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Czechoslovak Mathematical Journal, Vol. 20 (1970), No. 4, 680–695

Persistent URL: http://dml.cz/dmlcz/100990

Czechoslovak Mathematical Journal, 20 (95) 1970, Praha

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(Received August 11, 1969)

INTRODUCTION

In this second part of our paper we present the theory of L_p spaces for our integration theory of vector valued functions with respect to an operator valued measure countably additive in the strong operator topology, see [4].

In § 1 we define the L_1 norm of the measurable function which generalizes the classical L_1 norm, see the Corollary of Theorem 4, and also the notion of the semivariation $\hat{\boldsymbol{m}}$. While the classical L_1 norm is a finite non negative countably additive measure, absolutely continuous with respect to the initial measure, our L_1 norm is in general only a countably subadditive set function not continuous on $\mathfrak{S}(\mathscr{P})$ and not absolutely $\hat{\boldsymbol{m}}$ continuous. However, in the important special case when Y is a weakly complete Banach space, the finite L_1 norm of the measurable function is continuous on $\mathfrak{S}(\mathscr{P})$ and therefore also absolutely $\hat{\boldsymbol{m}}$ continuous, see Theorem 5.

In § 2 we show that the equivalent classes of measurable functions $L_1\mathfrak{M}(\boldsymbol{m})$ as well as integrable functions $L_1\mathfrak{I}(\boldsymbol{m})$, with finite L_1 norms form Banach spaces, in general different, see Theorems 7 and 9. These Banach spaces behave in general very badly, namely, no analogs of classical convergence and separability theorems are valid for them. Somewhat better is the behaviour of the Banach space $L_1\mathfrak{I}_s(\boldsymbol{m})$ which is the closure of the set of all simple integrable functions $\mathfrak{I}_s$ in $L_1\mathfrak{M}(\boldsymbol{m})$, see Theorems 13, 14 and 15. However, the most important is the Banach space $L_1(\boldsymbol{m})$ consisting of equivalent classes of those measurable functions whose L_1 norms are continuous on $\mathfrak{S}(\mathscr{P})$, see Theorems 8 and 9. By Theorem 9 $L_1(\boldsymbol{m}) \subset L_1\mathfrak{I}_s(\boldsymbol{m})$, and $L_1(\boldsymbol{m}) = L_1\mathfrak{I}_s(\boldsymbol{m})$ if and only if the semivariation $\hat{\boldsymbol{m}}$ is continuous on $\mathscr{P}$, see Theorem 11. If Y is a weakly complete Banach space, then by the important Theorem 10, $L_1\mathfrak{M}(\boldsymbol{m}) = L_1(\boldsymbol{m})$.

The importance of $L_1(\boldsymbol{m})$ lies in its good classical properties. Namely, complete analogs of classical Vitali and Lebesgue convergence theorems are valid for it, see Theorems 16 and 17. Owing to Theorem 19 only the space $L_1(\boldsymbol{m})$ can be separable.

Concerning the separability of $L_1(\boldsymbol{m})$, analogs of classical theorems are valid, see Theorems 20 and 21. Further, if $\boldsymbol{m}$ is a Baire or a regular Borel operator valued measure whose semivariation $\hat{\boldsymbol{m}}$ is continuous on $\mathscr{B}_0$ or $\mathscr{B}$ respectively, then $C_0(T, X)$ is dense in $L_1(\boldsymbol{m})$, see Theorems 22 and 23. This fact is of great importance in connection with the representation theorems of § 2 in [5]. It is worth noting that at the same time the space $L_1(\boldsymbol{m})$ is in general substantially wider then the Banach space of Bochner integrable functions $L_1(v(\boldsymbol{m}, .), X)$ where $v(\boldsymbol{m}, .)$ denotes the variation of the measure $\boldsymbol{m}$. This latter space is treated in [3].

In the short § 3 we indicate a similar theory of L_p spaces for $p \geqq 1$ and of Orlicz spaces.

There are some similarities between the presented theory and the theory developed in [11] and [12] for integration of scalar functions with respect to a finitely additive vector measure. In terminology and notation we follow [4] and [5]. In part III of our paper, which is being prepared, we shall treat Fubini type theorems.

