^1)\,A_1 + (\alpha_2^2\omega_1^3 + \beta_2^2\omega_3^1)\,A_2 + (\omega_1^3 + \beta\omega_3^1)\,A_3 + \\
&\qquad + \varphi(\omega_1^3 + \beta\omega_3^1)\,A_4\,,\\
dA_3 &= \omega_3^1 A_1 + \psi^*\omega_3^1 A_2 + (\alpha_3^3\omega_1^3 + \beta_3^3\omega_3^1)\,A_3 + \\
&\qquad + (\sigma_2^*\omega_1^3 + \sigma_1^*\omega_3^1)\,A_4\,,\\
dA_4 &= (\beta\omega_1^3 + \omega_3^1)\,A_1 + \varphi^*(\beta\omega_1^3 + \omega_3^1)\,A_2 + (\gamma_2^*\omega_1^3 + \gamma_1^*\omega_3^1)\,A_3 + \\
&\qquad + (\alpha_4^4\omega_1^3 + \beta_4^4\omega_3^1)\,A_4\,.
\end{aligned}
$$

Here the invariants α_i^i, β_i^i are related by

$$(1.12)\qquad \alpha_1^1 + \alpha_2^2 + \alpha_3^3 + \alpha_4^4 = 0\,,\quad \beta_1^1 + \beta_2^2 + \beta_3^3 + \beta_4^4 = 0\,.$$

From (1.11) we obtain

$$(1.13)\qquad d\omega_1^3 = Q(\omega_1^3 \wedge \omega_3^1)\,,\quad d\omega_3^1 = -Q^*(\omega_1^3 \wedge \omega_3^1)$$

where

$$Q = \beta_3^3 - \beta_1^1 + \sigma_1\beta - \sigma_2 + \psi\gamma_1^*\,,\quad Q^* = \alpha_1^1 - \alpha_3^3 + \sigma_1^*\beta - \sigma_2^* + \psi^*\gamma_1\,.\ ^{1})$$

Relations (1.13) together with equations of structure (1.3) are integrability conditions for (1.11), see [1], pp. 17–18. Since there are 13 independent quadratic relations in integrability conditions and since there are 19 unknown functions in (1.11) the couple of congruences is determined depending on 6 functions of 2 arguments.

From (1.11) there follows that

$$
\begin{aligned}
(1.14)\qquad & 1^\circ\ \ \Psi = A_3 + \psi A_4\,, && 3^\circ\ \ \Psi^* = A_1 + \psi^* A_2\,,\\
& 2^\circ\ \ \Phi = A_3 + \varphi A_4\,, && 4^\circ\ \ \Phi^* = A_1 + \varphi^* A_2
\end{aligned}
$$

are the points of intersection of the focal planes (having the tangent points A_1, A_2, A_3, A_4 with respect to the focal surfaces (A_1), (A_2), (A_3), (A_4)) with the rays (A_3, A_4), (A_1, A_2).

The quadric going through two consecutive rays (A_1, A_2) and (A_3, A_4) for $\omega_3^1 = 0$ and $\omega_1^3 = 0$ and through another ray (A_3, A_4) and (A_1, A_2), respectively, will be denoted by l and l^*, respectively. If we denote by $F_{12}(F_{34})$ the intersection point of the ray (A_1, A_2) $((A_3, A_4))$ with the polar of the point $A_4(A_2)$ with respect to $l(l^*)$ we obtain, respectively

$$(1.15)\qquad F_{12} = A_1 - A_2\,,\quad F_{34} = A_3 - A_4\,.$$

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[1]) Note that in [1], there is incorrectly exchanged the term $\beta_3^3 - \beta_1^1$ by $\alpha_1^1 - \alpha_3^3$.

