Czechoslovak Mathematical Journal

Karel Segeth

On universally optimal quadrature formulae involving values of derivatives of integrand

Czechoslovak Mathematical Journal, Vol. 19 (1969), No. 4, 605–675

Persistent URL: http://dml.cz/dmlcz/100927

Czechoslovak Mathematical Journal, 19 (94) 1969, Praha

ON UNIVERSALLY OPTIMAL QUADRATURE FORMULAE INVOLVING VALUES OF DERIVATIVES OF INTEGRAND

KAREL SEGETH, Praha

(Received October 29, 1968)

1. INTRODUCTION

Recently, many mathematicians have treated the optimization of numerical processes. They have used the methods of functional analysis to find numerical processes optimal in the sense of minimization of the norm of error in some Banach or Hilbert space. Let us mention at least the works of SOBOLEV [4], [9], and SCHOENBERG [7].

The problem of optimal quadrature formula for integration of continuous periodic functions is treated by BABUŠKA [1], [2]. We use the same approach as he did in his latest paper [3]. In addition, we shall examine the quadrature formulae involving values of derivatives of the integrand. In some cases we shall confine ourselves particularly to the quadrature formulae involving values of the second derivative of the integrand. We shall show certain optimal and universal properties of these formulae.

The concept of the n-periodic space of continuous 2π-periodic functions with continuous derivatives up to the order n is introduced as a natural generalization of the concept of periodic space of [2] and its basic properties are derived in Sec. 2. Two linear functionals, integral and quadrature formula (involving the values of derivatives of the integrand up to the order n), are introduced in the n-periodic space in Sec. 3. We confine ourselves to the quadrature formulae with equidistant abscissae in the following considerations except Sec. 9. Sec. 4 deals with the optimal quadrature formula. It is shown that the coefficients of the optimal quadrature formula are identical in all equidistant abscissae.

The class of strongly n-periodic spaces is introduced as a restriction of the concept of the n-periodic space in Sec. 5. One of principal properties of the strongly n-periodic space is the monotony of the sequence $\|e^{ikx}\|$, which is supposed to be non-decreasing with $k \to \infty$. This restriction enables us to show some further properties of the optimal quadrature formula in this strongly n-periodic space, e.g. that the coefficients at values of derivatives of the odd order equal zero. The following sections treat the 0-periodic and the strongly 2-periodic spaces. The limits of the coefficients of the optimal quadrature formula for the number of abscissae increasing to infinity are found in Sec. 6. Universally optimal properties of quadrature formulae using these limit coefficients are proved in Sec. 7. Moreover, the uniqueness of the formulae with these properties is proved in this section.

The question of the advantage of using the values of the second derivative of the integrand in the quadrature formula is discussed in Sec. 8. This comparison shows the classes of spaces where using values of the second derivative of the integrand is less efficient than using only values of the integrand. A numerical example illustrating these considerations is given in Sec. 10. We consider the general quadrature formula with arbitrary abscissae again and compare its error with the error of the optimal quadrature formula with equidistant abscissae in Sec. 9. This comparison shows that the error of the optimal quadrature formula may be very large as compared with the error of the formulae with arbitrary abscissae in a general periodic space. On the other side, the error of the optimal formula is proved to be comparable with the infimum of the error attainable in general in the strongly periodic space.

As compared with [3] we confine ourselves only to the computation of the integral of the function given, i.e. the 0-th Fourier coefficient. Statements concerning only the 0-periodic spaces are analogous to corresponding statements of [3] except some details. Their proofs are based on the corresponding proofs of [3].

2. n-PERIODIC SPACES

All our following considerations will start from the concept of a Hilbert space of 2π-periodic continuous functions with continuous derivatives up to certain order. Let us introduce these spaces as follows.

Definition 2.1. Let n be a non-negative integer. A Hilbert space H is said to be n-periodic if the following conditions are satisfied:

(i) Elements $f \in H$ are 2π-periodic continuous functions with continuous derivatives up to the order n.

(ii) $\|f^{(s)}\|_C \leqq B(H)\,\|f\|$ for $s = 0, 1, \ldots, n$ and all elements $f \in H$ where $\|\,.\,\|_C$ is the usual norm in the space C of continuous 2π-periodic functions, $\|\,.\,\|$ is the norm in the space H. The number $B(H)$ is independent of f.

(iii) Let $f \in H$, c real, and $g(x) = f(x + c)$. Then $g \in H$, $\|g\| = \|f\|$.

The functions e^{ikx} for any integer k satisfy (i) of Definition 2.1 for arbitrary non-negative integer n. Therefore they may be elements of n-periodic spaces. Let us introduce the following convenient notation.

Definition 2.2. Let H be an n-periodic space. An integer k is said to belong to the set $U = U(H)$ if and only if $e^{ikx} \in H$.

The relation of n-periodic spaces for various n is important in the following considerations.

Remark 2.1. Let $p \geqq n \geqq 0$ be integers, let H be a p-periodic space. Then the space H is n-periodic.

This statement follows immediately from Definition 2.1.

The basic properties of the introduced n-periodic space are given in the following theorem.

Theorem 2.1. *Let H be an n-periodic space not containing only zero function. Then $U(H) \neq \emptyset$ and the system $\{e^{ikx}\}$, $k \in U(H)$ is an orthogonal basis for the space H.*

Furthermore

(2.1)
$$\sum_{k \in U} k^{2n} \|e^{ikx}\|^{-2} < +\infty .$$

Proof. For $f \in H$ and all integers k let us write

(2.2)
$$J_k(f) = \int_0^{2\pi} e^{-ikt} f(t)\, dt .$$

Then the functional J_k is additive and homogeneous. With respect to (ii), J_k may be shown to be also bounded; therefore it is a linear functional on H. Thus there exist functions $\varphi_k \in H$ such that

(2.3)
$$J_k(f) = (f, \varphi_k)$$

holds for all $f \in H$ and integers k.

Further let us write

(2.4)
$$Q^{s,x}(f) = f^{(s)}(x)$$

for $f \in H$; $s = 0, 1, \ldots, n$, and x real. Then the functional $Q^{s,x}$ may be shown to be a linear functional on H, too. Thus there exist functions $\psi^{s,x} \in H$ such that

(2.5)
$$Q^{s,x}(f) = (f, \psi^{s,x})$$

holds for all $f \in H$; $s = 0, 1, \ldots, n$, and x real.

Let c be real, $f \in H$. Let us write

$$f_{(c)}(x) = f(x + c) .$$

Then according to (iii) $f_{(c)} \in H$, $\|f_{(c)}\| = \|f\|$. From [5] we use the equality

$$(f, g) = \tfrac{1}{4}(\|f + g\|^2 - \|f - g\|^2 + i\|f + ig\|^2 - i\|f - ig\|^2)\,.$$

Then with respect to (iii) we get

$$(f_{(c)}, g_{(c)}) = (f, g)\,.$$

Thus from (2.4), (2.5) we have

$$f(x) = (f, \psi^{0,x}) = (f_{(x)}, \psi^{0,0}) = (f, \psi^{0,0}_{(-x)})$$

for $f = f(t)$, $f \in H$, i.e.

$$\psi^{0,x} = \psi^{0,0}_{(-x)}\,.$$

Using the relations (2.2) to (2.5) we get

(2.6) $$\varphi_k(x) = (\varphi_k, \psi^{0,x}) = (\varphi_k, \psi^{0,0}_{(-x)}) = \overline{(\psi^{0,0}_{(-x)}, \varphi_k)} =$$

$$= \overline{\int_0^{2\pi} e^{-ikt}\psi^{0,0}(t - x)\,\mathrm{d}t} = e^{ikx}\,\overline{\int_0^{2\pi} e^{-ikz}\psi^{0,0}(z)\,\mathrm{d}z} = e^{ikx}\overline{J_k(\psi^{0,0})}$$

for any integer k. From (2.3) $\varphi_k \in H$. Therefore if $J_k(\psi^{0,0}) \neq 0$ then $e^{ikx} \in H$, i.e. $k \in U(H)$. Let $U = \emptyset$. Then $J_k(\psi^{0,0}) = 0$ for any integer k, i.e. $\varphi_k = 0$, and using (2.3) we have

(2.7) $$J_k(f) = 0$$

for all $f \in H$ and any integer k. Using (i) the relation $f \in C$ follows from the assumption $f \in H$. Thus also $f \in L^2$ so that (2.7) implies $f = 0$ for all $f \in H$. Thus we get a contradiction with the assumptions of the theorem. Therefore $U \neq \emptyset$ and the first statement of the theorem has been proved.

Let $k, s \in U$; $k \neq s$. From (2.2), (2.3), (2.6) we have

(2.8) $$(\varphi_k, \varphi_s) = \overline{J_k(\psi^{0,0})}\int_0^{2\pi} e^{i(k-s)t}\,\mathrm{d}t = 0\,.$$

We shall prove indirectly that $J_k(\psi^{0,0}) \neq 0$, $k \in U$. Let $k \in U$, $J_k(\psi^{0,0}) = 0$. Then (2.6) implies that $\varphi_k = 0$ and with respect to (2.3) the equality (2.7) holds for all $f \in H$.

In particular, $e^{ikx} \in H$ but from (2.2) $J_k(e^{ikx}) = 2\pi$. It is a contradiction with (2.7). Then using (2.8) we get that the system $\{e^{ikx}\}$, $k \in U$ is orthogonal in H and that

(2.9) $$\|\varphi_k\|^2 = 2\pi J_k(\psi^{0,0}) > 0\,, \quad k \in U\,.$$

Now let us prove that the system $\{e^{ikx}\}$, $k \in U$ is complete in H. Let $f \in H$ is such an element that

$$(2.10) \qquad (f, e^{ikx}) = 0, \quad k \in U.$$

Using (2.6), (2.9) we get

$$e^{ikx} = \frac{\varphi_k(x)}{J_k(\psi^{0,0})}, \quad k \in U.$$

Then using (2.10) we have

$$(f, \varphi_k) = 0, \quad k \in U.$$

From (2.6), (2.9) also

$$(f, \varphi_k) = 0$$

for any integer $k \notin U$. Therefore using the same conclusion as above we get $f \in L^2$, $f = 0$. Thus the system $\{e^{ikx}\}$, $k \in U$ is complete in H and orthogonal, i.e. it is an orthogonal basis for H. The second statement of the theorem has been proved.

Now let us prove that the series (2.1) converges. Let us write

$$\|\varphi_k\| = \lambda_k$$

then the system $\{\lambda_k^{-1}\varphi_k\}$, $k \in U$ is an orthonormal basis for the space H. Since $\psi^{n,0} \in H$ according to (2.5), we have

$$(2.11) \qquad \psi^{n,0} = \sum_{k \in U} c_k \lambda_k^{-1} \varphi_k$$

where from (2.4), (2.5)

$$c_k = (\psi^{n,0}, \lambda_k^{-1}\varphi_k) = \lambda_k^{-1}\overline{\varphi_k^{(n)}(0)}, \quad k \in U.$$

Substituting (2.6), (2.9) we get

$$c_k = \frac{1}{2\pi}\lambda_k(-ik)^n, \quad k \in U.$$

Then from (2.11) it follows

$$(2.12) \qquad \sum_{k \in U} |c_k|^2 = \frac{1}{4\pi^2}\sum_{k \in U} \lambda_k^2 k^{2n} < +\infty.$$

Since from (2.6), (2.9)

$$\|e^{ikx}\| = 2\pi\lambda_k^{-1}$$

the relation (2.1) follows immediately from (2.12). The proof of the theorem has been completed.

In regard of the statement of Theorem 2.1 we introduce a convenient notation. In addition, we always shall suppose that $U(H) \neq \emptyset$ in the following.

Definition 2.3. Let H be an n-periodic space, $k \in U(H)$. Let us write

$$\eta_k = \|e^{ikx}\| .$$

Let us prove a property of the elements of an n-periodic space, which we shall use in the following.

Lemma 2.1. *Let H be an n-periodic space, $f \in H$. Then the functions $f(x), f'(x), \ldots$ $\ldots, f^{(n)}(x)$ have Fourier series expansion convergent absolutely and uniformly with respect to x.*

Proof. Let $f \in H$. Then from Theorem 2.1

$$f(x) = \sum_{k \in U} a_k e^{ikx}$$

where

$$a_k = \eta_k^{-1}(f, e^{ikx}), \quad k \in U .$$

Differentiating formally, we get

$$f^{(s)}(x) = \sum_{k \in U} a_k (ik)^s e^{ikx} ; \quad s = 0, 1, \ldots, n . \tag{2.13}$$

The series

$$\sum_{k \in U} |a_k| \, |k|^n \tag{2.14}$$

does not depend on x and majorizes the series (2.13) for $s = 0, 1, \ldots, n$. Let us prove that the series (2.14) converges. We have

$$\begin{aligned} (\sum_{k \in U} |a_k| \, |k|^n)^2 &= (\sum_{k \in U} |a_k| \, \eta_k |k|^n \, \eta_k^{-1})^2 \\ &\leqq \sum_{k \in U} |a_k|^2 \, \eta_k^2 \sum_{k \in U} k^{2n} \eta_k^{-2} = \|f\|^2 \sum_{k \in U} k^{2n} \eta_k^{-2} \end{aligned} \tag{2.15}$$

since from Theorem 2.1 it follows

$$\|f\|^2 = \sum_{k \in U} |a_k|^2 \, \eta_k^2$$

and the series

$$\sum_{k \in U} k^{2n} \eta_k^{-2}$$

converges according to Theorem 2.1. Therefore the series (2.14) converges and all series (2.13) converge absolutely and uniformly with respect to x for $s = 0, 1, \ldots, n$. The lemma has been proved.

Let us show another important property enabling us to examine various phenomena in connection with the norm in an n-periodic space.

Theorem 2.2. *A space H is n-periodic if and only if it is possible to construct it in the following way:*

Let $U \neq \emptyset$ be a set of certain integers. Let $\eta_k > 0$, $k \in U$ be real numbers satisfying the condition

$$\sum_{k \in U} k^{2n} \eta_k^{-2} < +\infty . \tag{2.16}$$

If the set U is finite we construct a linear space of all trigonometric polynomials of the form

$$\sum_{k \in U} p_k e^{ikx}$$

where p_k, $k \in U$ are arbitrary complex numbers. We denote this space by H and introduce scalar product in the following way:

$$\begin{cases} (e^{ikx}, e^{isx}) = 0 \,; \quad k, s \in U \,; \quad k \neq s \,, \\ (e^{ikx}, e^{ikx}) = \eta_k^2 \,, \quad k \in U \,. \end{cases} \tag{2.17}$$

If the set U is infinite we construct a linear set S of all trigonometric polynomials of a finite degree and the form

$$\sum_{k \in U^*} p_k e^{ikx}$$

where $U^* \subset U$ is any finite subset of the set U and p_k, $k \in U^*$ are arbitrary complex numbers. We introduce scalar product of elements of the set S by (2.17). Now we construct a complete linear hull of the set S and denote this space by H. We extend continuously the scalar product (2.17) in the corresponding way.

Proof. 1. Let H be an n-periodic space. Then the statement of the theorem follows immediately from Theorem 2.1.

2. Let us prove that the Hilbert space constructed in this way is n-periodic on the above assumptions. Let us verify the conditions (i), (ii), (iii).

The system $\{e^{ikx}\}$, $k \in U$ may be readily shown to be an orthogonal basis for the space H. In addition, if $f \in H$ then

$$f(x) = \sum_{k \in U} a_k e^{ikx} . \tag{2.18}$$

With respect to (2.16) we may prove

$$f^{(s)}(x) = \sum_{k \in U} a_k (ik)^s e^{ikx} \,; \quad s = 0, 1, \ldots, n \tag{2.19}$$

in the same way as in the proof of Lemma 2.1. The series (2.19) converge absolutely and uniformly with respect to x. Thus if $f \in H$ then the functions $f^{(s)}(x)$; $s = 0, 1, \ldots, n$ are continuous and 2π-periodic. Then (i) has been proved.

Analogously to (2.15) we may estimate

$$\|f^{(s)}\|_C = \max_{x \in \langle 0, 2\pi \rangle} |f^{(s)}(x)| = \max_{x \in \langle 0, 2\pi \rangle} \left| \sum_{k \in U} a_k (ik)^s e^{ikx} \right|$$
$$\leqq \sum_{k \in U} |a_k| \, |k|^s \leqq \|f\| \, (\sum_{k \in U} k^{2s} \eta_k^{-2})^{1/2} \leqq \|f\| \, (\sum_{k \in U} k^{2n} \eta_k^{-2})^{1/2}$$

using (2.16), (2.19) since with respect to (2.17)

$$\|f\|^2 = \sum_{k \in U} |a_k|^2 \, \eta_k^2 \tag{2.20}$$

holds and the system $\{e^{ikx}\}$, $k \in U$ is an orthogonal basis for the space H. Thus (ii) has been proved since it is sufficient to put

$$B(H) = (\sum_{k \in U} k^{2n} \eta_k^{-2})^{1/2} \, .$$

Let c be real, $g(x) = f(x + c)$. Then using (2.18) we have

$$g(x) = \sum_{k \in U} a_k e^{ik(x+c)} = \sum_{k \in U} b_k e^{ikx}$$

where

$$b_k = a_k e^{ikc} \, , \quad k \in U \, .$$

Then from (2.20) we get that $g \in H$ and

$$\|g\|^2 = \sum_{k \in U} |b_k|^2 \, \eta_k^2 = \sum_{k \in U} |a_k|^2 \, \eta_k^2 = \|f\|^2 \, .$$

Thus (iii) has been proved and the proof of the theorem has been completed.

3. QUADRATURE FORMULAE IN AN n-PERIODIC SPACE

We shall approximate an integral of a 2π-periodic function over its period by quadrature formulae involving the values of the function and its derivatives up to the order n. First let us introduce these functionals.

Definition 3.1. Let H be an n-periodic space, $f \in H$. Let us write

$$J(f) = \frac{1}{2\pi} \int_0^{2\pi} f(x) \, \mathrm{d}x \, .$$

Remark 3.1. The integral $J(f)$ is the absolute term of the Fourier series expansion of the function $f(x)$. We shall be concerned solely with an approximation of the functional J in the following. The functional J does not depend on the choice of n. It is a linear functional on an n-periodic space for any non-negative integer n.

Definition 3.2. Let H be an n-periodic space, $f \in H$. Let j be a positive integer, let

$$0 < x_{1s} < x_{2s} < \ldots < x_{js} \leqq 2\pi \; ; \quad s = 0, 1, \ldots, n$$

be real numbers. Let us denote the quadrature formula with the abscissae x_{rs} by

$$Y(n, j; \{A_{rs}\}; \{x_{rs}\}) (f) = \sum_{s=0}^{n} j^{-(s+1)} \sum_{r=1}^{j} A_{rs} f^{(s)}(x_{rs}) \, .$$

The numbers A_{rs} are said to be coefficients of the formula.

Remark 3.2. The formula $Y(n, j; \{A_{rs}\}; \{x_{rs}\})$ is a linear functional on an n-periodic space H. It employs the values of the integrand and its derivatives up to the order n in j abscissae x_{rs}, which are generally different for the calculation of derivatives of different orders. The factor $j^{-(s+1)}$ does not cause loss of generality. Its sense will be shown in Sec. 4 where we shall determine the limits of the coefficients A_{rs} of some quadrature formulae.

Let $p \geqq n \geqq 0$, $j > 0$ be integers, let H be a p-periodic space. Let the coefficients A_{rs} and the abscissae x_{rs}; $r = 1, 2, \ldots, j$; $s = 0, 1, \ldots, n$ be given. Let us put

$$A^*_{rs} = A_{rs} \, , \quad x^*_{rs} = x_{rs} \; ; \quad s = 0, 1, \ldots, n \, ,$$
$$A^*_{rs} = 0 \; ; \quad s = n + 1, \ldots, p$$

and $r = 1, 2, \ldots, j$. Let the abscissae x^*_{rs}; $s = n + 1, \ldots, p$; $r = 1, 2, \ldots, j$ be chosen arbitrarily. Then the formula

$$Y(p, j; \{A^*_{rs}\}; \{x^*_{rs}\})$$

on the p-periodic space H is identical with the formula

$$Y(n, j; \{A_{rs}\}; \{x_{rs}\})$$

on the same space H considered as n-periodic in the sense of Remark 2.1.

Since J and $Y(n, j; \{A_{rs}\}; \{x_{rs}\})$ are linear functionals on the n-periodic space their realizing functions, which we shall very often use, may be found using the Riesz theorem.

Lemma 3.1. *Let H be an n-periodic space. Then*

$$J(f) = (f, u) \tag{3.1}$$

for all $f \in H$ where $u \in H$.

(3.2) $$u(x) = 0 \quad \text{if} \quad 0 \notin U\,, \qquad = \eta_0^{-2} \quad \text{if} \quad 0 \in U$$

is a realizing function of the functional J. This function does not depend on n.

Proof. We get the statement of the lemma substituting (3.2) into (3.1) and using Lemma 2.1 and Theorem 2.1.

Lemma 3.2. *Let H be an n-periodic space. Then*

(3.3) $$Y(n, j; \{A_{rs}\}; \{x_{rs}\})(f) = (f, v)$$

for all $f \in H$ where $v \in H$,

(3.4) $$v(x) = \sum_{k \in U} \eta_k^{-2} e^{ikx} \sum_{s=0}^{n} (-ik)^s j^{-(s+1)} \sum_{r=1}^{j} \exp(-ikx_{rs})\, \bar{A}_{rs}$$

is a realizing function of the functional $Y(n, j; \{A_{rs}\}; \{x_{rs}\})$.

Proof. We get the statement of the lemma substituting (3.4) into (3.3) and using Lemma 2.1 and Theorem 2.1.

Now we may easily determine another important quantity, the error of the quadrature formula in the sense of the norm in the n-periodic space.