1. THE L_1 NORM

Definition 1. *Let $\boldsymbol{g}$ be a measurable function and let $E \in \mathfrak{S}(\mathscr{P})$. Then the L_1 norm of the function $\boldsymbol{g}$ on the set E, which will be denoted by $\hat{\boldsymbol{m}}(\boldsymbol{g}, E)$ is a non negative not necessarily finite number defined by the equality:*

$$\hat{\boldsymbol{m}}(\boldsymbol{g}, E) = \sup \left\{ \left| \int_E \boldsymbol{f} \, \mathrm{d}\boldsymbol{m} \right| , \boldsymbol{f} \in \mathfrak{I}_s, \ |\boldsymbol{f}(t)| \leqq |\boldsymbol{g}(t)| \ \textit{for each} \ t \in E \right\} .$$

The L_1 norm of the function $\boldsymbol{g}$ is defined by $\hat{\boldsymbol{m}}(\boldsymbol{g}, T) = \sup_{E \in \mathfrak{S}(\mathscr{P})} \hat{\boldsymbol{m}}(\boldsymbol{g}, E)$.

Let us remind that $\mathfrak{I}_s$ denotes the set of all simple integrable functions, see Definition 1 in [4], and that $|.|$ denotes the norm in the Banach space.

From this definition we immediately obtain the following

Theorem 1. *Let $\boldsymbol{g}$ be a measurable function and let $E \in \mathfrak{S}(\mathscr{P})$. Then:*

a) *$\hat{\boldsymbol{m}}(\boldsymbol{g}, .)$ is a monotone and countably subadditive set function on $\mathfrak{S}(\mathscr{P})$ with $\hat{\boldsymbol{m}}(\boldsymbol{g}, \emptyset) = 0$.*

b) *$\hat{\boldsymbol{m}}(a\boldsymbol{g}, E) = |a| \, . \, \hat{\boldsymbol{m}}(\boldsymbol{g}, E)$ for each scalar a.*

c) $\inf_{t \in E} |\boldsymbol{g}(t)| \, . \, \hat{\boldsymbol{m}}(E) \leqq \hat{\boldsymbol{m}}(\boldsymbol{g}, E) \leqq \|\boldsymbol{g}\|_E \, . \, \hat{\boldsymbol{m}}(E)$.

d) *If $\boldsymbol{h}$ is a measurable function with $|\boldsymbol{h}(t)| \leqq |\boldsymbol{g}(t)|$ almost everywhere $\boldsymbol{m}$ on E, then $\hat{\boldsymbol{m}}(\boldsymbol{h}, E) \leqq \hat{\boldsymbol{m}}(\boldsymbol{g}, E)$.*

e) $\hat{\boldsymbol{m}}(\boldsymbol{g}, E) = \hat{\boldsymbol{m}}(\boldsymbol{g}, \{t \in E, |\boldsymbol{g}(t)| > 0\})$.

f) $\hat{\boldsymbol{m}}(\boldsymbol{g}, E) = 0$ *if and only if* $\hat{\boldsymbol{m}}(\{t \in E, |\boldsymbol{g}(t)| > 0\}) = 0$.

From assertion c) of this theorem it is obvious that if $|x| = 1$, then $\hat{m}(E) = \hat{m}(x \cdot \chi_E, E)$ for each set $E \in \mathfrak{S}(\mathscr{P})$. Hence the L_1 norm $\hat{m}(.,.)$ generalizes the notion of the semivariation $\hat{m}$. From this assertion we also have the following form of Tschebyscheff inequality:

Corollary. *Let g be a measurable function, let $E \in \mathfrak{S}(\mathscr{P})$ and let $a > 0$. Then $\hat{m}(\{t \in E, |g(t)| \geqq a\}) \leqq \hat{m}(g, E)/a$.*

From this inequality immediately follows that convergence in the L_1 norm implies convergence in the semivariation $\hat{m}$, see Lemma 4 below and section 1.3 in [4].

To prove the triangle inequality $\hat{m}(g + h, E) \leqq \hat{m}(g, E) + \hat{m}(h, E)$ we need the following theorem, which generalizes Theorem 14 from [4]. It may be proved in the same way.

Theorem 2. *Let g be a measurable function and let $E \in \mathfrak{S}(\mathscr{P})$. Then*

$$\hat{m}(g, E) = \sup \left\{ \left| \int_E f \, \mathrm{d}m \right|, f \in \mathfrak{I}, |f(t)| \leqq |g(t)| \text{ for each } t \in E \right\}.$$

As an immediate consequence we have:

Corollary. *Let f be an integrable function and let $E \in \mathfrak{S}(\mathscr{P})$. Then*

$$\left| \int_E f \, \mathrm{d}m \right| \leqq \hat{m}(f, E).$$

We note that this inequality is much better then that of Theorem 14 in [4].