From this there follows that the point

$$E_{12} = A_1 + A_2 \quad \text{and} \quad E_{34} = A_3 + A_4 \tag{1.16}$$

is harmonically conjugated to F_{12} or F_{34} according to A_1, A_3 or A_2, A_4, respectively. The uniquely determined unit point of the frame $\{A_i\}$ is then

$$I = E_{12} + E_{34} = A_1 + A_2 + A_3 + A_4 . \tag{1.17}$$

The geometric meaning of invariants $\varphi, \psi, \varphi^*, \psi^*$ can be easily derived from

$$\begin{aligned} \varphi &= DV(A_3, A_4, \Phi, E_{34}), \quad & \psi &= DV(A_3, A_4, \Psi, E_{34}), \\ \varphi^* &= DV(A_1, A_2, \Phi^*, E_{12}), \quad & \psi^* &= DV(A_1, A_2, \Psi^*, E_{12}) \end{aligned} \tag{1.18}$$

(where DV is used to denote the cross-ratio); (1.18) follows from (1.14) and (1.16).

Definition 1.1. The line congruence L in an n-dimensional projective space P_n $(n \geqq 3)$ is an arbitrary two-parameter system of lines $p = p(u, v)$ where the admissible values of the parameter u, v range over a certain two-dimensional domain Ω.

Then by the character m of the congruence L (see Švec [6], pp. 11–12) we understand the natural number defined in the following way.

Definition 1.2. Let $V_3(L)$ be the "point extension" of the congruence L, i.e. the three-dimensional manifold formed by all points of all lines of L. If m is the least dimension of the subspace $\tau(u, v)$ $(\subseteqq P_n)$ containing all tangent spaces of $V_3(L)$ at the points of a fixed line $p_0 = p(u_0, v_0) \in L$ then we say that p_0 posseses in L the character m. If the character of all lines $p(u, v) \in L$ is equal to the same number m, then m is said to be the character of L.

Remark 1.1. For the character $m = \dim \tau$ there is obviously $1 \leqq \dim \tau \leqq 5$ (see Švec [6], p. 12).

Denoting by M_1, M_2, M_3, M_4, M_5 and M_6, respectively the $K -$ points which are Klein-Segre images (in a P_5) of the lines $(A_1, A_2), (A_3, A_4), (A_2, A_3), (A_1, A_4), (A_1, A_3)$ and (A_4, A_2), respectively then $(A_1, A_2) = p \in L_1(p)$, $(A_3, A_4) = q \in L_2(q)$. M_1, M_2 generate in P_5 two-dimensional manifolds (M_1) and (M_2) since they depend also only on two parameters u, v.

The line correspondence $C : [1,2] \to [3,4]$ betwen the congruences L_1, L_2 induces in P_5 the point correspondence $C^* : M_1 \to M_2$. The line joining the corresponding points of the manifolds (M_1) and (M_2) is called their Rozenfeld image.

Definition 1.3. By the Rozenfeld image of a couple of congruences $L_1(p), L_2(q)$ of P_3 we understand the line congruence $L_{12}(s)$ formed by ali Rozenfeld images s belonging to L_1, L_2.

To $L_{12}(s)$ in P_5 we adjoin the frame formed by the analytic points $M_1, M_2, \ldots, M_6$ (brief notation: $\{M_i\}$).

Remark 1.2. It is a well-known fact (see [2], p. 196) that the character of Rozenfeld image of a couple of congruences L_1, L_2 of P_3 is equal to 3 iff L_1, L_2 is a T – couple. In general, the character m of a congruence is equal to 5.

In 1962 Professor J. KLAPKA posed the following question. What is the geometric meaning of the case in which the above congruence L_{12} of Klein space P_5 has the character $m = 4$? This question will be investigated in the following.

Theorem 1.1. *The character m of the congruence $L_{12}(s)$ derived from the couple of congruences $L_1(p), L_2(q)$ of P_3 is smaller than 5 iff the invariants of Ivlev of $L_1(p), L_2(q)$ satisfy*

$$\varphi\psi^* - \varphi^*\psi = 0\,. \tag{1.19}$$

Proof. 1° Necessity: Let $m < 5$. The subspace determining m is generated by the points