Lemma 3.3. *Let H be an n-periodic space. Then the square of the error of the formula $Y(n, j; \{A_{rs}\}; \{x_{rs}\})$ is equal to*

(3.5) $$\|J - Y(n, j; \{A_{rs}\}; \{x_{rs}\})\|^2 = \sum_{k \in U} \eta_k^{-2} |B_k|^2$$

where

(3.6) $$B_0 = j^{-1} \sum_{r=1}^{j} A_{r0} - 1\,,$$

$$B_k = \sum_{s=0}^{n} (ik)^s j^{-s-1} \sum_{r=1}^{j} \exp(ikx_{rs})\, A_{rs} \quad \text{for} \quad k \neq 0\,.$$

Proof. With respect to Remarks 3.1, 3.2

$$J - Y(n, j; \{A_{rs}\}; \{x_{rs}\})$$

is a linear functional on H. Then there exists the function $\varrho \in H$ such that

$$(J - Y(n, j; \{A_{rs}\}; \{x_{rs}\}))(f) = (f, \varrho)$$

for all $f \in H$ where using Lemmas 3.1, 3.2 we may write

$$(3.7) \qquad \varrho(x) = \eta_0^{-2} - \sum_{k \in U} \eta_k^{-2} e^{ikx} \sum_{s=0}^{n} (-ik)^s j^{-s-1} \sum_{r=1}^{j} \exp(-ikx_{rs}) \bar{A}_{rs} .$$

The term η_0^{-2} does not occur for $0 \notin U$. In addition,

$$\|J - Y(n, j; \{A_{rs}\}; \{x_{rs}\})\|^2 = (\varrho, \varrho) .$$

To complete the proof of the lemma we determine this scalar product from (3.7) using Theorem 2.1.

In the following sections we shall treat above all quadrature formulae with j equidistant abscissae identical for the evaluation of the value of the integrand and also its derivatives. We shall derive some properties of these quadrature formulae and we shall compare them (at least partly in conclusion in Sec. 9) with properties of quadrature formulae $Y(n, j; \{A_{rs}\}; \{x_{rs}\})$ having general arbitrary abscissae x_{rs}.

Let us introduce the following notation for the quadrature formulae with equidistant abscissae.

Definition 3.3. Let H be an n-periodic space. Let j be a positive integer. We put

$$(3.8) \qquad x_{rs} = \frac{2\pi r}{j} ; \quad r = 1, 2, \ldots, j ; \quad s = 0, 1, \ldots, n .$$

Then let us denote the quadrature formula with equidistant abscissae and coefficients A_{rs} by

$$\hat{I}(n, j; \{A_{rs}\}) = Y(n, j; \{A_{rs}\}; \{x_{rs}\}) .$$

Remark 3.3. It holds

$$\hat{I}(n, j; \{A_{rs}\})(f) = \sum_{s=0}^{n} j^{-s-1} \sum_{r=1}^{j} A_{rs} f^{(s)} \left(\frac{2\pi r}{j}\right)$$

for all f from the n-periodic space H. The formula $\hat{I}(n, j; \{A_{rs}\})$ is a linear functional on the n-periodic space H. It employs the values of the integrand and its derivatives up to the order n in j equidistant abscissae $2\pi r/j$; $r = 1, 2, \ldots, j$, which are identical for the evaluation of derivatives of all orders.

Let us further show a particular form of the realizing function of the functional $\hat{I}(n, j; \{A_{rs}\})$.

Lemma 3.4. *Let H be an n-periodic space. Then*

$$\hat{I}(n, j; \{A_{rs}\})(f) = (f, v)$$

for all $f \in H$ *where* $v \in H$,

$$v(x) = \sum_{k \in U} \eta_k^{-2} e^{ikx} \sum_{s=0}^{n} (-ik)^s j^{-s-1} \sum_{r=1}^{j} \exp\left(\frac{-2\pi rik}{j}\right) \bar{A}_{rs}$$

is a realizing function of the functional $\hat{I}(n, j; \{A_{rs}\})$.

Proof. The statement of the lemma follows immediately from Lemma 3.2 by using (3.8).

4. THE OPTIMAL QUADRATURE FORMULA IN AN n-PERIODIC SPACE

In this section we shall seek the optimal approximation of the functional J in the class of quadrature formulae of the form $\hat{I}(n, j; \{A_{rs}\})$. We shall prove the existence and the uniqueness of this optimal quadrature formula and some its general properties.

First let us introduce the concept of the optimal quadrature formula.

Definition 4.1. Let H be an n-periodic space. The quadrature formula $\hat{I}(n, j; \{K_{rs}(n, j)\})$ is said to be optimal in the n-periodic space H if and only if

$$\|J - \hat{I}(n, j; \{K_{rs}(n, j)\})\| \leqq \|J - \hat{I}(n, j; \{A_{rs}\})\|$$

is valid for n, j fixed and arbitrary coefficients A_{rs}.

For the sake of brevity we denote the coefficients $K_{rs}(n, j)$ of the optimal quadrature formula by K_{rs} if it is evident what values n, j take.

Further let us denote the norm of the error of the optimal quadrature formula in the n-periodic space H by

$$\hat{X}(n, j, H) = \|J - \hat{I}(n, j; \{K_{rs}(n, j)\})\| .$$

The following theorem states the existence and the uniqueness of the optimal approximation of the functional J in the n-periodic space.

Theorem 4.1. *Let H be an n-periodic space. The quadrature formula $\hat{I}(n, j; \{K_{rs}(n, j)\})$ is optimal in this space H if and only if*

$$\tilde{\varrho}^{(s)}\left(\frac{2\pi r}{j}\right) = 0 \; ; \quad r = 1, 2, \ldots, j \; ; \quad s = 0, 1, \ldots, n \tag{4.1}$$

where $\tilde{\varrho} \in H$ *and*

$$(J - \hat{I}(n, j; \{K_{rs}(n, j)\}))(f) = (f, \tilde{\varrho}) \tag{4.2}$$

holds for all $f \in H$, *i.e.* $\tilde{\varrho}(x)$ *is a realizing function of the functional* $J - \hat{I}(n, j; \{K_{rs}(n, j)\})$.

The optimal quadrature formula does exist and is determined uniquely.

Proof. Let n, j be fixed. Using the notation (2.4) we may write

$$\hat{I}(n, j; \{A_{rs}\}) = \sum_{s=0}^{n} \sum_{r=1}^{j} j^{-s-1} A_{rs} Q^{s, 2\pi r/j}$$

with respect to Remark 3.3. Therefore the functionals $\hat{I}(n, j; \{A_{rs}\})$ form a finite-dimensional subspace of the space of all linear functionals on the n-periodic space H for n, j fixed and arbitrary coefficients A_{rs}.

Let $u \in H$, $v \in H$ be realizing functions of the functionals J, $\hat{I}(n, j; \{A_{rs}\})$ respectively. Then we may convert the problem of seeking the optimal approximation of the functional J in the subspace of functionals $\hat{I}(n, j; \{A_{rs}\})$ into the equivalent problem of seeking the optimal approximation $\tilde{v}$ of the element u in finite-dimensional subspace $V \subset H$ of realizing functions v of all functionals $\hat{I}(n, j; \{A_{rs}\})$.

In a Hilbert space this element $\tilde{v}$ exists and is determined uniquely as a projection of u to the subspace V, i.e. by the equality

$$(u - \tilde{v}, v) = 0$$

for all $v \in V$. From (4.2) we may write

$$\tilde{\varrho}(x) = u(x) - \tilde{v}(x)\,. \tag{4.3}$$

Then $\tilde{\varrho} \in H$, $(\tilde{\varrho}, v) = 0$ for all $v \in V$. This is equivalent to the relation

$$\hat{I}(n, j; \{A_{rs}\})\,(\tilde{\varrho}) = 0$$

or

$$\sum_{s=0}^{n} j^{-s-1} \sum_{r=1}^{j} A_{rs} \tilde{\varrho}^{(s)}\left(\frac{2\pi r}{j}\right) = 0 \tag{4.4}$$

for arbitrary coefficients A_{rs}; $r = 1, 2, \ldots, j$; $s = 0, 1, \ldots, n$. However, (4.4) is equivalent to (4.1) with respect to the arbitrary choice of coefficients A_{rs}. Substituting into (4.1) from Lemmas 3.1, 3.4, we get a nonsingular system of $j(n + 1)$ linear algebraic equations for $j(n + 1)$ unknown coefficients $K_{rs}(n, j)$. The theorem has been proved.

An important property of the coefficients of the optimal quadrature formula follows from this theorem:

Theorem 4.2. *Let H be an n-periodic space. Let $\hat{I}(n, j; \{K_{rs}(n, j)\})$ be the optimal quadrature formula in this space H. Then*

$$K_{rs}(n, j) = K_{ps}(n, j)\,; \quad p, r = 1, 2, \ldots, j\,; \quad s = 0, 1, \ldots, n\,. \tag{4.5}$$

Thus the coefficients K_{rs} of the optimal quadrature formula are equal to each other in all abscissae $2\pi r/j$ for s fixed.

Proof. Let $u \in H$, $\tilde{v} \in H$ be realizing functions of the functionals J, $\hat{I}(n, j; \{K_{rs}(n, j)\})$ respectively. Let us introduce the function $\tilde{\varrho} \in H$ by the relation (4.3). Then (4.2) holds and

$$\hat{X}(n, j, H) = \|\tilde{\varrho}\| . \tag{4.6}$$

Now let us construct functionals

$$\hat{I}_q = \hat{I}(n, j; \{K_{rs}^{(q)}\}); \quad q = 0, 1, \ldots, j-1$$

where

$$K_{rs}^{(q)} = K_{ms} ; \quad r = 1, 2, \ldots, j ; \quad s = 0, 1, \ldots, n$$

and

$$m - 1 \equiv r + q - 1 \bmod j .$$

Thus we have

$$\hat{I}_0 = \hat{I}(n, j; \{K_{rs}\})$$

and the formulae $\hat{I}_q$ result from $\hat{I}(n, j; \{K_{rs}\})$ by successive cyclic replacement of coefficients in abscissae.

Further let us write

$$\hat{X}_q = \|J - \hat{I}_q\| ; \quad q = 0, 1, \ldots, j-1 . \tag{4.7}$$

In particular, then

$$\hat{X}_0 = \hat{X}(n, j, H) .$$

Using Lemmas 3.1, 3.4 we may verify

$$(J - \hat{I}_q)(f) = (f, \tilde{\varrho}_q) ; \quad q = 0, 1, \ldots, j-1 \tag{4.8}$$

for all $f \in H$ where $\tilde{\varrho}_q \in H$,

$$\tilde{\varrho}_q(x) = u(x) - \tilde{v}_q(x) = \tilde{\varrho}\left(x + \frac{2\pi q}{j}\right); \quad q = 0, 1, \ldots, j-1 . \tag{4.9}$$

In particular, again

$$\tilde{\varrho}_0 = \tilde{\varrho} . \tag{4.10}$$

From the relations (iii), (4.6) to (4.10) we get finally

$$\hat{X}_q = \|\tilde{\varrho}_q\| = \|\tilde{\varrho}\| = \hat{X}(n, j, H) ; \quad q = 0, 1, \ldots, j-1 .$$

With respect to the uniqueness of the optimal quadrature formula and the construction of the functionals $\hat{I}_q$, (4.5) holds. The theorem has been proved.

Since the optimal quadrature formula in the n-periodic space H is of the particular form we may confine ourselves only to quadrature formulae of this form, i.e. formulae with coefficients identical in all abscissae for s fixed, without loss of generality. Let us introduce the following notation.

Definition 4.2. Let H be an n-periodic space. Let j be a positive integer, let A_s; $s = 0, 1, \ldots, n$ be given numbers. We put

$$A_{rs} = A_s \; ; \quad r = 1, 2, \ldots, j \; ; \quad s = 0, 1, \ldots, n \, .$$

Then let us denote the quadrature formula with coefficients A_{rs} independent of r and equidistant abscissae x_r by

$$I(n, j; A_0, A_1, \ldots, A_n) = \hat{I}(n, j; \{A_{rs}\}) \, .$$

Remark 4.1. It holds

$$I(n, j; A_0, A_1, \ldots, A_n)(f) = \sum_{s=0}^{n} A_s j^{-s-1} \sum_{r=1}^{j} f^{(s)}\left(\frac{2\pi r}{j}\right)$$

for all f from the n-periodic space H. Putting

$$\left.\begin{aligned} A_{rs} &= A_s \\ x_{rs} &= \frac{2\pi r}{j} \end{aligned}\right\} \quad r = 1, 2, \ldots, j \; ; \quad s = 0, 1, \ldots, n$$

we may also write

$$I(n, j; A_0, A_1, \ldots, A_n) = Y(n, j; \{A_{rs}\}; \{x_{rs}\}) \, .$$

The formula $I(n, j; A_0, A_1, \ldots, A_n)$ is a linear functional on the n-periodic space H. It employs the values of the integrand and its derivatives up to the order n. The values of the derivative of the s-th order are summed over all j abscissae $2\pi r/j$; $r = 1, 2, \ldots, j$ and then this sum is multiplied by the coefficient A_s independent of r.

Let us further show a particular form of the realizing function of the functional $I(n, j; A_0, A_1, \ldots, A_n)$.

Lemma 4.1. *Let H be an n-periodic space. Then*

$$I(n, j; A_0, A_1, \ldots, A_n)(f) = (f, v)$$

for all $f \in H$ where $v \in H$,

$$v(x) = \sum_{s=0}^{n} \bar{A}_s \sum_{tj \in U} (-it)^s \, \eta_{tj}^{-2} e^{itjx} \tag{4.11}$$

is a realizing function of the functional $I(n, j; A_0, A_1, \ldots, A_n)$. The function $v(x)$ is periodic with the period $2\pi/j$.

Proof. With respect to Lemma 4.3 we have

$$v(x) = \sum_{s=0}^{n} \bar{A}_s \sum_{k \in U} (-ik)^s \, \eta_k^{-2} j^{-s-1} e^{ikx} \sum_{r=1}^{j} \exp\left(\frac{-2\pi rik}{j}\right) \tag{4.12}$$

where

$$\sum_{r=1}^{j} \exp\left(\frac{-2\pi rik}{j}\right) = 0 \quad \text{for} \quad k \neq tj,\ t \text{ integer}, \tag{4.13}$$
$$= j \quad \text{for} \quad k = tj,\ t \text{ integer}.$$

Substituting (4.13) into (4.12) we get immediately (4.11). The periodicity may be readily verified by computation from (4.11). Thus the lemma has been proved.

Further we may determine the error of the quadrature formula $I(n, j; A_0, A_1, \ldots, A_n)$ in the sense of the norm in the n-periodic space.

Lemma 4.2. *Let H be an n-periodic space, $0 \in U$. Then the square of the error of the quadrature formula $I(n, j; A_0, A_1, \ldots, A_n)$ equals*

$$\|J - I(n, j; A_0, A_1, \ldots, A_n)\|^2 = \eta_0^{-2}(1 - A_0 - \bar{A}_0)$$
$$+ \sum_{s=0}^{n} \sum_{m=0}^{n} A_s \bar{A}_m (-1)^m \, i^{s+m} \sum_{tj \in U} t^{s+m} \eta_{tj}^{-2}.$$

If $0 \notin U$ the first term does not occur.

Proof. With respect to Remark 4.1 the lemma may be proved analogously to Lemma 3.3. Using Lemma 4.1 we get

$$\varrho(x) = \eta_0^{-2} - \sum_{s=0}^{n} \bar{A}_s \sum_{tj \in U} (-it)^s \, \eta_{tj}^{-2} e^{itjx} \tag{4.14}$$

instead of (3.7) where the term η_0^{-2} does not occur for $0 \notin U$. To complete the proof of the lemma we again determine

$$\|J - I(n, j; A_0, A_1, \ldots, A_n)\|^2 = (\varrho, \varrho)$$

using Theorem 2.1.

Since in the following considerations we shall confine ourselves to quadrature formulae of the form $I(n, j; A_0, A_1, \ldots, A_n)$ let us introduce the optimal approximation of the functional J in the class of these formulae analogously to Definition 4.1.

Definition 4.3. Let H be an n-periodic space. The quadrature formula $I(n, j; K_0(n, j), K_1(n, j), \ldots, K_n(n, j))$ is said to be optimal in the n-periodic space H if and only if

$$\|J - I(n, j; K_0(n, j), K_1(n, j), \ldots, K_n(n, j))\| \leqq \|J - I(n, j; A_0, A_1, \ldots, A_n)\|$$

is valid for n, j fixed and arbitrary coefficients $A_0, A_1, \ldots, A_n$.

For the sake of brevity we denote the coefficients $K_s(n, j)$ of the optimal quadrature formula by K_s if it is evident what values n, j take.

Further let us denote the norm of the error of the optimal quadrature formula in the n-periodic space H by

$$X(n, j, H) = \|J - I(n, j; K_0(n, j), K_1(n, j), \ldots, K_n(n, j))\| .$$

Remark 4.2. In the following considerations (except Sec. 9) we shall confine ourselves only to quadrature formulae $I(n, j; A_0, A_1, \ldots, A_n)$ introduced in Definition 4.2. We shall always treat the optimal quadrature formula in the n-periodic space H as the formula $I(n, j; K_0(n, j), K_1(n, j), \ldots, K_n(n, j))$ introduced in Definition 4.3. The connection with the optimal quadrature formula $\hat{I}(n, j; \{K_{rs}(n, j)\})$ introduced in Definition 4.1 is evident from Theorem 4.2. Moreover, the relation

$$X(n, j, H) = \hat{X}(n, j, H)$$

follows from this theorem.

Let us state the existence and the uniqueness of the optimal approximation analogously to Theorem 4.1 (this time in the sense of Definition 4.3).

Theorem 4.3. *Let H be an n-periodic space. The quadrature formula $I(n, j; K_0(n, j), K_1(n, j), \ldots, K_n(n, j))$ is optimal in this space H if and only if*

$$\tilde{\varrho}^{(s)}(0) = 0 ; \quad s = 0, 1, \ldots, n \tag{4.15}$$

where $\tilde{\varrho} \in H$,

$$(J - I(n, j; K_0(n, j), K_1(n, j), \ldots, K_n(n, j)))(f) = (f, \tilde{\varrho}) \tag{4.16}$$

holds for all $f \in H$, i.e. $\tilde{\varrho}(x)$ is a realizing function of the functional $J - I(n, j; K_0(n, j), K_1(n, j), \ldots, K_n(n, j))$.

The optimal quadrature formula does exist and is determined uniquely.

If $0 \notin U$ then $K_s(n, j) = 0$; $s = 0, 1, \ldots, n$.

Proof. Functionals $I(n, j; A_0, A_1, \ldots, A_n)$ for fixed n, j and arbitrary coefficients A_s form a subspace of the finite-dimensional space of all functionals $\hat{I}(n, j; \{A_{rs}\})$ with arbitrary coefficients A_{rs}. With respect to Theorem 4.2 the optimal approximation of the functional J in the space of functionals $\hat{I}(n, j; \{A_{rs}\})$ is certain functional $I(n, j; K_0(n, j), K_1(n, j), \ldots, K_n(n, j))$ from the subspace of functionals $I(n, j; A_0, A_1, \ldots, A_n)$.

With respect to Theorem 4.1 this optimal quadrature formula exists and is determined uniquely by the condition (4.1) where $\tilde{\varrho} \in H$ is given in (4.16). Since realizing functions $\varrho \in H$ of all functionals $I(n, j; A_0, A_1, \ldots, A_n)$ are periodic with the period $2\pi/j$ (as Lemma 4.1 states) the condition (4.15) is equivalent to (4.1).

Let us write the nonsingular system (4.15) of $n + 1$ linear algebraic equations for $n + 1$ unknowns $K_0, K_1, \ldots, K_n$ explicitly. Using (4.14) and Lemma 2.1 we get

$$\tilde{\varrho}^{(s)}(x) = 0^s \eta_0^{-2} - \sum_{m=0}^{n} \overline{K}_m \sum_{tj \in U} (-1)^m j^s (it)^{m+s} \eta_{tj}^{-2} e^{itjx} ; \quad s = 0, 1, \ldots, n$$

where we put $0^0 = 1$. The term $0^s \eta_0^{-2}$ does not occur for $0 \notin U$. The system (4.15) has the form

$$\sum_{m=0}^{n} \overline{K}_m (-1)^m i^{m+s} \sum_{tj \in U} t^{m+s} \eta_{tj}^{-2} = 0^s \eta_0^{-2} ; \quad s = 0, 1, \ldots, n . \tag{4.17}$$

In particular, for $0 \notin U$

$$\sum_{m=0}^{n} \overline{K}_m (-1)^m i^{m+s} \sum_{tj \in U} t^{m+s} \eta_{tj}^{-2} = 0 ; \quad s = 0, 1, \ldots, n . \tag{4.18}$$

The system (4.18) has only a trivial solution. The theorem has been proved.

Now we may find the error of the optimal quadrature formula in the n-periodic space.

Theorem 4.4. *Let H be an n-periodic space, $0 \in U$. Then the square of the error of the optimal quadrature formula equals*

$$X^2(n, j, H) = \eta_0^{-2}\left(1 - \overline{K_0(n, j)}\right) .$$

If $0 \notin U$ then

$$X^2(n, j, H) = 0 .$$

Proof. The proof follows immediately from Lemma 4.2 by using the relations (4.17), (4.18) of the proof of Theorem 4.3.

Remark 4.3. The case of an n-periodic space H, $0 \notin U$ is trivial in virtue of Theorems 4.3, 4.4. The coefficients of the optimal quadrature formula equal 0 and this formula integrates all functions $f \in H$ exactly. Therefore in the following we shall confine ourselves only to those n-periodic spaces where $0 \in U$ and we shall not repeat the assumption $0 \in U$ explicitly.