Theorem 3. *Let f and g be measurable functions and let $E \in \mathfrak{S}(\mathscr{P})$. Then*

$$\hat{m}(f + g, E) \leqq \hat{m}(f, E) + \hat{m}(g, E),$$

and therefore also

$$\hat{m}(f + g, T) \leqq \hat{m}(f, T) + \hat{m}(g, T).$$

Proof. By assertion e) of Theorem 1 $\hat{m}(f + g, E) = \hat{m}(f + g, E')$ where $E' = \{t \in E, |f(t)| + |g(t)| > 0\}$. Let h be a simple integrable function fulfilling $|h(t)| \leqq |f(t) + g(t)|$ for each $t \in E'$. Then for each $t \in E'$

$$h(t) = \frac{h(t) \cdot |f(t)|}{|f(t)| + |g(t)|} + \frac{h(t) \cdot |g(t)|}{|f(t)| + |g(t)|}.$$

By Theorem 4 in [4] both summands are integrable functions and therefore by

Theorem 2

$$\left|\int_{E'} h \, \mathrm{d}\boldsymbol{m}\right| \leqq \left|\int_{E'} \frac{h(t) \, . \, |f(t)|}{|f(t)| + |g(t)|} \, \mathrm{d}\boldsymbol{m}\right| + \left|\int_{E'} \frac{h(t) \, . \, |g(t)|}{|f(t)| + |g(t)|} \, \mathrm{d}\boldsymbol{m}\right| \leqq \hat{\boldsymbol{m}}(f, E) + \hat{\boldsymbol{m}}(g, E) \, .$$

This proves the theorem.

In the same way as in Lemma 1 in [5] we obtain its following generalization:

Theorem 4. *For each measurable function* g *and for each set* $E \in \mathfrak{S}(\mathscr{P})$

$$\hat{\boldsymbol{m}}(g, E) = \sup_{|y^*| \leqq 1} \int_E |g| \, \mathrm{d}v(y^* \boldsymbol{m}, .) \, , \quad y^* \in Y^* \, .$$

If for each functional $y^* \in Y^*$ $\int_E |g| \, \mathrm{d}v(y^* \boldsymbol{m}, .) < +\infty$, *then also* $\hat{\boldsymbol{m}}(g, E) < +\infty$.

As an immediate consequence we have:

Corollary. *Let* Y *be the space of scalars of* X, *see examples* 1 *and* 5 *in section* 1.1 *in* [4], *or let* $\boldsymbol{m}$ *be a scalar measure and* $\boldsymbol{m}(E)\, x = \boldsymbol{m}(E) \, . \, x$, *see examples* 1 *and* 2 *in section* 1.1 *in* [4]. *Then* $\hat{\boldsymbol{m}}(g, E) = \int_E |g| \, \mathrm{d}v(\boldsymbol{m}, .)$ *for each measurable function* g *and for each set* $E \in \mathfrak{S}(\mathscr{P})$.

Thus we see that $\hat{\boldsymbol{m}}(., .)$ generalizes the classical L_1 norm. However, in general $\hat{\boldsymbol{m}}(g, E) \leqq \int_E |g| \, \mathrm{d}v(\boldsymbol{m}, .)$, and it very frequently happens that $\int_E |g| \, \mathrm{d}v(\boldsymbol{m}, .) = +\infty$ and $\hat{\boldsymbol{m}}(g, E) < +\infty$. Let for example T be the set of all non negative integers, $\mathscr{P}$ the σ-algebra of all subsets of T, X the space of scalars of Y, Y being an infinite dimensional Banach space. Then $\boldsymbol{m} : \mathscr{P} \to Y$ is a countably additive vector measure. Now $\hat{\boldsymbol{m}}(g, E) < +\infty$ if and only if the series $\sum_{k \in E} g(k) \, . \, \boldsymbol{m}(\{k\})$ is unconditionally convergent in Y, see IV.10.4 in [6], while on the other hand $\int_E |g| \, \mathrm{d}v(\boldsymbol{m}, .) < +\infty$ if and only if this series is absolutely convergent. Let us remind that by the theorem of Dvoretzky and Rogers, see section 3.4 in [15], every infinite dimensional Banach space contains an unconditionally convergent series which is not absolutely convergent.