$$\begin{aligned} &(1.20)\quad M_1\\ &\quad M_2\\ d^{I}M_1 &= \{(\alpha_1^1 + \alpha_2^2)\,M_1 - M_3 + \varphi M_4 + M_5 + \psi M_6\}\,\omega_1^3\,,\\ d^{II}M_1 &= \{(\beta_1^1 + \beta_2^2)\,M_1 + \beta\varphi M_4 + \beta M_5\}\,\omega_3^1\,,\\ d^{I}M_2 &= \{(\alpha_3^3 + \alpha_4^4)\,M_2 - \varphi^*\beta M_3 - \beta M_5\}\,\omega_1^3\,,\\ d^{II}M_2 &= \{(\beta_3^3 + \beta_4^4)\,M_2 - \varphi^* M_3 + M_4 - M_5 - \psi^* M_6\}\,\omega_3^1 \end{aligned}$$

where e.g. $d^{I}M_1 = (dM_1)_{\omega_3{}^1=0}$, $d^{II}M_1 = (dM_1)_{\omega_1{}^3=0}$. $m < 5$ if the rank of the matrix

$$\left\|\begin{matrix} 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ \alpha_1^1 + \alpha_2^2 & 0 & -1 & \varphi & 1 & \psi\\ \beta_1^1 + \beta_2^2 & 0 & 0 & \beta\varphi & \beta & 0\\ 0 & \alpha_3^3 + \alpha_4^4 & -\varphi^*\beta & 0 & -\beta & 0\\ 0 & \beta_3^3 + \beta_4^4 & -\varphi^* & 1 & -1 & -\psi^* \end{matrix}\right\|$$

is smaller than 6, i.e. if this matrix is singular. But this means that

$$\left\|\begin{matrix} -1 & \varphi & 1 & \psi\\ 0 & \beta\varphi & \beta & 0\\ -\varphi^*\beta & 0 & -\beta & 0\\ -\varphi^* & 1 & -1 & -\psi^* \end{matrix}\right\|$$

must be singular what gives the desired relation (1.19).

2° Sufficiency: Let there hold (1.19). The tangent space of the congruence L_{12} at $s = (M_1, M_2)$ is determined by

$$(1.21) \qquad (M_1, M_2, \mathrm{d}^{\mathrm{I}}M_1, \mathrm{d}^{\mathrm{II}}M_1, \mathrm{d}^{\mathrm{I}}M_2, \mathrm{d}^{\mathrm{II}}M_2) .$$

If we substitute (1.20) into (1.21) we obtain that

$$(M_1, M_2, \mathrm{d}^{\mathrm{I}}M_1, \mathrm{d}^{\mathrm{II}}M_1, \mathrm{d}^{\mathrm{I}}M_2, \mathrm{d}^{\mathrm{II}}M_2) = $$
$$= -\beta^2(\varphi\psi^* - \varphi^*\psi)(M_1, M_2, M_3, M_4, M_5, M_6) .$$

By assumption we have $\varphi\psi^* - \varphi^*\psi = 0$, but β and $(M_1, M_2, \ldots, M_6)$ are non-zero so that

$$(M_1, M_2, \mathrm{d}^{\mathrm{I}}M_1, \mathrm{d}^{\mathrm{II}}M_1, \mathrm{d}^{\mathrm{I}}M_2, \mathrm{d}^{\mathrm{II}}M_2) = 0 ,$$

i.e. the rank of this matrix is smaller than 6, which gives $m < 5$, q.e.d.

Theorem 1.2. *The character m of the congruence $L_{12}(s)$ of the couple of congruences $L_1(p)$, $L_2(q)$ from P_3 is smaller than 5 iff there is a common tangent linear complex in the corresponding lines $p \in L_1$, $q \in L_2$.*

Proof. 1° Necessity: Let $m < 5$. Then by Theorem 1.1, (1.19) is true. This implies by [1], p. 24 the existence of a common tangent linear complex in the mentioned corresponding lines.

2° Sufficiency: Let there exist a common tangent linear complex in the mentioned corresponding lines. Every tangent linear complex of $L_1(p)$ at $p = (A_1, A_2)$ has for Klein image the intersection of Klein quadric with the hyperplane of P_5 containing the plane determined by the points M_1, $\mathrm{d}^{\mathrm{I}}M_1$, $\mathrm{d}^{\mathrm{II}}M_1$. Further, every tangent linear complex of $L_2(q)$ at $q = (A_3, A_4)$ has for Klein image the hyperplane containing the plane determined by the points M_2, $\mathrm{d}^{\mathrm{I}}M_2$, $\mathrm{d}^{\mathrm{II}}M_2$. For the existence of a common tangent linear complex of congruences $L_1(p)$, $L_2(q)$ at P_3 in the corresponding lines p, q it is necessary and sufficient that these planes meet. But this is guaranteed (as it is obvious from the above) just by (1.19), q.e.d.