The coefficient of the optimal quadrature formula may be easily determined in the simplest case of the 0-periodic space H.

Theorem 4.5. *Let H be a 0-periodic space. Then the coefficient of the optimal quadrature formula equals*

$$K_0(0, j) = \frac{1}{\eta_0^2 \sum_{tj \in U} \eta_{tj}^{-2}} . \tag{4.19}$$

Proof. The coefficient K_0 may be readily determined from (4.17) if we consider the proof of Theorem 4.3. Obviously K_0 is real.

The calculation of coefficients of the optimal quadrature formula is more complicated in a general n-periodic space, $n > 0$.

5. OPTIMAL QUADRATURE FORMULA IN A STRONGLY n-PERIODIC SPACE

In the following we shall confine ourselves to n-periodic spaces satisfying some further assumptions to be able to examine the asymptotical behaviour of coefficients of the optimal quadrature formula and the error of quadrature formulae.

Definition 5.1. Let H be an n-periodic space. This space H is said to be strongly n-periodic if and only if the following conditions are satisfied:

(iv) $e^{ikx} \in H$ for all integers k, $\eta_k = \eta_{-k}$.

(v) Let k, j be integers, $|k| \geqq |j|$. Then $\eta_k \geqq \eta_j$.

(vi) $\eta_{kj}^2 \sum_{t=0}^{\infty} (t + k)^{2n} \eta_{(t+k)j}^{-2} \leqq D(H)$ for $k = 1, 2, \ldots, [n/2] + 1$ and all positive integers j.[1]) The number $D(H)$ does not depend on j.

For the sake of brevity we use the terms "periodic" or "strongly periodic" space if it is evident what value n takes.

Remark 5.1. Let H be a strongly n-periodic space. Then $U(H)$ is the set $\{\ldots, -1, 0, 1, \ldots\}$ of all integers as follows from Definition 5.1.

We are interested in the relation of strongly n-periodic spaces for various n as in the case of n-periodic space.

[1]) $[x]$ denotes the largest integer, for which $[x] \leqq x$ holds.

Remark 5.2. Let $p \geqq n \geqq 0$ be integers, let H be a strongly p-periodic space. Then the space H is strongly n-periodic.

The statement follows immediately from Definition 5.1 and Remark 2.1.

The condition (vi) of Definition 5.1 is independent of the conditions (iv), (v) of Definition 5.1 and the conditions (i) to (iii) of Definition 2.1 as we may show at least in the simplest case $n = 0$.

Theorem 5.1. *There exists a* 0-*periodic space* H *satisfying the conditions* (iv), (v) *of Definition* 5.1 *such that this space* H *is not strongly* 0-*periodic.*

Proof. We use Theorem 2.2 to construct the 0-periodic space H with these properties. Let U be a set of all integers. We put

$$\text{(5.1)} \qquad \eta_t^2 = \eta_{-t}^2 = 2^{2^{2s}}; \quad t = 2^{2^s}, 2^{2^s} + 1, \ldots, 2^{2^{s+1}} - 1 \quad \text{for any integer} \quad s \geqq 0\,,$$
$$\eta_{-1}^2 = \eta_0^2 = \eta_1^2 = 1\,.$$

Let us construct the corresponding Hilbert space of 2π-periodic continuous functions. The condition (2.16) of Theorem 2.2 is satisfied since we may estimate

$$\sum_{t=-\infty}^{\infty} \eta_t^{-2} = 3 + 2\sum_{t=2}^{\infty} \eta_t^{-2} = 3 + 2\sum_{s=0}^{\infty} \sum_{t=2^{2^s}}^{2^{2^{s+1}}-1} \eta_t^{-2} =$$
$$= 3 + 2\sum_{s=0}^{\infty} (2^{2^{s+1}} - 2^{2^s}) \times 2^{-2^{2s}} \leqq 3 + 2\sum_{s=0}^{\infty} 2^{-s} = 7 < +\infty\,.$$

Thus this space is 0-periodic. The validity of (iv), (v) follows immediately from (5.1).

Now let us put

$$\text{(5.2)} \qquad j_r = 2^{2^r} \quad \text{for any integer} \quad r \geqq 0$$

and let us show

$$\text{(5.3)} \qquad \eta_{j_r}^2 \sum_{t=1}^{\infty} \eta_{tj_r}^{-2} > 2^{2^r} - 1 \quad \text{for any integer} \quad r \geqq 0\,.$$

We get a contradiction with (vi) in this way. Thus the space H constructed in this way is not strongly 0-periodic. We may estimate

$$\eta_{j_r}^2 \sum_{t=1}^{\infty} \eta_{tj_r}^{-2} = \sum_{t=1}^{\infty} \eta_{j_r}^2 \eta_{tj_r}^{-2} \geqq \sum_{t=1}^{j_{r+1}/j_r - 1} \eta_{j_r}^2 \eta_{tj_r}^{-2} = 2^{2^r} - 1$$

for r fixed since from (5.1), (5.2) we have

$$j_{r+1}/j_r = 2^{2^r}$$

and

$$\eta_{j_r}^2 \eta_{tj_r}^{-2} = 1 \quad \text{for} \quad t = 1, 2, \ldots, 2^{2^r} - 1 .$$

Thus (5.3) has been proved and the theorem holds.

We may transform the series

$$(5.4) \qquad \sum_{t=-\infty}^{\infty} t^m \eta_{tj}^{-2} ; \quad m = 0, 1, \ldots, 2n$$

into a simpler form in a strongly n-periodic space H.

Lemma 5.1. *Let H be a strongly n-periodic space. Then the series (5.4) converge absolutely and uniformly with respect to j for all positive integers j. It holds*

$$(5.5) \qquad \sum_{t=-\infty}^{\infty} \eta_{tj}^{-2} = \eta_0^{-2} + 2 \sum_{t=1}^{\infty} \eta_{tj}^{-2} ,$$

$$\sum_{t=-\infty}^{\infty} t^m \eta_{tj}^{-2} = 0 \quad \text{for} \quad m = 1, 3, \ldots, 2n - 1 ,$$

$$\sum_{t=-\infty}^{\infty} t^m \eta_{tj}^{-2} = 2 \sum_{t=1}^{\infty} t^m \eta_{tj}^{-2} \quad \text{for} \quad m = 2, 4, \ldots, 2n$$

and all positive integers j where the series on the right-hand side converge absolutely and uniformly with respect to j.

Proof. From (v) we have

$$(5.6) \qquad |t|^m \eta_{tj}^{-2} \leqq t^{2n} \eta_{tj}^{-2} \leqq t^{2n} \eta_t^{-2}$$

for any integer t and any positive integer j. The series

$$(5.7) \qquad \sum_{t=-\infty}^{\infty} t^{2n} \eta_t^{-2}$$

converges with respect to Theorem 2.1, does not depend on j, and majorizes all the series (5.4). Hence the series (5.4) converge absolutely and uniformly with respect to j for all positive integers j. The rest of the statement follows immediately from (iv) by permuting the series.

With respect to properties of the strongly periodic space we may generally find the form of some coefficients of the optimal quadrature formula.

Theorem 5.2. *Let H be a strongly n-periodic space, let $I(n, j; K_0(n, j), K_1(n, j), \ldots$ $\ldots, K_n(n, j))$ be an optimal quadrature formula in this space H. Then*

$$(5.8) \qquad K_m(n, j) = 0 \quad \textit{for all} \quad m \leqq n \quad \textit{odd} .$$

Proof. Let us consider the system (4.17) of linear algebraic equations for the coefficients K_m; $m = 0, 1, \ldots, n$ of the optimal quadrature formula. Using Lemma 5.1 we may rewrite the equation (4.17) for $s = 0$ as

$$\bar{K}_0\left(\eta_0^{-2} + 2\sum_{t=1}^{\infty} \eta_{tj}^{-2}\right) + 2\sum_{\substack{m>0 \\ \text{even}}}^{n} \bar{K}_m i^m \sum_{t=1}^{\infty} t^m \eta_{tj}^{-2} = \eta_0^{-2}. \tag{5.9}$$

For $s > 0$ even we get

$$\sum_{\substack{m \geq 0 \\ \text{even}}}^{n} \bar{K}_m i^m \sum_{t=1}^{\infty} t^{m+s} \eta_{tj}^{-2} = 0. \tag{5.10}$$

Finally for s odd, we have

$$\sum_{\substack{m>0 \\ \text{odd}}}^{n} \bar{K}_m i^m \sum_{t=1}^{\infty} t^{m+s} \eta_{tj}^{-2} = 0. \tag{5.11}$$

The system (4.17) is decomposed into two systems, the system (5.10) with the equation (5.9) that together determine the coefficients K_m for m even and the system (5.11) that determines the coefficients K_m for m odd. The system (5.11) is homogeneous and nonsingular with respect to Theorem 4.3. Hence it has only a trivial solution. Therefore the theorem has been proved.

Remark 5.3. Let $n \geqq 0$ be an even integer, let H be a strongly $(n + 1)$-periodic space, let $I(n, j; K_0(n, j), K_1(n, j), \ldots, K_n(n, j))$ be the optimal quadrature formula in this space H regarded as strongly n-periodic in the sense of Remark 5.2. Then the formula

$$I(n + 1, j; K_0(n, j), K_1(n, j), \ldots, K_n(n, j), 0)$$

is optimal in the strongly $(n + 1)$-periodic space H. Therefore it holds

$$K_m(n + 1, j) = K_m(n, j)\,; \quad m = 0, 1, \ldots, n\,,$$
$$K_{n+1}(n + 1, j) = 0\,.$$

The statement follows immediately from Theorem 5.2.

We may determine the coefficients of the quadrature formula optimal in the strongly n-periodic space explicitly in two simplest cases, i.e. $n = 0$ and $n = 2$. We shall confine ourselves to these two cases of a strongly 0-periodic and a strongly 2-periodic space in Sec. 6, 7, 8. The case $n = 1$ may be converted into the case of a strongly 0-periodic space, the case $n = 3$ into the case $n = 2$.

Theorem 5.3. *Let H be a strongly 0-periodic space. Then*

$$K_0(0, j) = \frac{1}{1 + 2\eta_0^2 \sum_{t=1}^{\infty} \eta_{tj}^{-2}} \tag{5.12}$$

is a coefficient of the quadrature formula optimal in this space H.

Proof. The statement follows immediately from Theorem 4.5.

Theorem 5.4. *Let H be a strongly 2-periodic space. Let us put*

$$W^{-1} = \eta_0^2((\eta_0^{-2} + 2\sum_{t=1}^{\infty} \eta_{tj}^{-2}) \sum_{t=1}^{\infty} t^4 \eta_{tj}^{-2} - 2(\sum_{t=1}^{\infty} t^2 \eta_{tj}^{-2})^2) . \tag{5.13}$$

Then

$$K_0(2, j) = W \sum_{t=1}^{\infty} t^4 \eta_{tj}^{-2} , \tag{5.14}$$

$$K_1(2, j) = 0 , \tag{5.15}$$

$$K_2(2, j) = W \sum_{t=1}^{\infty} t^2 \eta_{tj}^{-2} \tag{5.16}$$

are the coefficients of the quadrature formula optimal in this space H.

Proof. Writing the system (4.17) (or the system (5.9), (5.10)) for $n = 2$ and solving it we get the statement of the theorem immediately from Theorem 4.3. The equality (5.15) follows from Theorem 5.2.

6. LIMITS OF COEFFICIENTS OF OPTIMAL QUADRATURE FORMULAE

It is apparent from Theorem 4.3 that coefficients of the optimal quadrature formula depend on particular n-periodic space H where we construct the formula. In general, if we consider a quadrature formula optimal in a certain n-periodic space then its error may be very large in another n-periodic space as compared with the error of the quadrature formula optimal in this latter space. From the practical point of view it is useful to seek such quadrature formulae the error of which is asymptotically comparable in some sense with the error of optimal quadrature formula universally, i.e. comparable with the error of optimal formulae in all particular n-periodic or strongly n-periodic spaces of some class of spaces.

First let us try to determine limits for $j \to \infty$ of coefficients of the quadrature formula optimal in 0-periodic and strongly 2-periodic spaces. We find out that these limits may be determined independently of the choice of space H. We shall use this fact in Sec. 7 to construct quadrature formulae with properties universal with respect to the choice of space H.

Now let us make an auxiliary statement.

Lemma 6.1. *Let H be an n-periodic space. Then*

$$\lim_{j\to\infty} \sum_{\substack{tj\in U \\ t\neq 0}} t^{2s}\eta_{tj}^{-2} = 0\,; \quad s = 0, 1, \ldots, n\,. \tag{6.1}$$

In addition, if the space H is strongly n-periodic we have

$$\lim_{j\to\infty} \sum_{t=1}^{\infty} t^{2s}\eta_{tj}^{-2} = 0\,; \quad s = 0, 1, \ldots, n\,. \tag{6.2}$$

Proof. Let the set $U(H)$ be finite. Then there exists a number $T \in U$ such that

$$|t| \leqq |T| \quad \text{for all} \quad t \in U\,.$$

Let $j > |T|$ then $tj \notin U$ for any integer $t \neq 0$ and thus

$$\sum_{\substack{tj\in U \\ t\neq 0}} t^{2s}\eta_{tj}^{-2} = 0\,; \quad s = 0, 1, \ldots, n$$

holds for all $j > |T|$. (6.1) follows immediately from this relation.

Now let $U(H)$ be infinite. All series

$$\sum_{k\in U} k^{2s}\eta_k^{-2}\,; \quad s = 0, 1, \ldots, n$$

converge absolutely since the series (2.1) majorizes them and converges with respect to Theorem 2.1. The absolute convergence is equivalent to the following statement: For arbitrary $\varepsilon > 0$ there exists a positive integer K so that

$$\varepsilon > \Big|\sum_{k\in U} k^{2s}\eta_k^{-2} - \sum_{\substack{k\in U \\ |k|\leq K}} k^{2s}\eta_k^{-2}\Big| = \sum_{\substack{k\in U \\ |k|>K}} k^{2s}\eta_k^{-2}\,; \quad s = 0, 1, \ldots, n\,. \tag{6.3}$$

Let j be a positive integer. We extract such k's from the set U for which $k = tj$ with a certain integer t. Then

$$\sum_{\substack{k\in U \\ |k|>K}} k^{2s}\eta_k^{-2} \geqq \sum_{\substack{tj\in U \\ |tj|>K}} t^{2s}j^{2s}\eta_{tj}^{-2} \geqq \sum_{\substack{tj\in U \\ |tj|>K}} t^{2s}\eta_{tj}^{-2}\,; \quad s = 0, 1, \ldots, n\,. \tag{6.4}$$

Now let $j > K$. Then from (6.3), (6.4) we get

$$\varepsilon > \sum_{\substack{tj\in U \\ |tj|>K}} t^{2s}\eta_{tj}^{-2} = \sum_{\substack{tj\in U \\ t\neq 0}} t^{2s}\eta_{tj}^{-2}\,; \quad s = 0, 1, \ldots, n\,. \tag{6.5}$$

Therefore for arbitrary $\varepsilon > 0$, there exists a positive integer K such that (6.5) is valid for all $j > K$. Hence (6.1) has been proved. (6.2) follows from the statement proved by using Lemma 5.1.

We may find the limit of the coefficient $K_0(0, j)$ supposing that the space H is only 0-periodic.

Theorem 6.1. *Let H be a 0-periodic space. Then*

$$\lim_{j \to \infty} K_0(0, j) = 1 . \tag{6.6}$$

Proof. With respect to Theorem 4.5 we may write

$$K_0(0, j) = \frac{1}{1 + \eta_0^2 \sum\limits_{\substack{tj \in U \\ t \neq 0}} \eta_{tj}^{-2}}$$

as follows from the absolute convergence of the series in (4.19). We get the limit (6.6) immediately, using Lemma 6.1.

Before we prove an analogous statement for strongly 2-periodic space let us show this auxiliary statement.

Lemma 6.2. *Let H be a strongly n-periodic space. Let*

$$\lim_{j \to \infty} \frac{\eta_j^2}{\eta_{2j}^2} = 0 . \tag{6.7}$$

Then

$$\lim_{j \to \infty} \frac{\eta_j^2}{\eta_{tj}^2} = 0 \tag{6.8}$$

for any integer $t \geqq 2$.

Proof. With respect to (v) we may write

$$\frac{\eta_j^2}{\eta_{2j}^2} \geqq \frac{\eta_j^2}{\eta_{tj}^2}$$

for any integer $t \geqq 2$. Passing to the limit for $j \to \infty$ and using (6.7), we get (6.8).

Theorem 6.2. *Let H be a strongly 2-periodic space. Then*

$$K_0(2, j) \leqq 1 , \quad K_2(2, j) \leqq 1 \tag{6.9}$$

for any positive integer j. Further

(6.10) $$\lim_{j\to\infty} K_0(2, j) = 1\,,$$

(6.11) $$\limsup_{j\to\infty} K_2(2, j) \leqq 1\,.$$

The relation

(6.12) $$\limsup_{j\to\infty} K_2(2, j) = 1$$

holds if and only if

(6.13) $$\liminf_{j\to\infty} \frac{\eta_j^2}{\eta_{2j}^2} = 0\,.$$

If the limit exists in one of the relations (6.12), (6.13) *then it exists in the other as well.*

Proof. Using Hölder inequality we have

(6.14) $$\left(\sum_{t=1}^{\infty} t^2\eta_{tj}^{-2}\right)^2 \leqq \sum_{t=1}^{\infty} \eta_{tj}^{-2} \sum_{t=1}^{\infty} t^4\eta_{tj}^{-2}\,.$$

We may rewrite (5.13) as

(6.15) $$W^{-1} = \sum_{t=1}^{\infty} t^4\eta_{tj}^{-2} + 2\eta_0^2\left(\sum_{t=1}^{\infty} \eta_{tj}^{-2} \sum_{t=1}^{\infty} t^4\eta_{tj}^{-2} - \left(\sum_{t=1}^{\infty} t^2\eta_{tj}^{-2}\right)^2\right) \geqq \sum_{t=1}^{\infty} t^4\eta_{tj}^{-2}\,.$$

We use Theorem 5.4 now. From (5.14), (6.15) we get the first part of (6.9). Analogously from (5.16), (6.15) it follows

$$K_2(2, j) \leqq \sum_{t=1}^{\infty} t^2\eta_{tj}^{-2}\left(\sum_{t=1}^{\infty} t^4\eta_{tj}^{-2}\right)^{-1} \leqq 1\,,$$

which completes the proof of (6.9).

With respect to (5.14), (6.15) we write

$$\frac{1}{K_0(2, j)} = 1 + 2\eta_0^2 \sum_{t=1}^{\infty} \eta_{tj}^{-2} - 2\eta_0^2 \sum_{t=1}^{\infty} t^2\eta_{tj}^{-2} \frac{\sum_{t=1}^{\infty} t^2\eta_{tj}^{-2}}{\sum_{t=1}^{\infty} t^4\eta_{tj}^{-2}}\,.$$

The inequality

$$0 \leqq \frac{\sum_{t=1}^{\infty} t^2\eta_{tj}^{-2}}{\sum_{t=1}^{\infty} t^4\eta_{tj}^{-2}} \leqq 1$$

may be easily verified. Then using Lemma 6.1 we have

$$\lim_{j\to\infty} \frac{1}{K_0(2, j)} = 1 ,$$

which proves (6.10). (6.11) follows immediately from (6.9).

Further with respect to (5.16), (6.15) we write

$$(6.16) \qquad \frac{1}{K_2(2, j)} = \frac{\sum_{t=1}^{\infty} t^4 \eta_{tj}^{-2}}{\sum_{t=1}^{\infty} t^2 \eta_{tj}^{-2}} + 2\eta_0^2 \sum_{t=1}^{\infty} \eta_{tj}^{-2} \frac{\eta_j^2 \sum_{t=1}^{\infty} t^4 \eta_{tj}^{-2}}{\eta_j^2 \sum_{t=1}^{\infty} t^2 \eta_{tj}^{-2}} - 2\eta_0^2 \sum_{t=1}^{\infty} t^2 \eta_{tj}^{-2} .$$

The limit of the last term is equal to 0 with respect to Lemma 6.1. Using (vi) we rewrite the central term as

$$2\eta_0^2 \sum_{t=1}^{\infty} \eta_{tj}^{-2} \frac{\eta_j^2 \sum_{t=1}^{\infty} t^4 \eta_{tj}^{-2}}{\eta_j^2 \sum_{t=1}^{\infty} t^2 \eta_{tj}^{-2}} \leqq 2\eta_0^2 \sum_{t=1}^{\infty} \eta_{tj}^{-2} \,.\, \eta_j^2 \sum_{t=1}^{\infty} t^4 \eta_{tj}^{-2} \leqq 2\eta_0^2\, D(H) \sum_{t=1}^{\infty} \eta_{tj}^{-2} .$$

In virtue of Lemma 6.1 the limit of this central term equals 0, too. Thus from (6.16) we get

$$(6.17) \qquad \liminf_{j\to\infty} \frac{1}{K_2(2, j)} = \liminf_{j\to\infty} \frac{\sum_{t=1}^{\infty} t^4 \eta_{tj}^{-2}}{\sum_{t=1}^{\infty} t^2 \eta_{tj}^{-2}} ,$$

which is equivalent to

$$(6.18) \qquad \limsup_{j\to\infty} K_2(2, j) = \limsup_{j\to\infty} \frac{\sum_{t=1}^{\infty} t^2 \eta_{tj}^{-2}}{\sum_{t=1}^{\infty} t^4 \eta_{tj}^{-2}} .$$

A. Let (6.13) hold. Then there exists a sequence $\{j_r\}$ of positive integers such that

$$\lim_{r\to\infty} \frac{\eta_{j_r}^2}{\eta_{2j_r}^2} = 0 .$$

Using this and Lemmas 5.1, 6.2 we get

$$(6.19) \qquad \lim_{r\to\infty} \frac{\sum_{t=1}^{\infty} t^2 \eta_{tj_r}^{-2}}{\sum_{t=1}^{\infty} t^4 \eta_{tj_r}^{-2}} = \lim_{r\to\infty} \frac{\sum_{t=1}^{\infty} t^2 \frac{\eta_{j_r}^2}{\eta_{tj_r}^2}}{\sum_{t=1}^{\infty} t^4 \frac{\eta_{j_r}^2}{\eta_{tj_r}^2}} = 1 .$$

Therefore 1 is a limit point of the sequence

$$\left\{\frac{\sum_{t=1}^{\infty} t^2 \eta_{tj}^{-2}}{\sum_{t=1}^{\infty} t^4 \eta_{tj}^{-2}}\right\}.$$

With respect to (6.11), (6.18) it is the largest limit point and hence (6.12) follows from (6.13).