As it was said in Introduction, the really interesting functions are those whose L_1 norms are continuous on $\mathfrak{S}(\mathscr{P})$. If $\int_T |g| \, \mathrm{d}v(\boldsymbol{m}, .) < +\infty$, then the L_1 norm $\hat{\boldsymbol{m}}(g, .)$ is clearly continuous on $\mathfrak{S}(\mathscr{P})$, but this is in general, as we stated above, a too strong restriction. In Theorem 5 below we prove the continuity of the L_1 norm under a much weaker restriction. To prove this theorem we need the next lemma which generalizes Theorem 5 in [4]. It may be proved in just the same way. We note that the assertion of this lemma will be substantially strengthened below in Theorem 8.

Lemma 1. *Let* g *be a measurable function and let its* L_1 *norm* $\hat{\boldsymbol{m}}(g, .)$ *be conti-*

nuous on $\mathfrak{S}(\mathscr{P})$, i.e., if $E_n \searrow \emptyset$, $E_n \in \mathfrak{S}(\mathscr{P})$, $n = 1, 2, \ldots$ then $\lim_{n \to \infty} \hat{\boldsymbol{m}}(\boldsymbol{g}, E_n) = 0$. Then $\boldsymbol{g}$ is an integrable function.

The following theorem generalizes the $*$-Theorem in section 1.1 in [4]. Using the preceding lemma it may be proved in just the same way.

Theorem 5. *Let Y contain no subspace isomorphic to the space c_0, for example let Y be a weakly complete Banach space, see Theorem 5 and section 6 in [2], let $\boldsymbol{g}$ be a measurable function and let $\hat{\boldsymbol{m}}(\boldsymbol{g}, T) < +\infty$. Then the L_1 norm $\hat{\boldsymbol{m}}(\boldsymbol{g}, .)$ is continuous on $\mathfrak{S}(\mathscr{P})$. In general for a measurable function $\boldsymbol{g}$ the continuity of its L_1 norm $\hat{\boldsymbol{m}}(\boldsymbol{g}, .)$ on $\mathfrak{S}(\mathscr{P})$ is equivalent with the existence of a finite non negative countably additive measure λ_g on $\mathfrak{S}(\mathscr{P})$ with the properties: $\lambda_g(E) \leqq \|\int \boldsymbol{g}\, \mathrm{d}\boldsymbol{m}\|(E) \leqq \hat{\boldsymbol{m}}(\boldsymbol{g}, E)$ and $\lim_{\lambda_g(E) \to 0} \hat{\boldsymbol{m}}(\boldsymbol{g}, E) = 0$, $E \in \mathfrak{S}(\mathscr{P})$.*

From here we have the following interesting

Corollary. *Let $\boldsymbol{g}$ be a measurable function and let its L_1 norm $\hat{\boldsymbol{m}}(\boldsymbol{g}, .)$ be continuous on $\mathfrak{S}(\mathscr{P})$. Then $\hat{\boldsymbol{m}}(\boldsymbol{g}, T) < +\infty$.*

Proof. By the second assertion of the preceding theorem there is a finite non negative countably additive measure λ_g on $\mathfrak{S}(\mathscr{P})$ with $\lim_{\lambda_g(E) \to 0} \hat{\boldsymbol{m}}(\boldsymbol{g}, E) = 0$, $E \in \mathfrak{S}(\mathscr{P})$. Take an $\varepsilon > 0$ such that $\lambda_g(E) < \varepsilon$ implies $\hat{\boldsymbol{m}}(\boldsymbol{g}, E) < 1$, $E \in \mathfrak{S}(\mathscr{P})$. Since the measure λ_g is finite, by the method of exhaustation, see Exercise 3, § 17 in [9], there is a set $G \in \mathfrak{S}(\mathscr{P})$ with $\lambda_g(E - G) = 0$ for each set $E \in \mathfrak{S}(\mathscr{P})$. Now by Lemma IV.9.7. in [6] there is a finite number of disjoint sets $E_i \in \mathfrak{S}(\mathscr{P})$, $i = 1, 2, \ldots, n$ with $\bigcup_{i=1}^{n} E_i = G$, each E_i being either an atom or $\lambda_g(E_i) < \varepsilon$. Since for each atom A it is clearly $\hat{\boldsymbol{m}}(\boldsymbol{g}, A) < +\infty$, hence the subadditivity of the set function $\hat{\boldsymbol{m}}(\boldsymbol{g}, .)$, see assertion a) of Theorem 1, proves the corollary.