Theorem 1.3. *The character m of the congruence $L_{12}(s)$ in P_5 of the couple of congruences $L_1(p)$, $L_2(q)$ of P_3 is smaller than 5 iff*

$$(1.22) \qquad DV[A_1, A_2, \Psi^*, \Phi^*] = DV[A_3, A_4, \Psi, \Phi] .$$

Proof. 1° Necessity: Let $m < 5$. Then by Theorem 1.1, (1.19) is true, i.e. $\varphi\psi^* - \varphi^*\psi = 0$. Hence we obtain

$$(1.19') \qquad \frac{\psi}{\varphi} = \frac{\psi^*}{\varphi^*} .$$

But $DV[A_1, A_2, \Psi^*, \Phi^*] = \psi^*/\varphi^*$, $DV[A_3, A_4, \Psi, \Phi] = \psi/\varphi$. From (1.19′) the equality of the both cross-ratios follows.

2° Sufficiency: Let there hold the equality (1.22). Then (1.19′) is valid and hence (1.19) follows which is necessary and sufficient for $m < 5$, q.e.d.

2

Let the line congruence with the projective connection be defined as in [4], p. 291 in the following way: Let Ω be a region in the 2-dimensional Euclidean space E_2. Let to each point $(u, v) \in \Omega$ correspond a 3-dimensional projective space $P_3 = P_3(u, v)$, the so called local space. Let us consider a line $p = p(u, v)$ in this space. If points $(u_1, v_1) \in \Omega$, $(u_2, v_2) \in \Omega$ are joined by an arc $\gamma \subset \Omega$, then a certain collineation K_γ between the spaces $P_3(u_1, v_1)$ and $P_3(u_2, v_2)$ is determined. The line $p(u, v)$ is in this case the generatrix of König's manifold of the type $\mathscr{P}^2_{1,3}$, namely, of the congruence $\mathscr{L}_1$ with the projective connection. Let $q = q(u, v)$ be another line of the space $P_3(u,v)$ which is skew to $p(u, v)$ and which forms the generatrix of another congruence $\mathscr{L}_2$ with the projective connection. We have obtained a couple $\mathscr{L}_1$, $\mathscr{L}_2$ of congruences with the projective connection; this couple is particular since the corresponding lines $p(u, v)$, $q(u, v)$ lie in the same local space $P_3(u, v)$.

Under these assumptions the moving frame $\{A_1, \ldots, A_4\}$ can be chosen in $P_3(u, v)$ in such a way that

$$p(u, v) = [A_1, A_2], \quad q(u, v) = [A_3, A_4]$$

and simultaneously the following system of equations is satisfied

$$\nabla A_i = \omega_i^j A_j, \quad i, j = 1, \ldots, 4 \tag{2.1}$$

where

$$\omega_i^j = a_i^j(u, v)\, du + b_i^j(u, v)\, dv \tag{2.2}$$

and

$$[A_1, A_2, A_3, A_4] = 1\,. \tag{2.3}$$

From (2.3) by means of (2.1) there follows

$$\omega_1^1 + \omega_2^2 + \omega_3^3 + \omega_4^4 = 0\,. \tag{2.4}$$

If the arc γ joins the points (u_1, v_1), (u_2, v_2) in Ω as before then its parametric equations can be assumed in the following way

$$u = u(t)\,, \quad v = v(t) \tag{2.5}$$

where the parameter t satisfies the conditions

$$(2.6) \qquad \begin{aligned} u(0) &= u_1\,, \quad v(0) = v_1\,,\\ u(1) &= u_2\,, \quad v(1) = v_2\,. \end{aligned}$$