B. Let (6.13) be invalid. We write

$$\liminf_{j \to \infty} \frac{\eta_j^2}{\eta_{2j}^2} = E > 0\,. \tag{6.20}$$

Now we use the relation (6.17). Let us write

$$\frac{\sum_{t=1}^{\infty} t^4 \eta_{tj}^{-2}}{\sum_{t=1}^{\infty} t^2 \eta_{tj}^{-2}} = 1 + \frac{12}{4 + \dfrac{\eta_{2j}^2}{\eta_j^2}} + Z \tag{6.21}$$

where

$$Z = \frac{\sum_{t=1}^{\infty} t^4 \eta_{tj}^{-2} - \left(1 + \dfrac{12}{4 + (\eta_{2j}^2/\eta_j^2)}\right) \sum_{t=1}^{\infty} t^2 \eta_{tj}^{-2}}{\sum_{t=1}^{\infty} t^2 \eta_{tj}^{-2}}\,.$$

The inequality

$$Z \geqq 0 \tag{6.22}$$

may be easily proved by verifying that the numerator is non-negative. Simplifying the numerator we get

$$\sum_{t=3}^{\infty} \eta_{tj}^{-2} \left(t^4 - \left(1 + \frac{12}{4 + (\eta_{2j}^2/\eta_j^2)}\right) t^2\right).$$

It may be shown that all the terms of this series are non-negative. Thus with respect to (6.17), (6.20) to (6.22) we have

$$\liminf_{j \to \infty} \frac{1}{K_2(2, j)} \geqq 1 + \frac{12}{4 + 1/E} > 1\,,$$

i.e.

$$\limsup_{j \to \infty} K_2(2, j) < 1\,,$$

which completes the equivalence.

Now let

$$(6.23) \qquad \lim_{j\to\infty} \frac{\eta_j^2}{\eta_{2j}^2} = 0$$

hold. Then using Lemmas 5.1, 6.1, 6.2, and (6.16) we get

$$(6.24) \qquad \lim_{j\to\infty} K_2(2, j) = 1$$

in the analogous way as (6.19).

On the other hand, let (6.24) exist. Using (6.21), (6.22) we have

$$\frac{\sum_{t=1}^{\infty} t^4 \eta_{tj}^{-2}}{\sum_{t=1}^{\infty} t^2 \eta_{tj}^{-2}} \geqq 1 + \frac{12}{4 + \frac{\eta_{2j}^2}{\eta_j^2}} \geqq 1 .$$

Then with respect to (6.18), (6.24)

$$\lim_{j\to\infty} \frac{12}{4 + \frac{\eta_{2j}^2}{\eta_j^2}} = 0$$

exists and the existence of (6.23) follows from this. The proof has been completed.

Thus we have shown that the limit of the coefficient $K_0(0, j)$ equals 1 in any 0-periodic space and the limit of the coefficient $K_0(2, j)$ equals 1 in any strongly 2-periodic space, too. We have found an upper bound for the superior limit of the coefficient $K_2(2, j)$ independently of the choice of strongly 2-periodic space. This bound is equal to 1. We have shown that there exists such a space where this bound is attainable. In the sense of Theorem 2.2 the condition (6.13) is satisfied e.g. by the strongly 2-periodic space with the norm $\eta_k^2 = e^{|k|}$.

From Sec. 7 it will be apparent why we do not examine 2-periodic spaces generally.

Let us introduce a notation for quadrature formulae using limit coefficients.

Definition 6.1. Let H be a 0-periodic space. The quadrature formula $I(0, j; 1)$ is said to be a limit formula. We denote its error by

$$\Phi(0, j, H) = \|J - I(0, j; 1)\| .$$

Definition 6.2. Let H be a 2-periodic space. The quadrature formula $I(2, j; 1, 0, 1)$ is said to be a limit formula. We denote its error by

$$\Phi(2, j, H) = \|J - I(2, j; 1, 0, 1)\| .$$

Remark 6.1. As compared with the coefficients of the optimal quadrature formulae, the coefficients of the limit formulae do not depend on the choice of the space H. The limit formula $I(0, j; 1)$ is a usual trapezoid rule. The limit formula $I(2, j; 1, 0, 1)$ is a certain generalization of the trapezoid rule. The sum of the values of the second derivative of the integrand in equidistant abscissae multiplied by j^{-3} appears in this formula.

We shall be concerned with examining the properties of the limit quadrature formulae in Sec. 7. Let us consider their errors here now.

Theorem 6.3. *Let H be a 0-periodic space. Then the square of the error of the limit quadrature formula equals*

$$\Phi^2(0, j, H) = \sum_{\substack{tj \in U \\ t \neq 0}} \eta_{tj}^{-2} .$$

Proof. The statement follows from Lemma 4.2 by using Definition 6.1.

Theorem 6.4. *Let H be a 2-periodic space. Then the square of the error of the limit quadrature formula equals*

$$\Phi^2(2, j, H) = \sum_{\substack{tj \in U \\ t \neq 0}} (t^2 - 1)^2 \eta_{tj}^{-2} .$$

Proof. The statement follows from Lemma 4.2 by using Definition 6.2.

7. UNIVERSAL PROPERTIES OF LIMIT QUADRATURE FORMULAE

Let us now examine the asymptotical behaviour of the errors $\Phi(0, j, H)$, $\Phi(2, j, H)$ of the limit quadrature formulae for $j \to \infty$ in comparison with the behaviour of the error of the corresponding optimal quadrature formula. Let us introduce the following concepts.

Definition 7.1. Let H be an n-periodic space, let $X(n, j, H)$ be an error of the optimal quadrature formula in this space.

The quadrature formula

$$I(n, j; A_0, A_1, \ldots, A_n) \tag{7.1}$$

is said to be asymptotically optimal in this space H if and only if

$$\|J - I(n, j; A_0, A_1, \ldots, A_n)\| = X(n, j, H)\,(1 + o(1))$$

holds for $j \to \infty$.

The formula (7.1) is said to be optimal in order in this space H if and only if

$$\|J - I(n, j; A_0, A_1, \ldots, A_n)\| = X(n, j, H)\, O(1)$$

holds for $j \to \infty$.

Further the formula (7.1) is said to be universally asymptotically optimal or universally optimal in order in a class of n-periodic spaces if and only if it is asymptotically optimal or optimal in order respectively in any n-periodic space H of this class.

Remark 7.1. Let H be an n-periodic space. If the quadrature formula $I(n, j; A_0, A_1, \ldots, A_n)$ is asymptotically optimal in this space H then it is also optimal in order in this space.

Analogously if the formula $I(n, j; A_0, A_1, \ldots, A_n)$ is asymptotically optimal in some class of n-periodic spaces then it is also optimal in order in this class of spaces.

Now let us show the universal optimality of the limit quadrature formulae.

Theorem 7.1. *The limit quadrature formula $I(0, j; 1)$ is universally asymptotically optimal in the class of all 0-periodic spaces.*

Proof. With respect to Definition 7.1 it is sufficient to prove

$$\lim_{j \to \infty} \frac{\Phi(0, j, H)}{X(0, j, H)} = 1 . \tag{7.2}$$

With respect to Theorems 4.4, 4.5, and 6.3 we have

$$\frac{\Phi^2(0, j, H)}{X^2(0, j, H)} = 1 + \eta_0^2 \sum_{\substack{tj \in U \\ t \neq 0}} \eta_{tj}^{-2} .$$

Further using Lemma 6.1 we get

$$\lim_{j \to \infty} \frac{\Phi^2(0, j, H)}{X^2(0, j, H)} = 1 .$$

The relation (7.2) follows from this equality.

Theorem 7.2. *The limit quadrature formula $I(2, j; 1, 0, 1)$ is universally optimal in order in the class of all strongly 2-periodic spaces.*

This formula is universally asymptotically optimal only in the class of strongly 2-periodic spaces such that

$$\lim_{j \to \infty} \frac{\eta_j^2}{\eta_{2j}^2} = 0 . \tag{7.3}$$

Proof. To prove the first statement of the theorem, it is sufficient to show

$$\limsup_{j\to\infty} \frac{\Phi(2, j, H)}{X(2, j, H)} < +\infty . \tag{7.4}$$

With respect to Theorems 4.4, 5.4, 6.4, and (iv) we write

$$\frac{\Phi^2(2, j, H)}{X^2(2, j, H)} \tag{7.5}$$

$$= \frac{\sum_{t=1}^{\infty} (t^2 - 1)^2 \eta_{tj}^{-2} \sum_{t=1}^{\infty} t^4 \eta_{tj}^{-2}}{\sum_{t=1}^{\infty} \eta_{tj}^{-2} \sum_{t=1}^{\infty} t^4 \eta_{tj}^{-2} - (\sum_{t=1}^{\infty} t^2 \eta_{tj}^{-2})^2} + 2\eta_0^2 \sum_{t=1}^{\infty} (t^2 - 1)^2 \eta_{tj}^{-2} .$$

Using Lemma 6.1 we get

$$\lim_{j\to\infty} 2\eta_0^2 \sum_{t=1}^{\infty} (t^2 - 1)^2 \eta_{tj}^{-2} = 0 . \tag{7.6}$$

Let us find the limit of the first term on the right-hand side of (7.5) that we denote by $Q(j)$. By a straightforward computation we may find out that the denominator of $Q(j)$ equals

$$\sum_{t=1}^{\infty} \eta_{tj}^{-2} \sum_{t=1}^{\infty} t^4 \eta_{tj}^{-2} - (\sum_{t=1}^{\infty} t^2 \eta_{tj}^{-2})^2$$

$$= \eta_j^{-2} (\sum_{t=1}^{\infty} \eta_{tj}^{-2} - 2 \sum_{t=1}^{\infty} t^2 \eta_{tj}^{-2} + \sum_{t=1}^{\infty} t^4 \eta_{tj}^{-2})$$

$$+ \sum_{t=2}^{\infty} \eta_{tj}^{-2} \sum_{t=2}^{\infty} t^4 \eta_{tj}^{-2} - (\sum_{t=2}^{\infty} t^2 \eta_{tj}^{-2})^2$$

$$\geqq \eta_j^{-2} (\sum_{t=1}^{\infty} \eta_{tj}^{-2} - 2 \sum_{t=1}^{\infty} t^2 \eta_{tj}^{-2} + \sum_{t=1}^{\infty} t^4 \eta_{tj}^{-2}) = \eta_j^{-2} \sum_{t=1}^{\infty} (t^2 - 1)^2 \eta_{tj}^{-2}$$

as it follows from the inequality analogous to (6.14). Hence we may estimate

$$Q(j) \leqq \eta_j^2 \sum_{t=1}^{\infty} t^4 \eta_{tj}^{-2} \leqq D(H) \tag{7.7}$$

using (vi). Therefore with respect to (7.5) to (7.7) we get

$$\limsup_{j\to\infty} \frac{\Phi^2(2, j, H)}{X^2(2, j, H)} \leqq D(H) < +\infty .$$

Then (7.4) follows from this inequality immediately.

A. Let (7.3) hold. Then with respect to (7.7) and Lemmas 5.1, 6.2 we have

$$\lim_{j\to\infty} Q(j) \leqq \lim_{j\to\infty} \eta_j^2 \sum_{t=1}^{\infty} t^4 \eta_{tj}^{-2} = 1 .$$

However, using Definition 7.1 and (7.5), (7.6) we get

$$1 \leqq \lim_{j\to\infty} Q(j) , \tag{7.8}$$

i.e.

$$\lim_{j\to\infty} \frac{\Phi^2(2, j, H)}{X^2(2, j, H)} = 1 .$$

From this relation it follows

$$\lim_{j\to\infty} \frac{\Phi(2, j, H)}{X(2, j, H)} = 1 . \tag{7.9}$$

B. It remains to complete the proof of the equivalence. Thus let

$$\limsup_{j\to\infty} \frac{\eta_j^2}{\eta_{2j}^2} = E > 0 .$$

Using proper manipulation we get

$$Q(j) = \frac{\sum_{t=1}^{\infty} t^4\eta_{tj}^{-2} \sum_{t=1}^{\infty} \eta_{tj}^{-2} - 2\sum_{t=1}^{\infty} t^4\eta_{tj}^{-2} \sum_{t=1}^{\infty} t^2\eta_{tj}^{-2} + (\sum_{t=1}^{\infty} t^4\eta_{tj}^{-2})^2}{\sum_{t=1}^{\infty} \eta_{tj}^{-2} \sum_{t=1}^{\infty} t^4\eta_{tj}^{-2} - (\sum_{t=1}^{\infty} t^2\eta_{tj}^{-2})^2}$$

$$+ \frac{(\sum_{t=1}^{\infty} t^2\eta_{tj}^{-2})^2 - (\sum_{t=1}^{\infty} t^2\eta_{tj}^{-2})^2}{\sum_{t=1}^{\infty} \eta_{tj}^{-2} \sum_{t=1}^{\infty} t^4\eta_{tj}^{-2} - (\sum_{t=1}^{\infty} t^2\eta_{tj}^{-2})^2} = 1 + \frac{(\sum_{t=1}^{\infty} t^4\eta_{tj}^{-2} - \sum_{t=1}^{\infty} t^2\eta_{tj}^{-2})^2}{\sum_{t=1}^{\infty} \eta_{tj}^{-2} \sum_{t=1}^{\infty} t^4\eta_{tj}^{-2} - (\sum_{t=1}^{\infty} t^2\eta_{tj}^{-2})^2}$$

$$\geqq 1 + \frac{(\sum_{t=2}^{\infty} t^2(t^2-1)\,\eta_{tj}^{-2})^2}{\sum_{t=1}^{\infty} \eta_{tj}^{-2} \sum_{t=1}^{\infty} t^4\eta_{tj}^{-2}} \geqq 1 + \frac{144\,(\eta_j^2/\eta_{2j}^2)^2}{\eta_j^2 \sum_{t=1}^{\infty} \eta_{tj}^{-2} \,.\, \eta_j^2 \sum_{t=1}^{\infty} t^4\eta_{tj}^{-2}} \geqq 1 + \frac{144}{D^2(H)} \left(\frac{\eta_j^2}{\eta_{2j}^2}\right)^2$$

where we use (vi). Further we may write

$$\limsup_{j\to\infty} Q(j) \geqq 1 + \frac{144E^2}{D^2(H)} > 1 ,$$

which with (7.5), (7.6) gives

$$\limsup_{j\to\infty} \frac{\Phi^2(2, j, H)}{X^2(2, j, H)} > 1 .$$

This relation implies the invalidity of (7.9), which completes the proof of the theorem.

Let us note that the relation

$$\limsup_{j\to\infty} \frac{\Phi(2, j, H)}{X(2, j, H)} = 1$$

is equivalent to (7.9) with respect to Definition 7.1. Hence examining the asymptotical optimality we may confine ourselves to (7.9).

Whereas it is possible to examine the universal properties of the limit quadrature formula in a general 0-periodic space, in the treatment of the limit quadrature formula $I(2, j; 1, 0, 1)$ we have to confine ourselves only to strongly 2-periodic spaces if we want to prove the universality. This fact is shown by the following theorem.

Theorem 7.3. *There exists a 2-periodic space H such that*

$$\limsup_{j\to\infty} \frac{\Phi(2, j, H)}{X(2, j, H)} = +\infty .$$

Proof. We use Theorem 2.2 to construct the 2-periodic space H with this property. Let U be a set of all integers. We put

$$\text{(7.10)} \qquad \begin{aligned} \eta_k^2 &= e^{3|k|}k^4 ; \quad |k| = 7^r , \\ \eta_k^2 &= e^{|k|}k^4 ; \quad |k| \neq 7^r \end{aligned}$$

for all non-negative integers r. Let us construct the corresponding Hilbert space H of 2π-periodic continuous functions with the first and second derivative continuous. The condition (2.16) of Theorem 2.2 is fulfilled since we may estimate

$$\sum_{k=-\infty}^{\infty} k^4 \eta_k^{-2} \leqq \sum_{k=-\infty}^{\infty} e^{-|k|} k^4 k^{-4} = 1 + \frac{2e^{-1}}{1 - e^{-1}} < +\infty .$$

Thus this space is 2-periodic. The validity of (iv) follows immediately from (7.10). It may be readily shown that the statement of Lemma 5.1 holds in this space H. It is sufficient to use

$$|t|^m \eta_{tj}^{-2} \leqq t^4 \eta_{tj}^{-2} \leqq e^{-|t|j} j^{-4} \leqq e^{-|t|} ; \quad m = 0, 1, 2, 3, 4$$

instead of (5.6) in the proof of that lemma since the series

$$\sum_{k=-\infty}^{\infty} e^{-|k|}$$

converges. Now we may show that Theorem 5.2 holds in this space H for $n = 2$ and the statement of Theorem 5.4 holds finally, too.

We put

$$j_s = 7^s \tag{7.11}$$

for any non-negative integer s. To prove the theorem it is sufficient to show

$$\limsup_{s\to\infty} \frac{\Phi(2, j_s, H)}{X(2, j_s, H)} = +\infty . \tag{7.12}$$

Analogously to the proof of Theorem 7.2, we get

$$\limsup_{s\to\infty} \frac{\Phi^2(2, j_s, H)}{X^2(2, j_s, H)} = \lim_{s\to\infty} Q(j_s)$$

where

$$Q(j) = \frac{\sum_{t=1}^{\infty} t^4\eta_{tj}^{-2}\left(\sum_{t=1}^{\infty} \eta_{tj}^{-2} - 2\sum_{t=1}^{\infty} t^2\eta_{tj}^{-2} + \sum_{t=1}^{\infty} t^4\eta_{tj}^{-2}\right)}{\sum_{t=1}^{\infty} \eta_{tj}^{-2}\sum_{t=1}^{\infty} t^4\eta_{tj}^{-2} - \left(\sum_{t=1}^{\infty} t^2\eta_{tj}^{-2}\right)^2} .$$

We estimate the denominator $R(j)$ of the fraction $Q(j)$ by

$$R(j) \leqq \frac{9}{\eta_j^2\eta_{2j}^2} + \frac{2}{\eta_j^2}\sum_{t=3}^{\infty} t^4\eta_{tj}^{-2} + \frac{17}{\eta_{2j}^2}\sum_{t=3}^{\infty} t^4\eta_{2j}^{2} + \left(\sum_{t=3}^{\infty} t^4\eta_{tj}^{-2}\right)^2 \tag{7.13}$$

where we use the inequality

$$\sum_{t=3}^{\infty} \eta_{tj}^{-2} \leqq \sum_{t=3}^{\infty} t^4\eta_{tj}^{-2} .$$

Further we estimate the numerator $P(j)$ of the fraction $Q(j)$ by

$$P(j) = \sum_{t=1}^{\infty} t^4\eta_{tj}^{-2}\sum_{t=2}^{\infty} (t^2 - 1)^2\, \eta_{tj}^{-2} \geqq \frac{144}{\eta_{2j}^4} . \tag{7.14}$$

Using (7.13), (7.14) we have

(7.15)

$$0 \leqq \frac{1}{Q(j)} \leqq \frac{1}{144}\left(9\frac{\eta_{2j}^2}{\eta_j^2} + 2\frac{\eta_{2j}^4}{\eta_j^2}\sum_{t=3}^{\infty} t^4\eta_{tj}^{-2} + 17\eta_{2j}^2\sum_{t=3}^{\infty} t^4\eta_{tj}^{-2} + \left(\eta_{2j}^2\sum_{t=3}^{\infty} t^4\eta_{tj}^{-2}\right)^2\right) .$$

With respect to (7.10), (7.11) we may estimate

$$\sum_{t=3}^{\infty} t^4 \eta_{tj_s}^{-2} \leqq j_s^{-4} \sum_{t=3}^{\infty} \exp(-tj_s) = j_s^{-4} \frac{\exp(-3j_s)}{1 - \exp(-j_s)}. \tag{7.16}$$

Finally using (7.10), (7.11), (7.15), (7.16) we get

$$\lim_{s \to \infty} \frac{1}{Q(j_s)} = 0.$$

This implies (7.12) and the theorem has been proved.

The limit quadrature formulae $I(0, j; 1)$, $I(2, j; 1, 0, 1)$ are the unique universally asymptotically optimal or universally optimal in order formulae respectively in the sense of the two following theorems.

Theorem 7.4. *The quadrature formula*

$$I(0, j; A_0(j)) \tag{7.17}$$

is universally asymptotically optimal in the class of all 0-periodic spaces if and only if there exists a positive integer L such that $A_0(j) = 1$ for all integers $j \geqq L$.

Proof. A. If there exists a positive integer L such that $A_0(j) = 1$ for all integers $j \geqq L$ then the formula (7.17) is universally asymptotically optimal in the class of all 0-periodic spaces with respect to Theorem 7.1.