It is worth noting that in general the L_1 norm $\hat{\boldsymbol{m}}(\boldsymbol{g}, .)$ is not continuous on $\mathfrak{S}(\mathscr{P})$ even if the semivariation $\hat{\boldsymbol{m}}$ is and $\boldsymbol{g}$ is an integrable function with $\hat{\boldsymbol{m}}(\boldsymbol{g}, T) < +\infty$ (clearly $\boldsymbol{g}$ is unbounded). For example, modify the measure $\boldsymbol{m}$ and the function $\boldsymbol{f}$ in Example 7″ in [4] in the following way: $\boldsymbol{m}'(\{k\}) = 1/k \,.\, \boldsymbol{m}(\{k\})$ and $\boldsymbol{f}'(k) = k \,.\, \boldsymbol{f}(k)$. In these cases the L_1 norm $\hat{\boldsymbol{m}}(\boldsymbol{g}, .)$ as a set function on $\mathfrak{S}(\mathscr{P})$ is not absolutely $\hat{\boldsymbol{m}}$ continuous. If we restrict in the same Example 7″ the measure $\boldsymbol{m}$ to the δ-ring $\mathscr{P}_0$ of all finite subsets of T, then its semivariation $\hat{\boldsymbol{m}}$ is clearly continuous on $\mathscr{P} = \mathscr{P}_0$, $\boldsymbol{f}$ is a bounded integrable function and nevertheless, its L_1 norm $\hat{\boldsymbol{m}}(\boldsymbol{f}, .)$ as a set function on $\mathfrak{S}(\mathscr{P})$ is not continuous on $\mathfrak{S}(\mathscr{P})$.

Let now T be a locally compact Hausdorff topological space and denote by $\mathfrak{Q}$ the set of all functions of the form $\boldsymbol{f} = \sum_{i=1}^{r} \varphi_i \boldsymbol{x}_i$, where φ_i is a scalar continuous function with compact support in T and $\boldsymbol{x}_i \in \boldsymbol{X}$, $i = 1, 2, \ldots, r$. Then the following theorem generalizes Theorem 1 from [5] and may be proved in just the same way (in the case of Borel measure $\boldsymbol{m}$ we use its regularity in the strong operator topology).

Theorem 6. *Let $\boldsymbol{m}$ be a Baire or a regular Borel operator valued measure, let $\boldsymbol{g}$ be a Baire or Borel measurable function and let $E \in \mathfrak{S}(\mathscr{B}_0)$ or $E \in \mathfrak{S}(\mathscr{B})$ respectively. Then*

$$\hat{\boldsymbol{m}}(\boldsymbol{g}, E) = \sup \left\{ \left| \int_E \boldsymbol{f} \, \mathrm{d}\boldsymbol{m} \right|, \boldsymbol{f} \in \mathfrak{Q}, \ |\boldsymbol{f}(t)| \leqq |\boldsymbol{g}(t)| \ \text{for each} \ t \in E \right\}.$$

If each function $\boldsymbol{f} \in C_0(T, X)$ is integrable on E, then obviously we may replace $\mathfrak{Q}$ in the preceding equality by $C_0(T, X)$.

2. L_1 SPACES

Definition 2. *We say that a sequence of measurable functions $\{\boldsymbol{g}_n\}_{n=1}^{\infty}$ converges in the L_1 norm, or in mean, to a measurable function $\boldsymbol{g}$ iff $\lim_{n\to\infty} \hat{\boldsymbol{m}}(\boldsymbol{g}_n - \boldsymbol{g}, T) = 0$.*

From assertion f) of Theorem 1 we immediately have:

Lemma 2. *For a measurable function $\boldsymbol{g}$, $\hat{\boldsymbol{m}}(\boldsymbol{g}, T) = 0$ if and only if $\boldsymbol{g} = 0$ almost everywhere $\boldsymbol{m}$.*

According to this lemma we shall use:

Definition 3. *We say that two measurable functions $\boldsymbol{f}$ and $\boldsymbol{g}$ are equivalent iff $\boldsymbol{f} = \boldsymbol{g}$ almost everywhere $\boldsymbol{m}$.*

From here and from Theorem 3 we immediately have:

Lemma 3. *If a sequence of measurable functions converges in the L_1 norm to two measurable functions, then these functions are equivalent.*

From the Corollary of Theorem 1 we immediately obtain:

Lemma 4. *If a sequence of measurable functions converges in the L_1 norm to a measurable function, then this sequence converges also in the semivariation $\hat{\boldsymbol{m}}$ to this function.*

Convergence in the semivariation $\hat{\boldsymbol{m}}$ was treated in section 1.3 in [4]. We are now prepared to prove an important:

Theorem 7. *Let $\{\boldsymbol{g}_n\}_{n=1}^{\infty}$ be a sequence of measurable functions fundamental in the L_1 norm. Then there is a measurable function $\boldsymbol{g}$ to which this sequence converges* *in the L_1 norm. If every g_n, $n = 1, 2, \ldots$ is an integrable function, then g is also an integrable function. If for every n the set function $\hat{m}(g_n, .)$ is continuous on $\mathfrak{S}(\mathscr{P})$, then also the set function $\hat{m}(g, .)$ is continuous on $\mathfrak{S}(\mathscr{P})$.*

Proof. Owing to the triangle inequality for the L_1 norm, see Theorem 3, it is sufficient to find a measurable function g and a subsequence $\{g_{n_k}\}_{k=1}^{\infty}$ of the sequence $\{g_n\}_{n=1}^{\infty}$ converging in the L_1 norm to the function g.