After substituting (2.5) into (2.2), Pfaff's forms ω_i^j are changed into the following differentials

$$(2.7) \qquad p_i^j(t)\,\mathrm{d}t = \{a_i^j[u(t), v(t)]\,u'(t) + b_i^j[u(t), v(t)]\,v'(t)\}\,\mathrm{d}t$$

and the whole system (2.1) represents the linear differential system

$$(2.8) \qquad \frac{\mathrm{d}A_i}{\mathrm{d}t} = p_i^j(t)\,A_j(t)\,.$$

If $B_1, \ldots, B_4$ are points satisfying (2.8) and dependent on t in such a way that for $t = 0$, $t = 1$ we have $B_i = A_i$, $B_i = B_i$, respectively, then we can write

$$(2.9) \qquad KB_i = A_i\,, \quad i = 1, \ldots, 4$$

where K is the mentioned collineation between $P_3(u_1, v_1)$ and $P_3(u_2, v_2)$. To each curve in Ω passing through the point (u_1, v_1) one can associate the following objects:

1. Four curves $A_i(t)$ through $A_i(u_1, v_1)$ and their developments into $P_3(u_1, v_1)$ as integral curves of (2.8) with the initial condition that they are passing through the points $A_i(u_1, v_1)$.

2. Ruled surfaces R_{ij} generated by $[A_i(t), A_j(t)]$. These are König manifolds of the type $\mathscr{P}^1_{1,3}$ by whose development into $P_3(u_1, v_1)$ we understand then the ruled surface with the basic curves obtained by developing the curves $A_i(t)$, $A_j(t)$.

The curve $\gamma \subset \Omega$ gives also a couple of ruled surfaces generated by the line $p(t) = [A_1(t), A_2(t)]$ of the congruence $\mathscr{L}_1$ with the projective connection and by the line $q(t) = [A_3(t), A_4(t)]$, respectively. Similarly the curve $\gamma \subset \Omega$ gives a couple of ruled surfaces which are developments of both surfaces of the couple into $P_3(t) = P_3[u(t), v(t)]$ of an arbitrary couple of lines $p(t) = p[u(t), v(t)]$, $q(t) = q[u(t), v(t)]$.

Now consider the situation similar to that of the study of couples of congruences L_1, L_2 in a projective space. A frame $\{A_i\}_{i=1}^4$ where A_1, A_2 were foci on $p = [A_1, A_2]$ of congruence L_1 and A_3, A_4 were foci of the corresponding line $q = [A_3, A_4]$ of the congruence L_2.

For this reason write differential equations of the developable surfaces of the both congruences. As it is well known, in each congruence developable surfaces generate two layers in the general case and one layer of the surface in case of a parabolic congruence. In the sequel we shall not consider parabolic congruences. Further assume that no layer of the developable surfaces of the congruence $\mathscr{L}_2$ corresponds to either of both layers of the developable surfaces of the congruence $\mathscr{L}_1$.

For the congruence $\mathscr{L}_1$ the equation of the developable surfaces has the determinant form

(2.8) $$|A_1, A_2, \nabla A_1, \nabla A_2| = 0\,,$$

and similarly for the congruence $\mathscr{L}_2$

(2.9) $$|A_3, A_4, \nabla A_3, \nabla A_4| = 0$$

which means by (2.1) and (2.3) that

(2.8′) $$\omega_1^3\omega_2^4 - \omega_1^4\omega_2^3 = 0\,,$$

and

(2.9′) $$\omega_3^1\omega_4^2 - \omega_4^1\omega_3^2 = 0\,,$$

respectively. From this we obtain using (2.2)

(2.10) $$(a_1^3a_2^4 - a_2^3a_1^4)\,\mathrm{d}u^2 + (a_1^3b_2^4 + a_2^4b_1^3 - a_2^3b_1^4 - a_1^4b_2^3)\,\mathrm{d}u\,\mathrm{d}v + \\ + (b_1^3b_2^4 - b_2^3b_1^4)\,\mathrm{d}v^2 = 0$$

and

(2.11) $$(a_3^1a_4^2 - a_4^1a_3^2)\,\mathrm{d}u^2 + (a_3^1b_2^4 + a_4^2b_3^1 - a_3^2b_4^1 - a_4^1b_3^2)\,\mathrm{d}u\,\mathrm{d}v + \\ + (b_3^1b_4^2 - b_3^2b_4^1)\,\mathrm{d}v^2 = 0\,,$$

respectively. Under our assumptions neither of equations (2.10), (2.11) has a multiple root $\mathrm{d}u : \mathrm{d}v$ and both equations have no common roots.