B. Let us have an infinite sequence $\{j_s\}$ such that

$$A_0(j_s) \neq 1$$

for any positive integer s, i.e.

$$|1 - A_0(j_s)|^2 > 0$$

for any positive integer s. We construct a 0-periodic space H such that

$$\limsup_{s \to \infty} \frac{\|J - I(0, j_s; A_0(j_s))\|^2}{X^2(0, j_s, H)} = +\infty \tag{7.18}$$

holds in this space. In this way we get a contradiction in the sense of Definition 7.1.

Let us confine ourselves to spaces H such that $U(H)$ is a set of all integers. From Lemma 4.2 we write

$$\|J - I(0, j; A_0(j))\|^2 = \eta_0^{-2}|1 - A_0(j)|^2$$
$$+ |A_0(j)|^2 \sum_{\substack{t=-\infty \\ t \neq 0}}^{\infty} \eta_{tj}^{-2} \geqq \eta_0^{-2}|1 - A_0(j)|^2.$$

Further using Definition 7.1 and Theorem 6.3 we have

$$X^2(0, j, H) \leqq \Phi^2(0, j, H) = \sum_{\substack{t=-\infty \\ t \neq 0}}^{\infty} \eta_{tj}^{-2} .$$

Hence we may estimate

$$\frac{\|J - I(0, j; A_0(j))\|^2}{X^2(0, j, H)} \geqq \frac{|1 - A_0(j)|^2}{\eta_0^2 \sum\limits_{\substack{t=-\infty \\ t \neq 0}}^{\infty} \eta_{tj}^{-2}} . \tag{7.19}$$

For non-negative integers k we put

$$P(k) = 1 , \quad 0 \leqq k \leqq j_1 ,$$

$$P(k) = \prod_{r=1}^{s} \min \left(|1 - A_0(j_r)|^2, 1\right) , \quad j_{s-1} < k \leqq j_s .$$

Then

$$P(k) \leqq 1 , \quad P(k + 1) \leqq P(k)$$

for any integer $k \geqq 0$.

Now we put

$$\eta_k^2 = \eta_{-k}^2 = \exp\left(\frac{ke^k}{P(k)}\right) \tag{7.20}$$

for any integer $k \geqq 0$. Let us construct the corresponding Hilbert space H of 2π-periodic continuous functions. The condition (2.16) of Theorem 2.2 is fulfilled since we may write

$$\sum_{k=-\infty}^{\infty} \eta_k^{-2} \leqq \sum_{k=-\infty}^{\infty} e^{-|k|} = 1 + \frac{2e^{-1}}{1 - e^{-1}} < +\infty$$

using the estimate

$$\eta_k^2 = \eta_{-k}^2 \geqq \exp\left(ke^k\right) \geqq e^k$$

for any integer $k \geqq 0$.

With respect to (7.20) we may rewrite (7.19) as

$$\frac{\|J - I(0, j; A_0(j))\|^2}{X^2(0, j, H)} \geqq \frac{|1 - A_0(j)|^2}{2\eta_0^2 \sum\limits_{t=1}^{\infty} \eta_{tj}^{-2}} \geqq \frac{je^j(1 - e^{-1})\, |1 - A_0(j)|^2}{2\eta_0^2 P(j)}$$

since we may verify

$$\sum_{t=1}^{\infty} \eta_{tj}^{-2} \leqq \exp\left(-\frac{je^j}{P(j)}\right) \sum_{t=1}^{\infty} e^{-(t-1)} = \frac{1}{1 - e^{-1}} \exp\left(-\frac{je^j}{P(j)}\right) .$$

Further we may show

$$\frac{|1 - A_0(j_s)|^2}{P(j_s)} \geqq 1$$

for any positive integer s. Therefore we get

$$\frac{\|J - I(0, j_s; A_0(j_s))\|^2}{X^2(0, j_s, H)} \geqq \frac{1 - e^{-1}}{2\eta_0^2} j_s \exp(j_s) \to +\infty \quad \text{for} \quad s \to \infty .$$

Thus we have proved (7.18) and the proof of the theorem has been completed.

Remark 7.2. If the quadrature formula (7.17) is universally optimal in order in the class of all 0-periodic spaces then it is also universally asymptotically optimal in this class of spaces.

In fact, let (7.17) be universally optimal in order in this class. Then we show that there exists a positive integer L such that $A_0(j) = 1$ for any $j \geqq L$ in the same way as in part B of the proof of Theorem 7.4. Further, using Theorem 7.4 we get that the quadrature formula (7.17) is universally asymptotically optimal.

With respect to Remark 7.1 the statement analogous to that of Theorem 7.4 holds for the formula universally optimal in order in the class of all 0-periodic spaces.

We use the following auxiliary statement to prove an analogous property of the limit quadrature formula $I(2, j; 1, 0, 1)$.

Lemma 7.1. *Let $P(k)$ be a positive non-increasing function defined for all integers $k \geqq 0$, let $P(0) = 1$. Let H be a Hilbert space of 2π-periodic continuous functions with the first and second derivative continuous. Let*

$$(7.21) \qquad \eta_k^2 = \eta_{-k}^2 = k^4 \exp\left(\frac{ke^k}{P(k)}\right) \quad \textit{for any integer} \quad k > 0\,, \qquad \eta_0^2 = 1\,.$$

Then the space H with the norm (7.21) is strongly 2-periodic and

$$(7.22) \qquad \sum_{t=2}^{\infty} t^4 \eta_{tj}^{-2} \leqq \frac{j^{-4}}{1 - e^{-1}} \exp\left(-\frac{2je^{2j}}{P(2j)}\right)$$

holds for any positive integer j.

Proof. The proof is analogous to the corresponding part of the proof of Theorem 7.4. We shall show that the condition (2.16) of Theorem 2.2 is satisfied in the Hilbert space constructed in this way. With respect to (7.21) we may estimate

$$\eta_k^2 k^{-4} = \eta_{-k}^2 k^{-4} = \exp\left(\frac{ke^k}{P(k)}\right) \geqq e^k$$

for any integer $k > 0$. Therefore

$$\sum_{k=-\infty}^{\infty} k^4 \eta_k^{-2} \leqq \sum_{k=-\infty}^{\infty} e^{-|k|} = 1 + \frac{2e^{-1}}{1 - e^{-1}} < +\infty .$$

From Theorem 2.2 it follows that the space H constructed in this way is 2-periodic. Let us verify the conditions (iv) to (vi) now. (iv) follows from (7.21) immediately since U is a set of all integers. (v) follows from (7.21) and the properties of the function $P(k)$. To prove (vi) we estimate

$$t^4 \frac{\eta_j^2}{\eta_{tj}^2} \leqq \exp\left(\frac{je^j}{P(j)} - \frac{tje^{tj}}{P(tj)}\right) \leqq e^{-(t-1)}$$

for any integer $t > 0$. Hence

$$\eta_j^2 \sum_{t=1}^{\infty} t^4 \eta_{tj}^{-2} \leqq \sum_{t=1}^{\infty} e^{-(t-1)} = \frac{1}{1 - e^{-1}} .$$

Analogously we get the estimate

(7.23)
$$\eta_{2j}^2 \sum_{t=2}^{\infty} t^4 \eta_{tj}^{-2} \leqq 16 \sum_{t=2}^{\infty} e^{-(t-2)} = \frac{16}{1 - e^{-1}} .$$

Therefore (vi) is satisfied if we put

$$D(H) = \frac{16}{1 - e^{-1}} .$$

Thus the space H constructed in this way is strongly 2-periodic. Substituting η_{2j}^2 from (7.21) into (7.23) we get (7.22). The lemma has been proved.

Theorem 7.5. *The quadrature formula*

(7.24)
$$I(2, j; A_0(j), A_1(j), A_2(j))$$

is universally optimal in order in the class of all strongly 2-periodic spaces if and only if there exists a positive integer L such that

(7.25)
$$A_0(j) = 1 , \quad A_1(j) = 0 , \quad A_2(j) = 1$$

for all integers $j \geqq L$.

Proof. A. If there exists a positive integer L such that (7.25) holds for all integers $j \geqq L$ then the formula (7.24) is universally optimal in order in the class of all strongly 2-periodic spaces with respect to Theorem 7.2.

B. Let us have an infinite sequence $\{j_s\}$ such that at least one of the following inequalities holds for any positive integer s:

$$A_0(j_s) \neq 1\,, \tag{7.26}$$

$$A_1(j_s) \neq 0\,, \tag{7.27}$$

$$A_2(j_s) \neq 1\,. \tag{7.28}$$

Thus an infinite sequence that we shall denote again by $\{j_s\}$ may be extracted from the original sequence $\{j_s\}$ so that invariably the same inequality of the inequalities (7.26) to (7.28) holds for any positive integer s. With respect to this fact we distinguish three cases and construct a strongly 2-periodic space H in each of these cases analogously to the proof of Theorem 7.4 so that

$$\lim_{s\to\infty} \frac{\|J - I(2, j_s; A_0(j_s), A_1(j_s), A_2(j_s))\|^2}{X^2(2, j_s, H)} = +\infty \tag{7.29}$$

holds in this space H. In this way we get a contradiction in the sense of Definition 7.1.

Using Lemmas 4.2, 5.1 we may write

$$\|J - I(2, j; A_0(j), A_1(j), A_2(j))\|^2 = \eta_0^{-2}|1 - A_0(j)|^2$$
$$+ 2\sum_{t=1}^{\infty} |A_1(j)|^2\, t^2 \eta_{tj}^{-2} + 2\sum_{t=1}^{\infty} |A_0(j) - A_2(j)\, t^2|^2\, \eta_{tj}^{-2}$$
$$\geqq \eta_0^{-2}|1 - A_0(j)|^2 + 2\eta_j^{-2}|A_1(j)|^2 + 2\eta_j^{-2}|A_0(j) - A_2(j)|^2$$

in the strongly 2-periodic space H.

Further using Definition 7.1 and Theorem 6.4 we get

$$X^2(2, j, H) \leqq \Phi^2(2, j, H) = 2\sum_{t=2}^{\infty} (t^2 - 1)^2\, \eta_{tj}^{-2} \leqq 2\sum_{t=2}^{\infty} t^4 \eta_{tj}^{-2}\,.$$

Therefore we may estimate

$$\frac{\|J - I(2, j; A_0(j), A_1(j), A_2(j))\|^2}{X^2(2, j, H)} \geqq \frac{|1 - A_0(j)|^2}{2\eta_0^2 \sum_{t=2}^{\infty} t^4 \eta_{tj}^{-2}} + \frac{|A_1(j)|^2}{\eta_j^2 \sum_{t=2}^{\infty} t^4 \eta_{tj}^{-2}} + \frac{|A_0(j) - A_2(j)|^2}{\eta_j^2 \sum_{t=2}^{\infty} t^4 \eta_{tj}^{-2}}\,. \tag{7.30}$$

Now let us distinguish three possibilities with respect to the relations (7.26) to (7.28).

1. There exists an infinite sequence $\{j_s\}$ such that (7.26) holds for all positive integers s. For all non-negative integers k we write

$$(7.31) \qquad P(k) = 1\,, \quad 0 \leqq k \leqq j_1\,,$$

$$P(k) = \prod_{r=1}^{s} \min\left(\left|1 - A_0(j_r)\right|^2, 1\right), \quad j_{s-1} < k \leqq j_s\,.$$

From (7.26) it follows

$$\left|1 - A_0(j_s)\right|^2 > 0$$

for any integer $s > 0$. Therefore the function $P(k)$ satisfies the assumptions of Lemma 7.1 and the strongly 2-periodic space H with the norm (7.21) may be constructed. With respect to (7.22), (7.31) we may rewrite (7.30) as

$$(7.32) \qquad \frac{\left\|J - I(2, j_s; A_0(j_s), A_1(j_s), A_2(j_s))\right\|^2}{X^2(2, j_s, H)}$$

$$\geqq (1 - e^{-1})\, j_s^5 \exp(2j_s) \frac{\left|1 - A_0(j_s)\right|^2}{P(2j_s)} \geqq (1 - e^{-1})\, j_s^5 \exp(2j_s)$$

for any positive integer s since from (7.31) we may estimate

$$(7.33) \qquad \frac{\left|1 - A_0(j_s)\right|^2}{P(2j_s)} \geqq 1$$

for any positive integer s. However, the last term on the right-hand side of (7.32) tends to $+\infty$ as $s \to \infty$. Supposing (7.26) we have constructed the strongly 2-periodic space H satisfying (7.29). Therefore the quadrature formula (7.24) is not universally optimal in order in the class of all strongly 2-periodic spaces with regard to (7.26).

2. There exists an infinite sequence $\{j_s\}$ such that (7.27) holds for all positive integers s. Analogously for all integers $k \geqq 0$ we write

$$P(k) = 1\,, \quad 0 \leqq k \leqq j_1\,,$$

$$P(k) = \prod_{r=1}^{s} \min\left(\left|A_1(j_r)\right|^2, 1\right), \quad j_{s-1} < k \leqq j_s\,.$$

From (7.27) it follows

$$\left|A_1(j_s)\right|^2 > 0$$

for any positive integer s. Analogously to the first case, we use Lemma 7.1 to construct a strongly 2-periodic space H again satisfying (7.29). We get the estimates analogous to (7.32), (7.33). Hence even in this case the quadrature formula (7.24) is not universally optimal in order in the class of all strongly 2-periodic spaces.

3. There exists an infinite sequence $\{j_s\}$ such that (7.28) holds for all positive integers s. We distinguish two further possibilities. Either

(7.34) $\quad A_0(j_s) \neq 1$ is valid for the infinite number of indices s

or

(7.35) $\quad A_0(j_s) \neq 1$ is valid for the finite number of indices s (including the case that $A_0(j_s) = 1$ holds for all positive integers s).

If (7.34) holds then this third case is converted into the first case. If (7.35) holds then an infinite sequence that we shall denote again by $\{j_s\}$ may be extracted from the original sequence $\{j_s\}$ so that

$$A_0(j_s) = 1\,, \quad A_2(j_s) \neq 1$$

for any positive integer s. From this assumption we get

$$|A_0(j_s) - A_2(j_s)|^2 = |1 - A_2(j_s)|^2 > 0$$

for any positive integer s. Analogously to the first case, for all integers $k \geqq 0$ we write

$$P(k) = 1\,, \quad 0 \leqq k \leqq j_1\,,$$

$$P(k) = \prod_{r=1}^{s} \min\left(|1 - A_2(j_r)|^2, 1\right), \quad j_{s-1} < k \leqq j_s\,.$$

Now we again use Lemma 7.1 to construct the strongly 2-periodic space H satisfying (7.29). We get estimates analogous to (7.32), (7.33). Hence the quadrature formula (7.24) is not universally optimal in order in the class of all strongly 2-periodic spaces even in this last case.

This conclusion completes the proof of the theorem.

8. ON THE EFFICIENCY OF USING THE VALUES OF THE SECOND DERIVATIVE OF THE INTEGRAND IN QUADRATURE FORMULA

In this section we again confine ourselves to quadrature formulae involving values of the integrand and its first and second derivative, i.e. formulae of the general form $I(2, j; A_0, A_1, A_2)$. With respect to Theorems 5.2, 5.4 the coefficient at values of the first derivative of the integrand in the optimal quadrature formula is equal to 0 in any strongly 2-periodic space. Similarly this coefficient in the limit quadrature formula is equal to 0 as well. Therefore we shall not employ the values of the first derivative of the integrand in the quadrature formulae in this section.

Thus there is a question here what is the efficiency of using the values of the second derivative of the integrand in the quadrature formula, i.e. whether it is more or less efficient and advantageous to use the values of the integrand and its second derivative in j abscissae than to use only the values of the integrand but in more than j abscissae.

The number of arithmetic operations necessary for the computation is usually chosen as a criterion for the work required for obtaining the numerical result. Let us simplify this criterion in such a way that we use a time necessary for the computation of the value of the integrand in an abscissa as a unit of the work required.

Thus let us assume the time necessary for the computation of the value of the integrand in an abscissa to be equal to 1. Further let us assume the time necessary for the computation of the value of the second derivative of the integrand in an abscissa to be equal to $\alpha - 1$ where $\alpha \geqq 1$ is a certain real number.

In particular, if $1 \leqq \alpha < 2$ then the computation of the value of the integrand in an abscissa is more time-consuming than the computation of the value of its second derivative; if $\alpha = 2$ then both computations are comparably time-consuming; and if $\alpha > 2$ then the former computation is less time-consuming than the latter.

Using the quadrature formula $I(2, j; A_0, 0, A_2)$ the total time required for obtaining the numerical result is equal to

$$j + (\alpha - 1) j = \alpha j .$$

Hence if we do not use the values of the second derivative of the integrand then we may use the values of the integrand in $[\alpha j]$ abscissae, i.e. use the formula $I(0, [\alpha j]; A_0^*)$, because then the time required for obtaining the numerical result (the numerical value of the integral) is equal to αj, too.

Comparing the errors of the quadrature formulae $I(2, j; A_0, 0, A_2)$, $I(0, [\alpha j]; A_0^*)$ in dependence on the parameter α, we may get the answer to the question of the efficiency of the employment of the values of the second derivative of the integrand in the quadrature formula. We shall try to make such comparisons in some particular cases, especially in the case of the optimal and limit quadrature formula.

First let us introduce the following notation.

Definition 8.1. Let H be a strongly n-periodic space. Let us write

$$L^+(\alpha, s, k) = \limsup_{j \to \infty} \eta_{kj}^2 \sum_{t=0}^{\infty} (t + k)^{2s} \eta_{(t+k)[\alpha j]}^{-2} ,$$

$$L^-(\alpha, s, k) = \liminf_{j \to \infty} \eta_{kj}^2 \sum_{t=0}^{\infty} (t + k)^{2s} \eta_{(t+k)[\alpha j]}^{-2}$$

for $\alpha \geqq 1$ real; $s = 0, 1, \ldots, n$, and $k = 1, 2, \ldots, [n/2] + 1$.

If $L^+(\alpha, s, k) = L^-(\alpha, s, k)$ then we denote this common value by $L(\alpha, s, k)$.

Remark 8.1. The existence of the numbers $L^+(\alpha, s, k)$, $L^-(\alpha, s, k)$ follows from (vi) of Definition 5.1. These numbers depend on the particular choice of the space H. The inequality $0 \leqq L^-(\alpha, s, k) \leqq L^+(\alpha, s, k) \leqq D(H)$ is valid for $\alpha \geqq 1$ real; $s = 0, 1, \ldots, n$, and $k = 1, 2, \ldots, [n/2] + 1$.

However, the number $L(\alpha, s, k)$ need not exist as the following theorem states.

Theorem 8.1. *There exists a strongly* 0*-periodic space H such that*

$$L^-(1, 0, 1) \neq L^+(1, 0, 1) \tag{8.1}$$

holds in this space H. Hence $L(1, 0, 1)$ *does not exist in this space H.*

Proof. We use Theorem 2.2 to construct the strongly 0-periodic space H with the property (8.1). Let U be a set of all integers. We set

$$\eta_k^2 = \eta_{-k}^2 = 2^{4s}\,; \quad k = 2^{2s}, 2^{2s} + 1, \ldots, 2^{2s+2} - 1 \quad \text{for any integer} \quad s \geqq 0\,, \tag{8.2}$$
$$\eta_0^2 = 1\,.$$

Let us construct the corresponding Hilbert space H of 2π-periodic continuous functions. With respect to (8.2) the conditions (iv), (v) are satisfied in this space. First let us prove (vi) from which the condition (2.16) follows.

We write

$$j_r = 2^{2r}$$

for all non-negative integers r. Let us choose a fixed integer $r \geqq 0$. Then we may easily compute

$$\eta_{j_r}^2 \sum_{t=1}^{\infty} \eta_{tj_r}^{-2} = \eta_{j_r}^2 \sum_{s=r}^{\infty} \sum_{tj_r \in \langle j_s, j_{s+1})} \eta_{tj_r}^{-2} = \eta_{j_r}^2 \sum_{s=r}^{\infty} 3 \times 2^{2s-2r} \eta_{j_s}^{-2} = 4 \tag{8.3}$$

where we use (8.2) and the fact that the interval $\langle j_s/j_r, j_{s+1}/j_r)$ contains exactly $3 \times 2^{2s-2r}$ integers t.

Let j be a positive integer. Then there exists a non-negative integer r such that $j_r \leqq j < j_{r+1}$. If $j = j_r$ then (8.3) holds. Hence let $j_r < j < j_{r+1}$, i.e.

$$j = j_{r+1} - k \tag{8.4}$$

where $0 < k < j_{r+1} - j_r$, i.e. $0 < k < 3 \times 2^{2r}$, is valid for this integer k. We get

$$\eta_j^2 \sum_{t=1}^{\infty} \eta_{tj}^{-2} = \eta_j^2 \sum_{s=r}^{\infty} \sum_{tj \in \langle j_s, j_{s+1})} \eta_{tj}^{-2} \leqq \eta_{j_r}^2 \sum_{s=r}^{\infty} 3 \times 2^{2s-2r} \eta_{j_s}^{-2} = 4 \tag{8.5}$$

where we use (8.2), (8.3) and the fact that the interval $\langle j_s/(j_{r+1} - k), j_{s+1}/(j_{r+1} - k))$ contains at most $3 \times 2^{2s-2r}$ integers t.

Putting $j = 1$ in (8.5) we get the validity of (2.16). Therefore the space H constructed in this way is 0-periodic with respect to Theorem 2.2. Furthermore the condition (vi) follows from (8.5). Hence the space H is strongly 0-periodic.