Let us take a subsequence $\{g_{n_k}\}_{k=1}^{\infty}$ in such a way that $\hat{m}(g_{n_k} - g_{n_{k+1}}, T) < 1/2^{2k}$ for every $k = 1, 2, \ldots$ If we put $A_k = \{t \in T, |g_{n_k}(t) - g_{n_{k+1}}(t)| > 1/2^k\}$, then $A_k \in \mathfrak{S}(\mathscr{P})$ and by the Corollary of Theorem 1, $\hat{m}(A_k) < 1/2^k$ for every k. Thus the sequence $\{g_{n_k}\}_{k=1}^{\infty}$ is almost uniformly $\hat{m}$ fundamental and therefore, see section 1.3 in [4], there is a measurable function g to which this sequence converges almost uniformly $\hat{m}$ and thus also almost everywhere m. We now prove that this sequence converges also in the L_1 norm to the function g.

Let us have a fixed k_0 and a non zero simple integrable function h fulfilling $|h(t)| \leqq |g(t) - g_{n_{k_0}}(t)|$ for every $t \in T$. Put $H = \{t \in T, |h(t)| > 0\}$, $c = \min\{|h(t)|, t \in H\}$ and $C = \max\{|h(t)|, t \in H\}$. Since the sequence $\{g_{n_k}\}_{k=1}^{\infty}$ converges almost uniformly $\hat{m}$ to the function g, there is a $k_h > k_0$ such that

$$\hat{m}\left(\overbrace{\left\{t \in T, |g_{n_{k_h}}(t) - g(t)| > \frac{c}{3}\right\}}^{A}\right) < \frac{1}{C \cdot 2^{k_0 - 2}}.$$

Obviously $A \in \mathfrak{S}(\mathscr{P})$. But then $|h(t)| \leqq 2 \cdot |g_{n_{k_h}}(t) - g_{n_{k_0}}(t)|$ for every $t \in H - A$, and therefore $|\int_{H-A} h \, dm| \leqq 2 \cdot \hat{m}(g_{n_{k_h}} - g_{n_{k_0}}, T) < 1/2^{k_0-2}$. Since $|\int_A h \, dm| \leqq C \cdot \hat{m}(A) < 1/2^{k_0-2}$, we have $|\int_T h \, dm| \leqq 1/2^{k_0-3}$. But k_0 and h were arbitrary, and thus we proved that the sequence $\{g_{n_k}\}_{k=1}^{\infty}$ converges in the L_1 norm to the function g.

If g_{n_k}, $k = 1, 2, \ldots$ are integrable functions, then the integrability of the function g follows from Theorem 16 in [4], since the sequence $\{g_{n_k}\}_{k=1}^{\infty}$ converges almost everywhere m to the function g and by the Corollary of Theorem 2 we have $|\int_E g_{n_k} \, dm - \int_E g_{n_{k_1}} \, dm| \leqq \hat{m}(g_{n_k} - g_{n_{k_1}}, T)$ for each set $E \in \mathfrak{S}(\mathscr{P})$.

The final assertion of the theorem is obvious.

Denote by $\tilde{\mathscr{P}}$ the collection of all sets $A \in \mathscr{P}$ for which $\lim_{n \to \infty} \hat{m}(A \cap E_n) = 0$ for any decreasing sequence of sets $E_n \searrow \emptyset$, $E_n \in \mathscr{P}$. Obviously $\tilde{\mathscr{P}}$ is a δ-subring of $\mathscr{P}$ and $A \cap E \in \tilde{\mathscr{P}}$ for any $A \in \tilde{\mathscr{P}}$ and $E \in \mathscr{P}$. We may say that $\tilde{\mathscr{P}}$ is the greatest δ-subring of $\mathscr{P}$ on which the semivariation $\hat{m}$ is continuous. Obviously, in general $\tilde{\mathscr{P}} \neq \mathscr{P}$. However, if Y is a weakly complete Banach space, then $\tilde{\mathscr{P}} = \mathscr{P}$, see the $*$-Theorem in section 1.1 in [4].