For the variable ratio $\mathrm{d}u : \mathrm{d}v$ the planes

(2.12) $$[\nabla A_1, A_1, A_2]\,,\quad [\nabla A_2, A_1, A_2]\,,\quad [\nabla A_3, A_3, A_4]\,,\quad [\nabla A_4, A_3, A_4]\,,$$

i.e. the planes

(2.13) $$\omega_1^3[A_1, A_2, A_3] + \omega_1^4[A_1, A_2, A_4]\,,\quad \omega_2^3[A_1, A_2, A_3] + \omega_2^4[A_1, A_2, A_4]\,, \\ \omega_3^1[A_1, A_3, A_4] + \omega_3^2[A_2, A_3, A_4]\,,\quad \omega_4^1[A_1, A_3, A_4] + \omega_4^2[A_2, A_3, A_4]$$

are fixed if and only if

(2.14) $$\omega_1^4 = \psi\omega_1^3\,,\quad \omega_2^4 = \varphi\omega_2^3\,,\quad \omega_3^2 = \psi^*\omega_3^1\,,\quad \omega_4^2 = \varphi^*\omega_4^1$$

where ψ, φ, ψ^*, φ^* are scalars (i.e. independent of $\mathrm{d}u : \mathrm{d}v$); simultaneously there is assumed that

(2.15) $$\omega_1^3\omega_2^3\omega_3^1\omega_4^1 \neq 0\,.$$

The system (2.1) has the form

$$\begin{aligned} \nabla A_1 &= \omega_1^1 A_1 + \omega_1^2 A_2 + \omega_1^3 (A_3 + \psi A_4), \\ \nabla A_2 &= \omega_2^1 A_1 + \omega_2^2 A_2 + \omega_2^3 (A_3 + \varphi A_4), \\ \nabla A_3 &= \omega_3^1 (A_1 + \psi^* A_2) + \omega_3^3 A_3 + \omega_3^4 A_4, \\ \nabla A_4 &= \omega_4^1 (A_1 + \varphi^* A_2) + \omega_4^3 A_3 + \omega_4^4 A_4. \end{aligned} \tag{2.16}$$

The planes (2.12) are the focal planes at A_1, A_2, A_3, A_4. There follows from (2.16) that the points

$$\Psi = A_3 + \psi A_4, \quad \Phi = A_3 + \varphi A_4, \tag{2.17}$$

and

$$\Psi^* = A_1 + \psi^* A_2, \quad \Phi^* = A_1 + \varphi^* A_2 \tag{2.18}$$

are the points of intersection of $q = [A_3, A_4]$ and $p = [A_1, A_2]$, respectively, with the focal planes at the points A_1, A_2 and A_3, A_4, respectively. The notation ψ, φ, ψ^*, $\varphi^*, \Psi, \Phi, \Psi^*$ and Φ^* is chosen by the similar way to that of [1], p. 20, equation (19).

A. Švec introduced in [4], § 12, p. 317 the notion of Klein space $\mathscr{K}(\mathscr{L}_1)$ with the projective connection associated to a congruence $\mathscr{L}_1$. But in our case $\mathscr{K}(\mathscr{L}_1) \equiv \equiv \mathscr{K}(\mathscr{L}_2)$ because Klein space depends only on the local space $P_3(u, v)$, as it is obvious from this definition of A. Švec:

Definition 2.1. To each point $(u, v) \in \Omega$ let there correspond a line $p(u, v) \in \mathscr{L}_1$ lying in the local space $P_3(u, v)$ containing a moving frame $\{A_i\}_{i=1}^4$. To (u, v) let us attach further a 5-dimensional projective space $P_5(u, v)$ with the corresponding hyperquadric $R(u, v)$ which is Klein image of the full line space with respect to $P_3(u, v)$. In $P_5(u, v)$, a moving frame determines a 6-tuple of points M_{ij} which are Klein images of $p_{ij} = [A_i, A_j]$, i.e. they are the points

$$M_{ij}, \quad i, j = 1, \ldots, 4; \quad i < j \quad (M_{ij} = -M_{ji}). \tag{2.19}$$