Now we write

$$(8.6) \qquad k_r = j_{r+1} - 1 = 2^{2r+2} - 1$$

for any non-negative integer r. Thus k_r is a particular case of (8.4) with $k = 1$ and a fixed r. Analogously to the proof of (8.3), using (8.2) and (8.6) we get

$$\eta_{k_r}^2 \sum_{t=1}^{\infty} \eta_{tk_r}^{-2} = \eta_{j_r}^2 \sum_{s=r}^{\infty} \sum_{tk_r \in \langle j_s, j_{s+1})} \eta_{tk_r}^{-2} = \eta_{j_r}^2 \Big(\sum_{tk_r \in \langle j_r, j_{r+1})} \eta_{tk_r}^{-2} + \sum_{s=r+1}^{\infty} \sum_{tk_r \in \langle j_s, j_{s+1})} \eta_{tk_r}^{-2} \Big)$$

$$\leqq \eta_{j_r}^2 \Big(\sum_{tk_r \in \langle j_r, j_{r+1})} \eta_{tk_r}^{-2} + \sum_{s=r+1}^{\infty} 3 \times 2^{2s-2r} \eta_{j_s}^{-2} \Big) = \eta_{j_r}^2 \sum_{tk_r \in \langle j_r, j_{r+1})} \eta_{tk_r}^{-2} + 1 = 2$$

for a fixed $r \geqq 0$ since it may be readily shown that the interval $\langle j_r/k_r, j_{r+1}/k_r)$ contains only one positive integer, namely the integer 1.

Finally using Definition 8.1 we have

$$L^-(1, 0, 1) \leqq \liminf_{r \to \infty} \eta_{k_r}^2 \sum_{t=1}^{\infty} \eta_{tk_r}^{-2} \leqq 2 < 4$$

$$= \lim_{r \to \infty} \eta_{j_r}^2 \sum_{t=1}^{\infty} \eta_{tj_r}^{-2} \leqq L^+(1, 0, 1)\,.$$

Thus the validity of (8.1) in the constructed strongly 0-periodic space H has been proved.

Let us further prove an auxiliary statement generalizing the statement of Lemma 6.1.

Lemma 8.1. *Let H be a strongly n-periodic space. Then the series*

$$(8.7) \qquad \sum_{t=0}^{\infty} (t + k)^{2s} \eta_{(t+k)[\alpha j]}^{-2}$$

converges uniformly with respect to j for all $\alpha \geqq 1$ real; $s = 0, 1, \ldots, n$, and $k = 1, 2, \ldots, [n/2] + 1$. Further, for these α, s, k

$$(8.8) \qquad \lim_{j \to \infty} \sum_{t=0}^{\infty} (t + k)^{2s} \eta_{(t+k)[\alpha j]}^{-2} = 0\,.$$

Proof. From (v) we have

$$(t + k)^{2s} \eta_{(t+k)[\alpha j]}^{-2} \leqq (t + k)^{2n} \eta_{(t+k)[\alpha j]}^{-2} \leqq (t + k)^{2n} \eta_{t+k}^{-2}$$

for integers $t \geqq 0$, $j > 0$, real $\alpha \geqq 1$; $s = 0, 1, \ldots, n$, and $k = 1, 2, \ldots, [n/2] + 1$. The series

$$\sum_{t=0}^{\infty} (t + k)^{2n} \eta_{t+k}^{-2} = \sum_{t=k}^{\infty} t^{2n} \eta_t^{-2}$$

converges with respect to Theorem 2.1, does not depend on j, and majorizes all the series (8.7). Therefore the series (8.7) converge uniformly with respect to j for $\alpha \geqq 1$ real; $s = 0, 1, \ldots, n$, and $k = 1, 2, \ldots, [n/2] + 1$. The first statement of the lemma has been proved.

With respect to the uniform convergence of the series (8.7), we get immediately (8.8) considering the condition

$$\lim_{k \to \infty} \eta_k^{-2} = 0\,,$$

which is necessary for the convergence of the series (8.7). Hence the lemma has been proved.

In the following considerations except Theorem 8.5 we shall confine ourselves only to certain classes of strongly 2-periodic spaces where the numbers $L(\alpha, s, k)$ exist. We shall make the comparison of the efficiency of employing the values of the second derivative of the integrand in the quadrature formula for these classes.

One of these classes is the following.

Definition 8.2. Let us denote by Π a class of strongly 2-periodic spaces H such that

$$\lim_{j \to \infty} \frac{\eta_j^2}{\eta_{[\alpha j]}^2} = \lim_{j \to \infty} \frac{\eta_{2j}^2}{\eta_{2[\alpha j]}^2} = 0 \tag{8.9}$$

holds for any real $\alpha > 1$ in each of these spaces.

An example of the strongly 2-periodic space belonging to the class Π is presented in the following theorem.

Theorem 8.2. *Let H be a Hilbert space of 2π-periodic continuous functions with the first and second derivative continuous. Let*

$$\eta_k^2 = \sum_{t=0}^{\infty} \gamma_t k^{2t} \tag{8.10}$$

for any integer k where γ_t are real non-negative coefficients such that

$$\lim_{t \to \infty} \gamma_t^{1/t} = 0\,, \tag{8.11}$$

$\gamma_0 \neq 0$, *and $\gamma_t > 0$ holds for the infinite number of indices t.*

Then the space H with the norm (8.10) *is strongly* 2-*periodic and belongs to the class* Π.

Proof. First let us show the validity of (8.9) for η_k defined in (8.10), (8.11). We write

$$f(x) = \sum_{t=0}^{\infty} \gamma_t x^{2t} . \tag{8.12}$$

Then with respect to (8.11), $f(x)$ is an entire function. Particularly, using (8.10) we may write

$$f(k) = \eta_k^2$$

for any integer k.

We rewrite the first part of (8.9) as

$$\lim_{j \to \infty} \frac{f(j)}{f([\alpha j])} = 0 , \quad \alpha > 1 . \tag{8.13}$$

With respect to the assumption, $f(x)$ is increasing for $x > 0$. Thus

$$0 \leqq \frac{f(j)}{f([\alpha j])} \leqq \frac{f(j)}{f(\alpha j - 1)} , \quad \alpha > 1 \tag{8.14}$$

and any positive integer j. To prove (8.13) it is sufficient to show

$$\lim_{j \to \infty} \frac{f(j)}{f(\alpha j - 1)} = 0 , \quad \alpha > 1 \tag{8.15}$$

using (8.14). We choose a positive integer N and write

$$f(j) = \sum_{t=0}^{N} \gamma_t j^{2t} + \sum_{t=N+1}^{\infty} \gamma_t j^{2t} , \tag{8.16}$$

$$f(\alpha j - 1) = \sum_{t=0}^{N} \gamma_t \left(\alpha - \frac{1}{j}\right)^{2t} j^{2t} + \sum_{t=N+1}^{\infty} \gamma_t \left(\alpha - \frac{1}{j}\right)^{2t} j^{2t} \tag{8.17}$$

using (8.12).

There exists a positive integer L such that

$$\alpha - \frac{1}{j} > 1 \quad \text{for any integer} \quad j \geqq L . \tag{8.18}$$

Hence we may estimate

$$\sum_{t=N+1}^{\infty} \gamma_t \left(\alpha - \frac{1}{j}\right)^{2t} j^{2t} \geqq \left(\alpha - \frac{1}{j}\right)^{2N+2} \sum_{t=N+1}^{\infty} \gamma_t j^{2t} , \quad j \geqq L . \tag{8.19}$$

Substituting (8.16), (8.17) into (8.19) we may write

$$f(j) \leqq \left(\alpha - \frac{1}{j}\right)^{-2N-2} f(\alpha j - 1) + \sum_{t=0}^{N} \gamma_t j^{2t}, \quad j \geqq L.$$

From this we get

$$\text{(8.20)} \qquad \frac{f(j)}{f(\alpha j - 1)} \leqq \left(\alpha - \frac{1}{j}\right)^{-2N-2} + \frac{\sum_{t=0}^{N} \gamma_t j^{2t}}{f(\alpha j - 1)}, \quad j \geqq L.$$

With respect to (8.12) the second term on the right-hand side of (8.20) tends to 0 as $j \to \infty$ so that passing to the limit in (8.20) we have

$$0 \leqq \lim_{j\to\infty} \frac{f(j)}{f(\alpha j - 1)} \leqq \alpha^{-2N-2}$$

for $\alpha > 1$ and any positive integer N. From this the condition (8.15) follows immediately and so does (8.13).

Analogously, using (8.12) we may rewrite the second part of (8.9) as

$$\text{(8.21)} \qquad \lim_{j\to\infty} \frac{f(2j)}{f(2[\alpha j])} = 0, \quad \alpha > 1.$$

Again we may estimate

$$\text{(8.22)} \qquad 0 \leqq \frac{f(2j)}{f(2[\alpha j])} \leqq \frac{f(2j)}{f(2\alpha j - 2)}, \quad \alpha > 1$$

and any positive integer j. To prove (8.21) it is sufficient to show

$$\text{(8.23)} \qquad \lim_{j\to\infty} \frac{f(2j)}{f(2\alpha j - 2)} = 0, \quad \alpha > 1$$

using (8.22).

We proceed in the same manner as in the previous case. We again use (8.18) and get the estimate

$$\sum_{t=N+1}^{\infty} \gamma_t \left(2\alpha - \frac{2}{j}\right)^{2t} j^{2t} \geqq \left(\alpha - \frac{1}{j}\right)^{2N+2} \sum_{t=N+1}^{\infty} \gamma_t 2^{2t} j^{2t}, \quad j \geqq L$$

analogous to (8.19). We shall prove (8.23) and thus (8.21) analogously in the further procedure. Hence we have proved (8.9) with regard to (8.10).

The validity of (iv), (v) follows immediately from (8.10), (8.11). From (8.9), which has been proved, it follows

$$\lim_{j\to\infty} \frac{\eta_{kj}^2}{\eta_{tj}^2} = 0 \tag{8.24}$$

for any integer $t > k$ and $k = 1, 2$. Using Lemma 8.1 we get both (vi) and the condition (2.16) of Theorem 2.2 in virtue of (8.24). Then the rest of the statement follows from Theorem 2.2 and the space H belongs to the class Π with respect to (8.13), (8.21).

Further let us prove two auxiliary statements concerning spaces of the class Π.

Lemma 8.2. *Let H be a strongly 2-periodic space, $H \in \Pi$. Then*

$$\lim_{j\to\infty} \frac{\eta_{2j}^2}{\eta_{[\alpha j]}^2} = +\infty \quad \textit{for} \quad 1 \leqq \alpha < 2,$$

$$= 1 \quad \textit{for} \quad \alpha = 2,$$

$$= 0 \quad \textit{for} \quad \alpha > 2.$$

Proof. A. Let $1 \leqq \alpha < 2$. Then it is sufficient to show

$$\lim_{j\to\infty} \frac{\eta_{[\alpha j]}^2}{\eta_{2j}^2} = 0\,. \tag{8.25}$$

Let us write $k = [\alpha j]$, $\beta = 2/\alpha$. Then $[\beta k] \leqq 2j$ and

$$\beta > 1\,. \tag{8.26}$$

From this fact, using (v) we get

$$\frac{\eta_{[\alpha j]}^2}{\eta_{2j}^2} \leqq \frac{\eta_k^2}{\eta_{[\beta k]}^2} \tag{8.27}$$

where the right-hand side of (8.27) tends to 0 as $j \to \infty$ with respect to Definition 8.2 and (8.26). From this fact the relation (8.25) follows.

B. If $\alpha = 2$ then the statement is obvious.

C. Let $\alpha > 2$, let us write $\beta = \alpha/2$. Then $[\alpha j] \geqq 2[\beta j]$ and

$$\beta > 1\,. \tag{8.28}$$

From this fact, using (v) we get

$$\frac{\eta_{2j}^2}{\eta_{[\alpha j]}^2} \leqq \frac{\eta_{2j}^2}{\eta_{2[\beta j]}^2} \tag{8.29}$$

where the right-hand side of (8.29) tends to 0 as $j \to \infty$ with respect to Definition 8.2 and (8.28). The lemma has been proved.

Lemma 8.3. *Let H be a strongly 2-periodic space, $H \in \Pi$. Then*

$$L(1, s, k) = k^{2s}, \tag{8.30}$$

$$L(\alpha, s, k) = 0 \quad for \quad \alpha > 1 \tag{8.31}$$

and $k = 1, 2$; $s = 0, 1, 2$.

Proof. With respect to Definition 8.1 and Lemma 8.1 we may write

$$L(\alpha, s, k) = \sum_{t=k}^{\infty} t^{2s} \lim_{j\to\infty} \frac{\eta_{kj}^2}{\eta_{t[\alpha j]}^2} \tag{8.32}$$

for $\alpha \geqq 1$; $k = 1, 2$; $s = 0, 1, 2$.

Let $\alpha = 1$. Then from the condition (8.9) of Definition 8.2 we get the validity of the assumption (6.7) of Lemma 6.2. From the statement (6.8) of this lemma the equality (8.30) follows for $k = 1$.

Analogously we have

$$0 \leqq \frac{\eta_{2j}^2}{\eta_{tj}^2} \leqq \frac{\eta_{2j}^2}{\eta_{3j}^2} \leqq \frac{\eta_{2j}^2}{\eta_{2[3j/2]}^2} \tag{8.33}$$

for any integer $t > 2$ using (v). The right-hand side of (8.33) tends to 0 as $j \to \infty$ with respect to (8.9) of Definition 8.2. From this fact the equality (8.30) follows for $k = 2$.

From (v) we get

$$0 \leqq \frac{\eta_{kj}^2}{\eta_{t[\alpha j]}^2} \leqq \frac{\eta_{kj}^2}{\eta_{k[\alpha j]}^2} \tag{8.34}$$

for any integer $t \geqq k$; $k = 1, 2$, and $\alpha > 1$. For these k, t the right-hand side of (8.34) tends to 0 as $j \to \infty$ with respect to (8.9) of Definition 8.2. From this fact the equality (8.31) follows.

The proof has been completed.

Let us introduce the following concepts convenient for further considerations.

Definition 8.3. Let $m > 2$ be an integer. Let us denote by Π_m the class of strongly 2-periodic spaces H such that

$$\eta_k^2 = p_m(k^2) \quad \text{for any integer } k \tag{8.35}$$

holds in each of these spaces where

$$p_m(x) = \sum_{t=0}^{m} a_t x^t \tag{8.36}$$

is a polynomial of the degree m with real non-negative coefficients $a_0, a_1, \ldots, a_m$; $a_0 \neq 0$.

We shall find the values of $L(\alpha, s, k)$ for the spaces of the class Π_m, too.

Lemma 8.4. *Let $m > 2$ be an integer, let H be a strongly 2-periodic space, $H \in \Pi_m$. Then*

$$\begin{aligned} L(\alpha, s, 1) &= \alpha^{-2m}\zeta(2(m - s)), \\ L(\alpha, s, 2) &= 2^{2m}\alpha^{-2m}(\zeta(2(m - s)) - 1) \end{aligned} \tag{8.37}$$

for $\alpha \geqq 1$ and $s = 0, 1, 2$ where

$$\zeta(x) = \sum_{t=1}^{\infty} t^{-x}$$

is the Riemann zeta-function.

Proof. We again use the relation (8.32). From (8.35), (8.36) we may calculate

$$\lim_{j \to \infty} \frac{\eta_{kj}^2}{\eta_{t[\alpha j]}^2} = k^{2m} t^{-2m} \alpha^{-2m} \tag{8.38}$$

for $t \geqq k$; $k = 1, 2$; $\alpha \geqq 1$. Substituting (8.38) into (8.32) we get (8.37). The lemma has been proved.

Now we may proceed and show the spaces belonging to the class Π_m.

Theorem 8.3. *Let H be a Hilbert space of 2π-periodic functions continuous with the first and second derivative continuous. Let*

$$\eta_k^2 = p_m(k^2) \quad \textit{for any integer } k$$

where

$$p_m(x) = \sum_{t=0}^{m} a_t x^t$$

is a polynomial of the degree $m > 2$ with real non-negative coefficients $a_0, a_1, \ldots$ $\ldots, a_m$; $a_0 \neq 0$. Then the space H with this norm is strongly 2-periodic and belongs to the class Π_m.

Proof. The conditions (8.35), (8.36) of Definition 8.3 are evidently satisfied. Hence it is sufficient to show this Hilbert space to be strongly 2-periodic. The conditions (iv), (v) follow from (8.35), (8.36), too. In the same manner as in the proof of

Lemma 8.4 we now prove (8.37). Then from (8.37) the condition (vi) follows and from (iv), (vi) we get the condition (2.16) of Theorem 2.2. With respect to Theorem 2.2 and the conditions (iv) to (vi) this Hilbert space is strongly 2-periodic. Furthermore it belongs to the class Π_m.

Now we proceed to the comparison of the errors of the quadrature formulae on the one hand using and on the other hand not using the values of the second derivative of the integrand. First we shall make the asymptotical comparison of the errors of the corresponding optimal quadrature formulae.

Theorem 8.4. *Let H be a strongly 2-periodic space. If $H \in \Pi_m$ for a certain integer $m > 2$ then*

$$\lim_{j\to\infty} \frac{X(2, j, H)}{X(0, [\alpha j], H)} > 1 \quad \textit{for} \quad \alpha \geqq 2\,. \tag{8.39}$$

If $H \in \Pi$ then

$$\begin{aligned}\lim_{j\to\infty} \frac{X(2, j, H)}{X(0, [\alpha j], H)} &= 0 \quad \textit{for} \quad 1 \leqq \alpha < 2\,, \\ &= 3 \quad \textit{for} \quad \alpha = 2\,, \\ &= +\infty \quad \textit{for} \quad \alpha > 2\,.\end{aligned} \tag{8.40}$$

Proof. From Theorem 4.4 we have

$$X^2(2, j, H) = \eta_0^{-2}(1 - K_0(2, j))\,, \tag{8.41}$$

$$X^2(0, [\alpha j], H) = \eta_0^{-2}(1 - K_0(0, [\alpha j]))\,. \tag{8.42}$$

Substituting $n = 2$ into (5.9) we get

$$K_0(2, j) = \frac{\eta_0^{-2} + 2K_2(2, j)\sum_{t=1}^{\infty} t^2 \eta_{tj}^{-2}}{\eta_0^{-2} + 2\sum_{t=1}^{\infty} \eta_{tj}^{-2}}\,. \tag{8.43}$$

Substituting (8.43) into (8.41) and (5.12) into (8.42) we have

$$\frac{X^2(2, j, H)}{X^2(0, [\alpha j], H)} = \frac{\sum_{t=1}^{\infty} (1 - t^2 K_2(2, j))\, \eta_{tj}^{-2}}{\sum_{t=1}^{\infty} \eta_{t[\alpha j]}^{-2}} \cdot \frac{\eta_0^{-2} + 2\sum_{t=1}^{\infty} \eta_{t[\alpha j]}^{-2}}{\eta_0^{-2} + 2\sum_{t=1}^{\infty} \eta_{tj}^{-2}} \tag{8.44}$$

where in virtue of Lemma 8.1 the last fraction on the right-hand side of (8.44) tends to 1 as $j \to \infty$ and $\alpha \geqq 1$. Hence it is sufficient to find the limit of the first fraction on the right-hand side of (8.44), its numerator and denominator being first multiplied by η_{2j}^2.

The numerator $N(j)$ of this fraction is equal to

$$N(j) = \eta_{2j}^2 \sum_{t=1}^{\infty} (1 - t^2 K_2(2, j))\, \eta_{tj}^{-2} .$$

Let us determine its limit for $j \to \infty$. We may write

$$N(j) = \eta_{2j}^2 (1 - K_2(2, j))\, \eta_j^{-2} + \eta_{2j}^2 \sum_{t=2}^{\infty} (1 - t^2 K_2(2, j))\, \eta_{tj}^{-2} .$$

Substituting (5.13), (5.16) into $K_2(2, j)$ and using Lemma 8.1, we get

$$\lim_{j\to\infty} N(j) = \lim_{j\to\infty} \left(\frac{\eta_{2j}^2 \sum_{t=2}^{\infty} t^4 \eta_{tj}^{-2} - \eta_{2j}^2 \sum_{t=2}^{\infty} t^2 \eta_{tj}^{-2}}{\eta_j^2 \sum_{t=1}^{\infty} t^4 \eta_{tj}^{-2}} + \eta_{2j}^2 \sum_{t=2}^{\infty} \eta_{tj}^{-2} - \frac{\eta_j^2 \sum_{t=1}^{\infty} t^2 \eta_{tj}^{-2} \cdot \eta_{2j}^2 \sum_{t=2}^{\infty} t^2 \eta_{tj}^{-2}}{\eta_j^2 \sum_{t=1}^{\infty} t^4 \eta_{tj}^{-2}} \right).$$

With respect to Definition 8.1 and Lemmas 8.3, 8.4 this relation may be rewritten as

(8.45)
$$\lim_{j\to\infty} N(j) = \frac{L(1, 2, 2) + L(1, 0, 2)\, L(1, 2, 1) - L(1, 1, 2) - L(1, 1, 1)\, L(1, 1, 2)}{L(1, 2, 1)} .$$

Let us denote by $D(j)$ the denominator of the first fraction on the right-hand side of (8.44) where its numerator and denominator were first multiplied by η_{2j}^2. Then using Definition 8.1 and Lemmas 8.3, 8.4 we have

$$\lim_{j\to\infty} D(j) = \lim_{j\to\infty} \frac{\eta_{2j}^2}{\eta_{[\alpha j]}^2} + L(\alpha, 0, 2) . \tag{8.46}$$

Now we distinguish two cases.