The following important theorem strengthens Lemma 1, see also the Corollary of Theorem 5.

Theorem 8. *Let g be a measurable function and let its L_1 norm $\hat{m}(g, .)$ be continuous on $\mathfrak{S}(\mathscr{P})$. Then there is a sequence of $\tilde{\mathscr{P}}$-simple functions converging in the L_1 norm to g.*

Proof. Since g is a measurable function, there is a sequence of simple integrable functions $\{g_n\}_{n=1}^{\infty}$ converging at each point $t \in T$ to $g(t)$ and such that the sequence $\{|g_n(t)|\}_{n=1}^{\infty}$ is non decreasing for each $t \in T$, see section 1.2 in [4]. We prove that the sequence $\{g_n\}_{n=1}^{\infty}$ converges in the L_1 norm to g.

Let $\varepsilon > 0$. Since $G_k = \{t \in T, |g(t)| > 1/k\} \in \mathfrak{S}(\mathscr{P})$, $k = 1, 2, \ldots$ is an increasing sequence of sets with $\bigcup_{k=1}^{\infty} G_k = \{t \in T, |g(t)| > 0\} \in \mathfrak{S}(\mathscr{P})$, the continuity of the L_1 norm $\hat{m}(g, .)$ on $\mathfrak{S}(\mathscr{P})$ implies the existence of an integer k_0 such that $\hat{m}(g - - g \,.\, \chi_{G_{k_0}}, T) < \varepsilon/6$. By the Corollaries of Theorems 1 and 5 $\hat{m}(G_{k_0}) \leqq k_0 \,.\, \hat{m}(g, T) < < +\infty$. Further, by the second part of Theorem 5 there is a finite non negative countably additive measure λ_g on $\mathfrak{S}(\mathscr{P})$ such that $\lim_{\lambda_g(E) \to 0} \hat{m}(g, E) = 0$, $E \in \mathfrak{S}(\mathscr{P})$. Take $\delta > 0$ in such a way that $\lambda_g(E) < \delta$ implies $\hat{m}(g, E) < \varepsilon/6$, $E \in \mathfrak{S}(\mathscr{P})$. According to Egoroff's theorem there is a set $A \in \mathfrak{S}(\mathscr{P})$ such that $\lambda_g(A) < \delta$ and on $G_{k_0} - A$ the sequence $\{g_n\}_{n=1}^{\infty}$ converges uniformly to g. Hence there is an integer n_0 such that $\|g_n - g\|_{G_{k_0} - A} < \varepsilon/(6k_0 \hat{m}(g, T) + 6)$ for every $n \geqq n_0$.

Since $|g_n(t)| \leqq |g(t)|$ for every $t \in T$ and every $n = 1, 2, \ldots$, by assertion d) of Theorem 1 $\hat{m}(g_n, E) \leqq \hat{m}(g, E)$ for each set $E \in \mathfrak{S}(\mathscr{P})$ and each n. From here and from the preceding inequalities, by subadditivity of the L_1 norm $\hat{m}(., .)$, we immediately have the inequality $\hat{m}(g - g_n, T) < \varepsilon$ for every $n \geqq n_0$. This proves that the sequence $\{g_n\}_{n=1}^{\infty}$ converges in the L_1 norm to g.

Since $|g_n(t)| \leqq |g(t)|$ for every $t \in T$ and every $n = 1, 2, \ldots$, and since the L_1 norm $\hat{m}(g, .)$ is continuous on $\mathfrak{S}(\mathscr{P})$, by assertion d) of Theorem 1 each g_n is in fact a $\tilde{\mathscr{P}}$-simple function. Thus the theorem is proved.

Definition 4. *By $\mathscr{L}_1\mathfrak{M}(m)$ or $\mathscr{L}_1\mathfrak{I}(m)$ we denote the set of all measurable or integrable functions g respectively, with $\hat{m}(g, T) < +\infty$. By $\mathscr{L}_1\mathfrak{I}_s(m)$ we denote the closure in the L_1 norm of the set of all simple integrable functions $\mathfrak{I}_s$ in $\mathscr{L}_1\mathfrak{M}(m)$. By $\mathscr{L}_1(m)$ we denote the set of all functions $g \in \mathscr{L}_1\mathfrak{M}(m)$ whose L_1 norms $\hat{m}(g, .)$ are continuous on $\mathfrak{S}(\mathscr{P})$ (By Theorem 8 $\mathscr{L}_1(m)$ is the closure of all $\tilde{\mathscr{P}}$-simple functions in $\mathscr{L}_1\mathfrak{M}(m)$). If we consider equivalent classes of functions, then we shall write L_1 instead of $\mathscr{L}_1$.*