The couple of the corresponding lines

$$p_{12}(u, v) \in \mathscr{L}_1, \quad p_{34}(u, v) \in \mathscr{L}_2 \tag{2.20}$$

is mapped onto $R(u, v)$ as the couple of points determining the line

$$s(u, v) = [M_{12}(u, v), M_{34}(u, v)] \tag{2.21}$$

which will be considered as the generating line of the 2-parametric system $\mathscr{L}_{12}$ which may be considered as a generalization of Rozenfeld transformation for a couple of congruences $\mathscr{L}_1, \mathscr{L}_2$ with the projective connection.

To be able to consider $\mathscr{L}_{12}$ as König manifold we complete the definition:

Definition 2.1. It suffices to define the connection for those spaces $P_5(u, v)$ determined by

$$\nabla M_{ij} = \omega_i^k M_{kj} + \omega_j^k M_{ik} \tag{2.22}$$

which are in fact Švec's equations given in [4], p. 318.

The significance of this choice follows from the fact that it is possible the 1-parametric systems (curves, ruled surfaces etc.) to transfer the notion of their development into a fixed space P_5.

Using the foregoing considerations let us start to solve the following problem:

a) Is it possible to define the character m of the line $s(u, v)$ of Rozenfeld transformation $\mathscr{L}_{12}$ of the couple $\mathscr{L}_1, \mathscr{L}_2$ of congruences with the projective connection by a way similar to the case of the couple of congruences of a projective space?

b) Does a similar relation exist betwen the character m and double ratio of the point quadruplets

$$A_1, A_2, \Psi^*, \Phi^*, \quad A_3, A_4, \Psi, \Phi \tag{2.23}$$

as in the case of a couple of congruences in a projective space as it was found in the first paragraph of the present paper?

a) **Definition 2.2.** By the character of the line (2.21) of Rozenfeld image $\mathscr{L}_{12}$ of the couple $\mathscr{L}_1, \mathscr{L}_2$ of the congruences with the connection we understand the dimension of the tangent space $\tau(u, v)$ of the line $s(u, v)$, i.e. of the subspace in $P_5(u, v)$, given by the points

$$M_{12}, M_{34}, \nabla_u M_{12}, \nabla_v M_{12}, \nabla_u M_{34}, \nabla_v M_{34} \tag{2.24}$$

where e.g. $\nabla_u M_{12} = (\nabla M_{12})_{dv=0}$.

The question b) is answered by

Theorem 2.1. *The character of the line $s = [M_{12}, M_{34}] \in \mathscr{L}_{12}$ being the image under Rozenfeld transformation of the couple $\mathscr{L}_1, \mathscr{L}_2$ of congruences with the projective connection is $m = 5$ or $m < 5$ if the double ratios of both quadruplets of the points (2.23) are different or equal, respectively.*

Proof. By (2.19), (2.22) and (2.1) we have

$$\begin{aligned}(2.25)\quad \nabla M_{12} &= (\omega_1^1 + \omega_2^2)\, M_{12} + \omega_2^3 M_{13} + \varphi\omega_2^3 M_{14} - \omega_1^3 M_{23} - \psi\omega_1^3 M_{24}\,,\\ \nabla M_{34} &= \qquad\qquad - \omega_4^1 M_{13} + \omega_3^1 M_{14} - \varphi^*\omega_4^1 M_{23} +\\ &\quad + \psi^*\omega_3^1 M_{24} + (\omega_3^3 + \omega_4^4)\, M_{34}\,,\end{aligned}$$

i.e.