A. Let $H \in \Pi_m$. Then from (8.45) we get

$$\lim_{j\to\infty} N(j) = 2^{2m} \frac{\zeta(2m)\, \zeta(2m - 4) - \zeta^2(2m - 2)}{\zeta(2m - 4)} \tag{8.47}$$

using Lemma 8.4. Further with respect to (8.35), (8.36) we have

$$\lim_{j\to\infty} \frac{\eta_{2j}^2}{\eta_{[\alpha j]}^2} = 2^{2m} \alpha^{-2m} . \tag{8.48}$$

Therefore from (8.46), using (8.48) and Lemma 8.4 we get

$$\lim_{j\to\infty} D(j) = 2^{2m}\alpha^{-2m}\zeta(2m)\,. \tag{8.49}$$

From (8.47), (8.49), using (8.44) we have

$$\lim_{j\to\infty} \frac{X^2(2, j, H)}{X^2(0, [\alpha j], H)} = \alpha^{2m}\,\frac{\zeta(2m)\,\zeta(2m-4) - \zeta^2(2m-2)}{\zeta(2m)\,\zeta(2m-4)}\,. \tag{8.50}$$

By computation we may verify

$$\zeta(2m)\,\zeta(2m-4) - \zeta^2(2m-2) = \sum_{t=1}^{\infty}\sum_{u=1}^{\infty}(u^4 - t^2u^2)\,t^{-2m}u^{-2m}$$

$$= \sum_{s=1}^{\infty}\sum_{r=s+1}^{\infty}(s^2 - r^2)^2\,s^{-2m}r^{-2m} > 9 \times 2^{-2m} > 0\,.$$

Now for any integer $m > 2$ we may estimate (see e.g. [6])

$$\zeta(2m)\,\zeta(2m-4) \leqq \zeta^2(2) = \frac{\pi^4}{36}\,. \tag{8.51}$$

Hence from (8.50) we get finally

$$\lim_{j\to\infty} \frac{X^2(2, j, H)}{X^2(0, [\alpha j], H)} \geqq \frac{324}{\pi^4}\left(\frac{\alpha}{2}\right)^{2m}.$$

For $\alpha \geqq 2$ the relation (8.39) follows from this inequality.

B. Let $H \in \Pi$. Then from (8.45) we get

$$\lim_{j\to\infty} N(j) = 9 \tag{8.52}$$

using Lemma 8.3.

Now let $1 \leqq \alpha < 2$. Then using Lemmas 8.2, 8.3 we may rewrite (8.46) as

$$\lim_{j\to\infty} D(j) = +\infty\,. \tag{8.53}$$

Finally from (8.52), (8.53) using (8.44) we get the first part of (8.40). Further let $\alpha = 2$. Then using Lemmas 8.2, 8.3 we may rewrite (8.46) as

$$\lim_{j\to\infty} D(j) = 1\,. \tag{8.54}$$

Analogously, from (8.52), (8.54) using (8.44) we get the second part of (8.40). Finally let $\alpha > 2$. Then analogously, using Lemmas 8.2, 8.3 we have

$$\lim_{j\to\infty} D(j) = 0\,. \tag{8.55}$$

From (8.52), (8.55) using (8.44) we get the last part of (8.40), too. The proof of the theorem has been completed.

Let us show a similar comparison of the errors of the limit quadrature formulae.

Theorem 8.5. *Let H be a strongly 2-periodic space. Then*

$$\frac{\Phi(2, j, H)}{\Phi(0, [\alpha j], H)} > 1 \tag{8.56}$$

for $\alpha \geqq 2$ and all positive integers j.

If $H \in \Pi$ then

$$\lim_{j \to \infty} \frac{\Phi(2, j, H)}{\Phi(0, [\alpha j], H)} = 0 \quad \text{for} \quad 1 \leqq \alpha < 2, \tag{8.57}$$
$$= 3 \qquad \text{for} \quad \alpha = 2,$$
$$= +\infty \quad \text{for} \quad \alpha > 2.$$

Proof. With respect to Theorems 6.3, 6.4 we may write

$$\frac{\Phi^2(2, j, H)}{\Phi^2(0, [\alpha j], H)} = \frac{\sum_{t=2}^{\infty} \eta_{tj}^{-2} + \sum_{t=2}^{\infty} t^2(t^2 - 2)\, \eta_{tj}^{-2}}{\sum_{t=1}^{\infty} \eta_{t[\alpha j]}^{-2}} \tag{8.58}$$

where we may estimate

$$\frac{\Phi^2(2, j, H)}{\Phi^2(0, [\alpha j], H)} \geqq \frac{\sum_{t=2}^{\infty} \eta_{tj}^{-2}}{\sum_{t=1}^{\infty} \eta_{t[\alpha j]}^{-2}}. \tag{8.59}$$

From (v) we have

$$\eta_{t[\alpha j]}^2 \geqq \eta_{2tj}^2$$

for any positive integer t and $\alpha \geqq 2$. Hence we may rewrite (8.59) as

$$\frac{\Phi^2(2, j, H)}{\Phi^2(0, [\alpha j], H)} \geqq \frac{\sum_{t=2}^{\infty} \eta_{tj}^{-2}}{\sum_{t=1}^{\infty} \eta_{2tj}^{-2}} \tag{8.60}$$

where

$$\sum_{t=1}^{\infty} \eta_{2tj}^{-2} = \sum_{\substack{t=2 \\ \text{even}}}^{\infty} \eta_{tj}^{-2}. \tag{8.61}$$

Using Lemma 5.1 we get

$$(8.62) \qquad \sum_{t=2}^{\infty} \eta_{tj}^{-2} = \sum_{\substack{t=2 \\ \text{even}}}^{\infty} \eta_{tj}^{-2} + \sum_{\substack{t=3 \\ \text{odd}}}^{\infty} \eta_{tj}^{-2} .$$

Substituting (8.61), (8.62) into (8.60) we get immediately (8.56).

Now let $H \in \Pi$. We multiply the numerator and the denominator of the fraction on the right-hand side of (8.58) by η_{2j}^2. Then the limit of the numerator $N(j)$ of this fraction for $j \to \infty$ is equal to (8.52) as we may verify using Definition 8.1 and Lemma 8.3. Analogously the limit of the denominator $D(j)$ of this fraction for $j \to \infty$ is equal to (8.46) as we again may verify using Definition 8.1 and Lemma 8.3.

From the relations (8.52) to (8.55) we get the statement (8.57) using (8.58). The theorem has been proved.

Finally let us make a comparison of the error $X(2, j, H)$ of the optimal quadrature formula and the error $\Phi(0, [\alpha j], H)$ of the limit formula.

Theorem 8.6. *Let H be a strongly 2-periodic space. If $H \in \Pi_m$ for a certain integer $m > 2$ then*

$$(8.63) \qquad \lim_{j \to \infty} \frac{X(2, j, H)}{\Phi(0, [\alpha j], H)} > 1 \quad \text{for} \quad \alpha \geqq 2 .$$

If $H \in \Pi$ then

$$\lim_{j \to \infty} \frac{X(2, j, H)}{\Phi(0, [\alpha j], H)} = 0 \quad \text{for} \quad 1 \leqq \alpha < 2 ,$$

$$= 3 \qquad \text{for} \quad \alpha = 2 ,$$

$$= +\infty \quad \text{for} \quad \alpha > 2 .$$

Proof. Let us write

$$\frac{X(2, j, H)}{\Phi(0, [\alpha j], H)} = \frac{X(2, j, H)}{X(0, [\alpha j], H)} \frac{X(0, [\alpha j], H)}{\Phi(0, [\alpha j], H)} .$$

With respect to Theorem 7.1 and Definition 7.1 we get

$$(8.64) \qquad \lim_{j \to \infty} \frac{X(2, j, H)}{\Phi(0, [\alpha j], H)} = \lim_{j \to \infty} \frac{X(2, j, H)}{X(0, [\alpha j], H)} .$$

Then the statement of the theorem follows from Theorem 8.4.

Remark 8.2. The estimate (8.39) of Theorem 8.4 may be readily improved if we use a better bound

$$(8.65) \qquad \zeta(2m)\, \zeta(2m - 4) \leqq \zeta(6)\, \zeta(2) = \frac{\pi^8}{5670}$$

instead of (8.51). Then we get

$$\lim_{j\to\infty} \frac{X(2, j, H)}{X(0, [\alpha j], H)} > 2{\cdot}319 \tag{8.66}$$

for $H \in \Pi_m$, $m > 2$, and $\alpha \geqq 2$. Further, treating (8.58) in the analogous way as in the proof of Theorem 8.4 and using the same notation we have

$$\lim_{j\to\infty} N(j) = L(1, 0, 2) - 2\,L(1, 1, 2) + L(1, 2, 2)$$

and (8.46). With respect to Lemma 8.4 and the relation

$$\zeta(2m) \leqq \zeta(6) = \frac{\pi^6}{945} \tag{8.67}$$

we get

$$\lim_{j\to\infty} \frac{\Phi(2, j, H)}{\Phi(0, [\alpha j], H)} > 2{\cdot}974$$

for $H \in \Pi_m$, $m > 2$, and $\alpha \geqq 2$ instead of the estimate (8.56) of Theorem 8.5. Finally, from (8.64), (8.66) we have

$$\lim_{j\to\infty} \frac{X(2, j, H)}{\Phi(0, [\alpha j], H)} > 2{\cdot}319$$

for $H \in \Pi_m$, $m > 2$, and $\alpha \geqq 2$ instead of the estimate (8.63) of Theorem 8.6.

The principal qualitative behaviour of these three quotients is given in Theorems 8.4, 8.5, and 8.6. The above improvement is only quantitative.

On the other hand, using the same conclusions and estimates as above we may easily compute that

$$\lim_{j\to\infty} \frac{X(2, j, H)}{X(0, [\alpha j], H)} > 3\,,$$

$$\lim_{j\to\infty} \frac{X(2, j, H)}{\Phi(0, [\alpha j], H)} > 3$$

holds for $H \in \Pi_m$, $m > 2$, and $\alpha \geqq 2{\cdot}180$. Further

$$\lim_{j\to\infty} \frac{\Phi(2, j, H)}{\Phi(0, [\alpha j], H)} > 3$$

holds for $H \in \Pi_m$, $m > 2$, and $\alpha \geqq 2{\cdot}006$.

The assumption $m > 2$ is very essential here. If we confine ourselves only to $m \geqq M$ for some integer $M > 2$ all the estimates may be improved since we have

$$\zeta(2m)\,\zeta(2m - 4) \leqq \zeta(2M)\,\zeta(2M - 4) \leqq \zeta(6)\,\zeta(2)\,,$$

$$\zeta(2m) \leqq \zeta(2M) \leqq \zeta(6)$$

instead of (8.65), (8.67) respectively. Thus we may consider the space $H \in \Pi$ as a "limit case" of spaces $H \in \Pi_m$ for increasing m.

The conclusion that the quadrature formula $I(0, [\alpha j]; 1)$ is more advantageous than the formula $I(2, j; 1, 0, 1)$ in the case $\alpha \geqq 2$, i.e. in the case when the computation of the value of the second derivative of the integrand in an abscissa is not less time-consuming as compared with the computation of the value of the integrand, follows for any j from Theorem 8.5. Thus this conclusion concerning the limit quadrature formulae holds for arbitrary j.

A similar conclusion concerning the optimal quadrature formulae is valid only asymptotically for $j \to \infty$. With respect to Theorem 8.4 it has been shown at least for certain classes of strongly 2-periodic spaces that the quadrature formula $I(0, [\alpha j]; K_0)$ is more advantageous than the formula $I(2, j; K_0, 0, K_2)$ in the asymptotical sense in the case $\alpha \geqq 2$, i.e. in the case when the computation of the value of the second derivative of the integrand in an abscissa is not less time-consuming as compared with the computation of the value of the integrand.

Theorem 8.6 states the same fact concerning the quadrature formulae $I(0, [\alpha j]; 1)$ and $I(2, j; K_0, 0, K_2)$.

A numerical example in Sec. 10 illustrates these conclusions using the computed values of integrals.

9. QUADRATURE FORMULAE WITH GENERAL ABSCISSAE

In Sec. 3 to 8 we confined ourselves to the quadrature formulae with equidistant abscissae. With regard to this fact we have found the optimal quadrature formula in an n-periodic space and have derived some its properties.

Let us return to the quadrature formulae with general arbitrary abscissae and let us try to compare the error of the formula optimal in the n-periodic space H (with equidistant abscissae) and the formula with general abscissae at least in some cases. We shall see that the error of the quadrature formula with general abscissae may be less than that of the optimal formula.

Let us introduce the following notation.

Definition 9.1. Let H be an n-periodic space. Let us denote the infimum of the errors of the quadrature formulae $Y(n, j; \{A_{rs}\}; \{x_{rs}\})$ for fixed j and arbitrary coefficients A_{rs} and abscissae x_{rs}; $r = 1, 2, \ldots, j$; $s = 0, 1, \ldots, n$ by

$$\Omega(n, j, H) = \inf_{A_{rs}, x_{rs}} \left\| J - Y(n, j; \{A_{rs}\}; \{x_{rs}\}) \right\| .$$

Now let us compare $\Omega(n, j, H)$ and $X(n, j, H)$ in the 0-periodic and 2-periodic space.

Theorem 9.1. *There exists a* 0-*periodic space H such that*

$$\limsup_{j\to\infty} \frac{X(0, j, H)}{j^\alpha \Omega(0, j, H)} = +\infty$$

holds for arbitrary $\alpha > 0$.

Proof. We use Theorem 2.2 to construct the 0-periodic space H with this property. Let U be a set of all integers. We set

$$\begin{aligned} \eta_k^2 &= e^{4|k|}\,; \quad |k| = 5^r\,, \\ \eta_k^2 &= e^{|k|}\,; \quad |k| \neq 5^r \end{aligned} \tag{9.1}$$

for any non-negative integer r. Let us construct the corresponding Hilbert space of 2π-periodic continuous functions. The condition (2.16) of Theorem 2.2 is satisfied since we may estimate

$$\sum_{k=-\infty}^{\infty} \eta_k^{-2} \leqq \sum_{k=-\infty}^{\infty} e^{-|k|} = 1 + \frac{2e^{-1}}{1 - e^{-1}} < +\infty\,.$$

Thus the space constructed in this way with the norm (9.1) is 0-periodic. Using Theorems 4.4, 4.5, and (9.1) we get

$$X^2(0, j, H) = \frac{2\sum_{t=1}^{\infty} \eta_{tj}^{-2}}{1 + 2\eta_0^2 \sum_{t=1}^{\infty} \eta_{tj}^{-2}}\,. \tag{9.2}$$

Now let us put

$$j_q = 5^q \tag{9.3}$$

for any non-negative integer q. Then $2tj_q \neq 5^r$ for all integers $t > 0$; $q, r \geqq 0$ so that using (9.1) we have

$$\sum_{t=1}^{\infty} \eta_{2tj_q}^{-2} = O(\exp(-2j_q))\,, \quad q \to \infty\,. \tag{9.4}$$

Thus from (9.2), (9.4)

$$X^2(0, 2j_q, H) = O(\exp(-2j_q))\,, \quad q \to \infty\,. \tag{9.5}$$

We now construct an upper bound of the quantity $\Omega(0, j, H)$. Let us consider the quadrature formula $Y(0, 2j; \{A_{rs}\}; \{x_{rs}\})$ where we set

$$\begin{aligned} A_{r0} &= 1\,; \quad r = 1, 2, \ldots, 2j\,, \\ x_{r0} &= \frac{2\pi}{j}\left(\frac{r+1}{2} - \frac{3}{4}\right); \quad r = 1, 3, \ldots, 2j - 1\,, \\ &= \frac{\pi r}{j}\,; \quad r = 2, 4, \ldots, 2j\,. \end{aligned} \tag{9.6}$$

Thus supposing (9.6) we may write

$$Y(0, 2j; \{A_{rs}\}; \{x_{rs}\})(f) = \frac{1}{2j}\sum_{p=1}^{j}\left(f\left(\frac{2\pi}{j}(p - \tfrac{3}{4})\right) + f\left(\frac{2\pi p}{j}\right)\right)$$

for all $f \in H$. Let us now use Lemma 3.3. Substituting (9.6) into (3.6) and using (4.13), we get

$$\|J - Y(0, 2j; \{A_{rs}\}; \{x_{rs}\})\|^2 = \tfrac{1}{4}\sum_{\substack{t=-\infty \\ t\neq 0}}^{\infty} |1 + e^{3it\pi/2}|^2\, \eta_{tj}^{-2} \tag{9.7}$$

$$= \tfrac{1}{4}\left(|1 - i|^2\, \eta_j^{-2} + |1 + i|^2\, \eta_{-j}^{-2} + \sum_{|t|\geqq 3} |1 + e^{3it\pi/2}|^2\, \eta_{tj}^{-2}\right)$$

with regard to (3.5), (9.6). Supposing again (9.1), (9.3) we may estimate

$$\sum_{|t|\geqq 3} |1 + e^{3it\pi/2}|^2\, \eta_{tj_q}^{-2} \leqq \frac{8\exp(-3j_q)}{1 - \exp(-j_q)} = O(\exp(-3j_q))\,, \quad q \to \infty\,. \tag{9.8}$$

Finally, from (9.1), (9.3), (9.7), (9.8) we get

$$\|J - Y(0, 2j_q; \{A_{rs}\}; \{x_{rs}\})\|^2 \leqq O(\exp(-4j_q)) \tag{9.9}$$

$$+ O(\exp(-3j_q)) = O(\exp(-3j_q))\,, \quad q \to \infty\,.$$

Thus from (9.5), (9.9) we have

$$\limsup_{j\to\infty} \frac{X(0, j, H)}{j^\alpha \Omega(0, j, H)} \geqq \limsup_{q\to\infty} \frac{X(0, 2 \times 5^q, H)}{2^\alpha 5^{\alpha q}\Omega(0, 2 \times 5^q, H)} = +\infty\,, \quad \alpha > 0\,.$$

The theorem has been proved.

Theorem 9.2. *There exists a 2-periodic space H such that*

$$\limsup_{j\to\infty} \frac{X(2, j, H)}{j^\alpha \Omega(2, j, H)} = +\infty \tag{9.10}$$

holds for arbitrary $\alpha > 0$.

Proof. The proof is analogous to that of Theorem 9.1. We use Theorem 2.2 to construct the 2-periodic space with the property (9.10). Let U be a set of all integers. We put

$$\eta_k^2 = e^{6|k|}k^4\,; \quad |k| = 7^r \quad \text{for any integer} \quad r \geqq 0\,, \tag{9.11}$$

$$\eta_k^2 = e^{2|k|}k^4\,; \quad |k| = 3 \times 7^r \quad \text{for any integer} \quad r \geqq 0\,,$$

$$\eta_k^2 = e^{|k|}k^4 \quad \text{for all the other integers } k\,.$$

Let us construct the corresponding Hilbert space of 2π-periodic continuous functions with the first and second derivative continuous. The condition (2.16) of Theorem 2.2 is satisfied since we may estimate

$$\sum_{k=-\infty}^{\infty} k^4 \eta_k^{-2} \leqq \sum_{k=-\infty}^{\infty} e^{-|k|} = 1 + \frac{2e^{-1}}{1 - e^{-1}} < +\infty .$$

Thus the space H constructed in this manner with the norm (9.11) is 2-periodic. From (9.11) the validity of (iv) follows. Then we may prove (5.5) for $n = 2$ in this space in the same way as in the proof of Lemma 5.1 considering the convergent series

$$\sum_{t=-\infty}^{\infty} e^{-|t|}$$

instead of (5.7).

With respect to (5.5) we may prove (5.8) in the same way as in the proof of Theorem 5.2 and (5.13) to (5.16) in the same way as in the proof of Theorem 5.4.

We set

$$j_q = 7^q \tag{9.12}$$

for any non-negative integer q and let us prove

$$\lim_{q\to\infty} \frac{\Phi(2, 2j_q, H)}{X(2, 2j_q, H)} = 1 . \tag{9.13}$$

For this prupose it is sufficient to follow the proof of Theorem 7.2 and use

$$Q(2j_q) \leqq \eta_{2j_q}^2 \sum_{t=1}^{\infty} t^4 \eta_{2tj_q}^{-2} \tag{9.14}$$

instead of (7.7). Since

$$2tj_q \neq 7^r , \quad 2tj_q \neq 3 \times 7^r \tag{9.15}$$

for any integers $t > 0$; $r, q \geqq 0$, from (9.11), (9.14) we get

$$\lim_{q\to\infty} Q(2j_q) \leqq 1 .$$

The relation (9.13) follows from this inequality and (7.8).

To prove (9.10) it is sufficient to show

$$\limsup_{q\to\infty} \frac{X(2, 2j_q, H)}{2^\alpha j_q^\alpha \Omega(2, 2j_q, H)} = +\infty \tag{9.16}$$

for any $\alpha > 0$. Hence with respect to (9.13) the relation

$$(9.17)\qquad \limsup_{q\to\infty} \frac{\Phi(2, 2j_q, H)}{2^\alpha j_q^\alpha \Omega(2, 2j_q, H)} = +\infty$$

for any $\alpha > 0$ is sufficient for the validity of (9.16).

Using Theorem 6.4, (9.11), (9.12), and (9.15) we get

$$(9.18)\qquad \Phi^2(2, 2j_q, H) \geqq 2^{-4} j_q^{-4} \frac{\exp(-4j_q)}{1 - \exp(-2j_q)} = O(j_q^{-4} \exp(-4j_q)), \quad q \to \infty$$

where we use the inequality

$$(t^2 - 1)^2 t^{-4} \geqq \tfrac{1}{2}$$

for any integer $t \geqq 2$.

Now we construct an upper bound of the quantity $\Omega(2, j, H)$. Let us consider the quadrature formula $Y(2, 2j; \{A_{rs}\}; \{x_{rs}\})$ where we set

$$(9.19)\qquad \begin{aligned} A_{r0} &= A_{r2} = 1; \quad r = 1, 2, \dots, 2j, \\ A_{r1} &= 0; \quad r = 1, 2, \dots, 2j, \\ x_{rs} &= \frac{2\pi}{j}\left(\frac{r+1}{2} - \frac{1}{8}\right); \quad r = 1, 3, \dots, 2j - 1, \\ &= \frac{\pi r}{j}; \quad r = 2, 4, \dots, 2j, \quad \text{and} \quad s = 0, 1, 2. \end{aligned}$$

Thus supposing (9.19) we may write

$$\begin{aligned} Y(2, 2j; \{A_{rs}\}; \{x_{rs}\})(f) &= \frac{1}{2j} \sum_{p=1}^{j} \left(f\left(\frac{2\pi}{j}\left(p - \frac{1}{8}\right)\right) + f\left(\frac{2\pi p}{j}\right)\right) \\ &+ \frac{1}{8j^3} \sum_{p=1}^{j} \left(f''\left(\frac{2\pi}{j}\left(p - \frac{1}{8}\right)\right) + f''\left(\frac{2\pi p}{j}\right)\right) \end{aligned}$$

for all $f \in H$. Let us now use Lemma 3.3. Substituting (9.19) into (3.6) and using (4.13) we get

$$(9.20)\qquad \begin{aligned} &\|J - Y(2, 2j; \{A_{rs}\}; \{x_{rs}\}\|^2 \\ &= \tfrac{1}{4} \sum_{\substack{t=-\infty \\ t\neq 0}}^{\infty} (\tfrac{1}{4}t^2 - 1)^2 \eta_{tj}^{-2} |1 + e^{2it\pi/8}|^2 \\ &= \tfrac{9}{32}|1 + 2^{-1/2} + 2^{-1/2} i|^2 \eta_j^{-2} + \tfrac{25}{32}|1 - 2^{-1/2} + 2^{-1/2} i|^2 \eta_{3j}^{-2} \\ &+ \tfrac{1}{2} \sum_{t=5}^{\infty} (\tfrac{1}{4}t^2 - 1)^2 |1 + e^{2it\pi/8}|^2 \eta_{tj}^{-2} \end{aligned}$$

with regard to (3.5), (9.19), (iv). Supposing again (9.12) we may estimate

$$(9.21)\qquad \sum_{t=5}^{\infty} (\tfrac{1}{4}t^2 - 1)^2 \left|1 + e^{2it\pi/8}\right|^2 \eta_{tj_q}^{-2}$$

$$\leqq j_q^{-4} \frac{\exp(-5j_q)}{4(1 - e^{-1})} = O(j_q^{-4} \exp(-5j_q)), \quad q \to \infty$$

with respect to (9.11). Finally, from (9.11), (9.12), (9.20), (9.21) we get

$$(9.22)\qquad \left\|J - Y(2, 2j_q; \{A_{rs}\}; \{x_{rs}\})\right\|^2 \leqq O(j_q^{-4} \exp(-5j_q)), \quad q \to \infty.$$

Hence from (9.18), (9.22) we have (9.17) supposing (9.19). From the relation (9.17) the statement (9.10) follows. Therefore the proof has been completed.

Remark 9.1. Theorems 9.1, 9.2 hold for any real α but the statement in the case $\alpha \leqq 0$ is trivial.

From Theorems 9.1, 9.2 it follows that the error of the optimal quadrature formula as compared with the infimum of the error attainable in general is asymptotically large in the 0-periodic or 2-periodic space.

The comparison of $\Omega(0, j, H)$ and $\Phi(0, j, H)$ is somewhat more favourable if we confine ourselves only to strongly 0-periodic spaces.

Theorem 9.3. *Let H be a strongly* 0*-periodic space. Then*

$$(9.23)\qquad \limsup_{j \to \infty} \frac{\Phi(0, j, H)}{j^{1/2}\Omega(0, j, H)} < +\infty.$$

Proof. Let us write

$$(9.24)\qquad T^2(0, j; \{A_{rs}\}; \{x_{rs}\}) = \left\|J - Y(0, j; \{A_{rs}\}; \{x_{rs}\})\right\|^2 \eta_j^2.$$

Then using Lemma 3.3 and (v) we may write

$$(9.25)\qquad T^2(0, j; \{A_{rs}\}; \{x_{rs}\}) = \eta_j^2 \sum_{k=-\infty}^{\infty} |B_k|^2 \eta_k^{-2} \geqq \sum_{k=-j}^{j} |B_k|^2$$

where B_k's are given in (3.6). Let us find a lower bound of the infimum of the quantity

$$\sum_{k=-j}^{j} |B_k|^2$$

in dependence on A_{r0}, x_{r0}. We introduce the finite-dimensional Hilbert space H_j of functions f of the form

$$(9.26)\qquad f(x) = \sum_{k=-j}^{j} f_k e^{ikx}$$

with the scalar product

(9.27) $$(f, g)_j = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,\overline{g(x)}\,\mathrm{d}x = \sum_{k=-j}^{j} f_k \bar{g}_k$$

for $f, g \in H_j$,

$$g(x) = \sum_{k=-j}^{j} g_k e^{ikx}\,.$$

From (9.27) we get the norm

(9.28) $$\|f\|_j^2 = \frac{1}{2\pi}\int_0^{2\pi} |f(x)|^2\,\mathrm{d}x = \sum_{k=-j}^{j} |f_k|^2$$

for all $f \in H_j$.

Let us consider the functionals J, $Y(0, j; \{A_{rs}\}; \{x_{rs}\})$ on the space H_j. They are linear, and using (9.26), (9.27) and by straightforward computation we may verify

(9.29) $$J(f) = (f, u_j)_j$$

for all $f \in H_j$ where $u_j \in H_j$,

(9.30) $$u_j(x) = 1\,,$$

and

(9.31) $$Y(0, j; \{A_{rs}\}; \{x_{rs}\})\,(f) = (f, v_j)_j$$

for all $f \in H_j$ where $v_j \in H_j$,

(9.32) $$v_j(x) = j^{-1} \sum_{k=-j}^{j} e^{ikx} \sum_{r=1}^{j} \exp\left(-ikx_{r0}\right) \bar{A}_{r0}\,.$$

Finally, from the relations (9.28) to (9.32) we compute

(9.33) $$\|J - Y(0, j; \{A_{rs}\}; \{x_{rs}\})\|_j^2 = \|\varrho_j\|_j^2 = \sum_{k=-j}^{j} |B_k|^2$$

where B_k's are given in (3.6) and $\varrho_j \in H_j$,

(9.34) $$\varrho_j(x) = u_j(x) - v_j(x)\,,$$

i.e. with respect to (9.26) we may write

(9.35) $$\varrho_j(x) = \sum_{k=-j}^{j} p_k e^{ikx}\,.$$

Thus let us find the minimum of (9.33) for all coefficients A_{r0} and abscissae x_{r0}. We write

(9.36) $$\Omega_j^2 = \inf_{A_{r0},x_{r0}} \|J - Y(0,j;\{A_{rs}\};\{x_{rs}\})\|_j^2 .$$

Then with respect to the fact that H_j is finite-dimensional there exist the coefficients A_{r0}^* and the abscissae x_{r0}^* such that

(9.37) $$\Omega_j^2 = \|J - Y(0,j;\{A_{rs}^*\};\{x_{rs}^*\})\|_j^2 .$$

Further, $\varrho_j^* \in H_j$ of the form (9.34), (9.35) exists so that

(9.38) $$(J - Y(0,j;\{A_{rs}^*\};\{x_{rs}^*\}))(f) = (f,\varrho_j^*)_j$$

holds for all $f \in H_j$ where with respect to (9.33), (9.36) to (9.38)

(9.39) $$\sum_{k=-j}^{j} |B_k|^2 \geqq \Omega_j^2 = \|\varrho_j^*\|_j^2$$

is valid for all coefficients A_{r0} and all abscissae x_{r0}.

Considering $\|\varrho_j\|_j^2$ as a function of $2j$ variables A_{r0}, x_{r0}; $r = 1, 2, \ldots, j$ and using (9.36) to (9.38) we have necessary conditions for the extremum $\|\varrho_j^*\|_j^2$ of this function in the form

(9.40) $$\varrho_j^*(x_{r0}^*) = \varrho_j^{*\prime}(x_{r0}^*) = 0\,; \quad r = 1, 2, \ldots, j .$$

Therefore the abscissae x_{r0}^* are double zeros of the function $\varrho_j^*(x)$. Then with respect to (9.35) we get

(9.41) $$\varrho_j^*(x) = C|h(x)|^2$$

where C is a constant and $|h|^2 \in H_j$,

$$h(x) = \prod_{r=1}^{j} (\exp(ix) - \exp(ix_{r0}^*)) .$$

Hence using the relations (9.38) to (9.41) and the equality

$$\|\varrho_j^*\|_j^2 = |C|^2 \, \||h|^2\|_j^2$$

we have

(9.42) $$\Omega_j^2 = \|\varrho_j^*\|_j^2 = (J - Y(0,j;\{A_{rs}^*\};\{x_{rs}^*\}))(\varrho_j^*) = \frac{(J(|h|^2))^2}{\||h|^2\|_j^2} .$$

Since $|h|^2 \in H_j$ we may write

(9.43) $$|h(x)|^2 = \sum_{k=-j}^{j} h_k e^{ikx}$$

with respect to (9.26). Further we may show

$$(9.44) \qquad |h_k| \leqq h_0 \,; \quad k = -j, -j+1, \ldots, j$$

since from (9.27) we have

$$h_k = \frac{1}{2\pi} \int_0^{2\pi} |h(x)|^2 \, e^{-ikx} \, dx$$

and $|h|^2 > 0$.

Finally, from (9.28), (9.42) to (9.44) we get

$$(9.45) \qquad \Omega_j^2 \geqq (2j+1)^{-1}$$

for arbitrary coefficients A_{r0} and arbitrary abscissae x_{r0}. Then from (9.25), (9.39), (9.45) we have

$$(9.46) \qquad T(0, j; \{A_{rs}\}; \{x_{rs}\}) \geqq (2j+1)^{-1/2}$$

and from (9.24), (9.46) we get

$$(9.47) \qquad \|J - Y(0, j; \{A_{rs}\}; \{x_{rs}\})\| \geqq \eta_j^{-1}(2j+1)^{-1/2}$$

for arbitrary coefficients A_{r0} and arbitrary abscissae x_{r0}. With respect to Theorem 6.3 and the conditions (iv), (vi) we may write

$$(9.48) \qquad \Phi^2(0, j, H) = 2 \sum_{t=1}^{\infty} \eta_{tj}^{-2} \leqq 2\eta_j^{-2} \, D(H) \, .$$

Then (9.23) follows from (9.47), (9.48). The theorem has been proved.

Remark 9.2. The relation

$$\limsup_{j \to \infty} \frac{X(0, j, H)}{j^{1/2} \Omega(0, j, H)} < +\infty$$

follows immediately from Theorem 9.3. Thus the error of the optimal quadrature formula is comparable, apart from the factor $j^{1/2}$, with the infimum of the error attainable in general in the strongly 0-periodic space. The same holds for the error of the limit quadrature formula.

We shall prove a less general statement in the strongly 2-periodic space. Let us introduce the following notation.

Definition 9.2. Let H be an n-periodic space. Let $Y(n, j; \{A_{rs}\}; \{x_{rs}\})$ be a quadrature formula in this space. We set

$$(9.49) \qquad x_{rs} = x_r \,; \quad r = 1, 2, \ldots, j \,; \quad s = 0, 1, \ldots, n$$

and denote by

$$\Omega^*(n, j, H) = \inf_{A_{rs}, x_r} \|J - Y(n, j; \{A_{rs}\}; \{x_{rs}\})\|$$

the infimum of the errors of the quadrature formulae $Y(n, j; \{A_{rs}\}; \{x_{rs}\})$ for fixed j, arbitrary coefficients A_{rs}, and abscissae given by (9.49), i.e. abscissae common for evaluating the values of both the integrand and its derivatives.

Remark 9.3. Let H be a 0-periodic space. Then

$$\Omega(0, j, H) = \Omega^*(0, j, H).$$

Now we compare $\Omega^*(2, j, H)$ and $\Phi(2, j, H)$ in the strongly 2-periodic space.

Theorem 9.4. *Let H be a strongly 2-periodic space. Then*

$$\limsup_{j \to \infty} \frac{\Phi(2, j, H)}{j^{1/2} \Omega^*(2, j, H)} < +\infty. \tag{9.50}$$

Proof. Let

$$Y(2, j; \{A_{rs}\}; \{x_{rs}\}) \tag{9.51}$$

be a quadrature formula in the strongly 2-periodic space H. Let x_r; $r = 1, 2, \ldots, j$ be real numbers satisfying the condition

$$0 < x_1 < x_2 < \ldots < x_j \leqq 2\pi.$$

We set

$$x_{rs} = x_r; \quad r = 1, 2, \ldots, j; \quad s = 0, 1, 2. \tag{9.52}$$

We suppose the validity of (9.52) for the quadrature formula (9.51) in the whole proof. The proof is analogous to that of Theorem 9.3.

Let us write

$$T^2(2, j; \{A_{rs}\}; \{x_{rs}\}) = \|J - Y(2, j; \{A_{rs}\}; \{x_{rs}\})\|^2 \eta_{2j}^2. \tag{9.53}$$

Then using Lemma 3.3 and (v) we may rewrite this as

$$T^2(2, j; \{A_{rs}\}; \{x_{rs}\}) = \eta_{2j}^2 \sum_{k=-\infty}^{\infty} |B_k|^2 \eta_k^{-2} \geqq \sum_{k=-2j}^{2j} |B_k|^2 \tag{9.54}$$

where B_k's are given in (3.6). Let us find a lower bound of the infimum of the quantity

$$\sum_{k=-2j}^{2j} |B_k|^2$$

in dependence on the coefficients A_{rs} and the abscissae x_r. In an analogous manner to (9.26), (9.27), we introduce the finite-dimensional Hilbert space H_j of functions f of the form

$$f(x) = \sum_{k=-2j}^{2j} f_k e^{ikx} \tag{9.55}$$

with the scalar product

$$(f, g)_j = \frac{1}{2\pi} \int_0^{2\pi} f(x) \overline{g(x)} \, dx = \sum_{k=-2j}^{2j} f_k \bar{g}_k \tag{9.56}$$

for $f, g \in H_j$,

$$g(x) = \sum_{k=-2j}^{2j} g_k e^{ikx} .$$

Let us consider the functionals J, $Y(2, j; \{A_{rs}\}; \{x_{rs}\})$ on the space H_j. They are linear, and by straightforward computation from (9.55), (9.56) we may find their realizing functions $u_j, v_j \in H_j$ respectively, which are given analogously to (9.30), (9.32).

Moreover, by computation we get

$$\|J - Y(2, j; \{A_{rs}\}; \{x_{rs}\})\|_j^2 = \|\varrho_j\|_j^2 = \sum_{k=-2j}^{2j} |B_k|^2 , \tag{9.57}$$

which is analogous to (9.33), where B_k's are given in (3.6) and $\varrho_j \in H_j$,

$$\varrho_j(x) = u_j(x) - v_j(x) , \tag{9.58}$$

i.e. with respect to (9.55) we have

$$\varrho_j(x) = \sum_{k=-2j}^{2j} p_k e^{ikx} . \tag{9.59}$$

Thus let us find the minimum of (9.57) for all coefficients A_{rs} and all abscissae x_r. We write

$$\Omega_j^2 = \inf_{A_{rs}, x_r} \|J - Y(2, j; \{A_{rs}\}; \{x_{rs}\})\|_j^2 . \tag{9.60}$$

Then with respect to the fact that H_j is finite-dimensional there exist the coefficients A^*_{rs} and the abscissae $x^*_r = x^*_{rs}$ such that analogously to (9.39),

$$\sum_{k=-2j}^{2j} |B_k|^2 \geqq \Omega_j^2 = \|\varrho^*_j\|_j^2 \tag{9.61}$$

holds for all coefficients A_{rs} and all abscissae x_r where $\varrho_j^* \in H_j$ is a function of the form (9.58), (9.59) given by

$$(9.62) \qquad (J - Y(2, j; \{A_{rs}^*\}; \{x_{rs}^*\})(f) = (f, \varrho_j^*)_j$$

for all $f \in H_j$.

Considering $\|\varrho_j\|_j^2$ as a function of $4j$ variables A_{rs}, x_r; $r = 1, 2, \ldots, j$; $s = 0, 1, 2$ and using (9.60) to (9.62) we have necessary conditions for the extremum $\|\varrho_j^*\|_j^2$ of this function in the form

$$(9.63) \qquad \varrho_j^{*(s)}(x_r^*) = 0\,; \quad s = 0, 1, 2, 3\,; \quad r = 1, 2, \ldots, j$$

analogous to (9.40). Therefore the abscissae x_r^* are quadruple zeros of the function $\varrho_j^*(x)$. Then with respect to (9.59)

$$(9.64) \qquad \varrho_j^*(x) = C|h(x)|^4$$

where C is a constant and $|h|^4 \in H_j$,

$$h(x) = \prod_{r=1}^{j} (\exp(ix) - \exp(ix_r^*))\,,$$

which corresponds to (9.41). Hence using (9.61) to (9.64) we get

$$(9.65) \qquad \Omega_j^2 = \|\varrho_j^*\|_j^2 = \frac{(J(|h|^4))^2}{\||h|^4\|_j^2} \geqq (4j + 1)^{-1}$$

for arbitrary coefficients A_{rs} and arbitrary abscissae x_r proceeding in the analogous way as in (9.42) to (9.45). Then from (9.53), (9.54), (9.61), (9.65) we have

$$(9.66) \qquad \|J - Y(2, j; \{A_{rs}\}; \{x_{rs}\})\| \geqq \eta_{2j}^{-1}(4j + 1)^{-1/2}$$

for arbitrary coefficients A_{rs} and arbitrary abscissae x_r.

With respect to Theorem 6.4 and the conditions (iv), (vi) we may write

$$(9.67) \qquad \Phi^2(2, j, H) = 2\sum_{t=2}^{\infty} (t^2 - 1)^2\, \eta_{tj}^{-2} \leqq 2\eta_{2j}^{-2}\, D(H)\,.$$

Then (9.50) follows from (9.66), (9.67). The theorem has been proved.

Remark 9.4. The relation

$$\limsup_{j\to\infty} \frac{X(2, j, H)}{j^{1/2}\Omega^*(2, j, H)} < +\infty$$

follows immediately from Theorem 9.4. Thus the error of the optimal quadrature formula is comparable, apart from the factor $j^{1/2}$, with the infimum of the error attainable in general in the strongly 2-periodic space if we suppose that the abscissae are common for evaluating the values of both the integrand and its derivatives (first and second). The same holds for the error of the limit quadrature formula.

While the error of the optimal quadrature formula is incomparable with the infimum of the error attainable in the 0-periodic and the 2-periodic space the situation is more favourable in the strongly 0-periodic or the strongly 2-periodic space as was shown in Theorems 9.3, 9.4.

Tab. 1

$f(x) = e^{3\sin x}$

j	$N(0, j) \times 10^{17}$	$N(2, j) \times 10^{17}$
8	162 740 472 455 793	21 491 193
16	7 163 765	0
24	0	0

$f(x) = e^{10\sin x}$

j	$N(0, j) \times 10^{14}$	$N(2, j) \times 10^{14}$
8	23 219 296 635 336 995	18 030 560 320 883
16	6 010 050 031 314	112
24	51 278 311	0
32	37	0
40	0	0

$f(x) = e^{50\sin x}$

j	$N(0, j) \times 10^{21}$	$N(2, j) \times 10^{21}$
8	35 483 337 836 564 182	15 147 478 502 324 658
16	4 520 016 724 736 825	7 875 422 083 042
24	194 454 867 617 997	60 599 912
32	2 625 086 827 743	12
40	12 014 357 949	0
48	20 199 971	0
56	13 480	0
64	4	0
72	0	0

10. A NUMERICAL EXAMPLE

Let us illustrate the results of Sec. 8, in particular the statement of Theorem 8.5, with a numerical example.

The values of the limit quadrature formulae $I(0, j; 1)(f)$ and $I(2, j; 1, 0, 1)(f)$ were numerically computed for

$$f(x) = \exp(3 \sin x), \quad \exp(10 \sin x), \quad \exp(50 \sin x)$$

and $j = 8, 16, \ldots, 72$. The rounded-off values of

$$N(0, j) = |J(f) - I(0, j; 1)(f)|,$$
$$N(2, j) = |J(f) - I(2, j; 1, 0, 1)(f)|$$

are given in Tab. 1. The computation was carried out on the ICT 1905 computer using double arithmetic. It is apparent that

$$N(2, j) \approx 3N(0, 2j)$$

holds for all j and f. It is in agreement with Theorem 8.5 and Remark 8.2.

References

[1] *I. Babuška:* Über die optimale Berechnung der Fourierschen Koeffizienten, Aplikace matematiky *11*, pp. 113—122 (1968).

[2] *I. Babuška:* Problems of optimization and numerical stability in computations, Aplikace matematiky *13*, pp. 3—26 (1968).

[3] *I. Babuška:* Über universal optimale Quadraturformeln, Aplikace matematiky *13*, pp. 304—338, 388—404 (1968).

[4] *I. Babuška, S. L. Sobolev:* Оптимизация численных методов, Aplikace matematiky *10*, pp. 96—129 (1965).

[5] *L. Collatz:* Funktionalanalysis und numerische Mathematik, Springer-Verlag, Berlin—Göttingen—Heidelberg 1964.

[6] *I. S. Gradštein, I. M. Ryžik:* Таблицы интегралов, сумм, рядов и произведений, 4th ed., Гос. изд. физ.-мат. лит., Москва 1962.

[7] *I. J. Schoenberg:* Spline interpolation and best quadrature formulae, Bull. Am. Math. Soc. *70* (1964).

[8] *K. Segeth:* On quadrature formulae involving values of derivatives, ZAMM *48* (1968).

[9] *S. L. Sobolev:* Лекции по теории кубатурных формул I, II, Новосибирск 1964, 1965.

Author's address: Praha 1, Žitná 25, ČSSR (Matematický ústav ČSAV).