$$(2.26)\quad \begin{aligned} \nabla_u M_{12} &= (a_1^1 + a_2^2) M_{12} + a_2^3 M_{13} + \varphi a_2^3 M_{14} - a_1^3 M_{23} - \psi a_1^3 M_{24}, \\ \nabla_v M_{12} &= (b_1^1 + b_2^2) M_{12} + b_2^3 M_{13} + \varphi b_2^3 M_{14} - b_1^3 M_{23} - \psi b_1^3 M_{24}, \\ \nabla_u M_{34} &= - a_4^1 M_{13} + a_3^1 M_{14} - \varphi^* a_4^1 M_{23} + \\ &\quad + \psi^* a_3^1 M_{24} + (a_3^3 + a_4^4) M_{34}, \\ \nabla_v M_{34} &= - b_4^1 M_{13} + b_3^1 M_{14} - \varphi^* b_4^1 M_{23} + \\ &\quad + \psi^* b_3^1 M_{24} + (b_3^3 + b_4^4) M_{34}. \end{aligned}$$

As the points $M_{12}, M_{13}, M_{14}, M_{23}, M_{24}, M_{34}$ do not belong to the same hyperplane of the space $P_5(u, v)$ (as they are Klein images of the edges of the frame $\{A_i\}_{i=1}^4$ not belonging to the same linear complex) the dimension of the space $\tau(u, v)$ is equal to the rank of the following matrix

$$(2.27)\quad \left\| \begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ a_1^1 + a_2^2 & a_2^3 & \varphi a_2^3 & -a_1^3 & -\psi a_1^3 & 0 \\ b_1^1 + b_2^2 & b_2^3 & \varphi b_2^3 & -b_1^3 & -\psi b_1^3 & 0 \\ 0 & -a_4^1 & a_3^1 & -\varphi^* a_4^1 & \psi^* a_3^1 & a_3^3 + a_4^4 \\ 0 & -b_4^1 & b_3^1 & -\varphi^* b_4^1 & \psi^* b_3^1 & b_3^3 + b_4^4 \end{matrix} \right\|$$

minus 1. The maximal rank of this matrix equals to six, i.e. the character m of the line $s(u, v) \in \mathscr{L}_{12}$ is at most 5. For $m = 5$ and $m < 5$, respectively it is necessary and sufficient that

$$D \neq 0 \quad \text{or} \quad D = 0, \quad \text{respectively},$$

where

$$(2.28)\quad D = \begin{vmatrix} a_2^3 & \varphi a_2^3 & -a_1^3 & -\psi a_1^3 \\ b_2^3 & \varphi b_2^3 & -b_1^3 & -\psi b_1^3 \\ -a_4^1 & a_3^1 & -\varphi^* a_4^1 & \psi^* a_3^1 \\ -b_4^1 & b_3^1 & -\varphi^* b_4^1 & \psi^* b_3^1 \end{vmatrix} = (a_1^3 b_2^3 - a_2^3 b_1^3)(a_3^1 b_4^1 - a_4^1 b_3^1)(\psi\varphi^* - \varphi\psi^*).$$

The first two terms of the product on the right hand side of (2.28) cannot be zero. If the first or second terms are zero then

$$(2.29)\quad \omega_1^3 \wedge \omega_2^3 = 0 \quad \text{or} \quad \omega_3^1 \wedge \omega_4^1 = 0, \quad \text{respectively}.$$

For example the first of these equations expresses that ω_1^3 and ω_2^3 are linearly dependent Pfaff forms, i.e. for example for $\omega_1^3 \neq 0$ we have $\omega_2^3 = \lambda\omega_1^3$, where λ is a scalar. Then the first equation (2.25) yields

$$(2.30)\quad \nabla M_{12} = (\omega_1^1 + \omega_2^2) M_{12} + \omega_1^3[\lambda(M_{13} + \varphi M_{14}) - (M_{23} + \psi M_{24})]$$

so that the point ∇M_{12} (for fixed u, v and variables $\mathrm{d}u$, $\mathrm{d}v$) does not generate a plane but a line, the case which we eliminated from our considerations. The second equation from (2.29) leads to a similar result. Therefore there remains the equation

$$\psi\varphi^* - \varphi\psi^* = 0$$

which is identical with (1.19). This gives the equality between the both double ratios. Thus the theorem is proved.

We see that for a couple $\mathscr{L}_1$, $\mathscr{L}_2$ of congruences with the projective connection and for its Rozenfeld image the same theorem holds as for a couple of congruences L_1, L_2 in a projective space.

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