Obviously, each of the set of functions mentioned above is a linear space. Moreover, from Theorems 7 and 8 we immediately have the following important

Theorem 9. *The spaces $L_1\mathfrak{M}(m)$, $L_1\mathfrak{I}(m)$, $L_1\mathfrak{I}_s(m)$ and $L_1(m)$ are Banach spaces and $L_1\mathfrak{M}(m) \supset L_1\mathfrak{I}(m) \supset L_1\mathfrak{I}_s(m) \supset L_1(m)$.*

It is easy to verify that in Example 7″ of § 2 in [4] all the above stated inclusions are proper. On the other hand from Theorem 5 we immediately have the following important result:

Theorem 10. *Let* $\boldsymbol{Y}$ *contain no subspace isomorphic to the space* $\boldsymbol{c}_0$, *for example let* $\boldsymbol{Y}$ *be a weakly complete Banach space. Then* $\mathscr{L}_1\mathfrak{M}(\boldsymbol{m}) = \mathscr{L}_1(\boldsymbol{m})$.

The following theorem is obvious from Theorem 8.

Theorem 11. $\mathscr{L}_1\mathfrak{I}_s(\boldsymbol{m}) = \mathscr{L}_1(\boldsymbol{m})$ *if and only if the semivariation* $\hat{\boldsymbol{m}}$ *is continuous on* $\mathscr{P}$, *i.e., if and only if* $\tilde{\mathscr{P}} = \mathscr{P}$.

If we restrict the measure $\boldsymbol{m}$ in Example 7″ in [4] to the δ-ring of all finite subsets of T, then its variation $v(\boldsymbol{m}, .)$ is finite on $\mathscr{P}$, while $L_1\mathfrak{I}(\boldsymbol{m}) \neq L_1\mathfrak{I}_s(\boldsymbol{m}) = L_1(\boldsymbol{m})$ (to the integrable function f constructed there, there is no sequence of simple integrable functions converging in the semivariation $\hat{\boldsymbol{m}}$, and therefore, see Lemma 4 above, no sequence of simple integrable functions converging in the L_1 norm).

The situations of Examples 2 and 3 from § 3 in [3] is worth to mentioning. For the so called second Dunford integral (Example 2, integration of vector functions with respect to a scalar measure), $L_1\mathfrak{M}(\boldsymbol{m}) = L_1(\boldsymbol{m})$ is just the space of Bochner integrable functions. In this Example 2 we found an example of an integrable function with infinite L_1 norm.

When considering the integration of scalar functions with respect to a vector measure, see Example 3 in [4], each integrable function has a finite L_1 norm and $\mathscr{L}_1\mathfrak{I}(\boldsymbol{m}) = \mathscr{L}_1(\boldsymbol{m})$. These facts may be proved in just the same way as IV.10.4 and IV.10.5 in [6], see also [10].

Theorem 12. *Let* φ *be an* $\boldsymbol{m}$ *essentially bounded scalar measurable function and let* $f \in \mathscr{L}_1\mathfrak{M}(\boldsymbol{m})$, $\mathscr{L}_1\mathfrak{I}(\boldsymbol{m})$, $\mathscr{L}_1\mathfrak{I}_s(\boldsymbol{m})$ *or* $\mathscr{L}_1(\boldsymbol{m})$. *Then* $\varphi . f \in \mathscr{L}_1\mathfrak{M}(\boldsymbol{m})$, $\mathscr{L}_1\mathfrak{I}(\boldsymbol{m})$, $\mathscr{L}_1\mathfrak{I}_s(\boldsymbol{m})$ *or* $\mathscr{L}_1(\boldsymbol{m})$ *respectively.*

Proof. The cases $f \in \mathscr{L}_1\mathfrak{M}(\boldsymbol{m})$ and $f \in \mathscr{L}_1(\boldsymbol{m})$ follow from assertion c) of Theorem 1. In the case $f \in \mathscr{L}_1\mathfrak{I}(\boldsymbol{m})$ we must use also Theorem 4 from [4]. In the case $f \in \mathscr{L}_1\mathfrak{I}_s(\boldsymbol{m})$ we make use of the well known fact that to each bounded scalar measurable function there is a sequence of simple ($\mathscr{P}$-simple) functions converging to it uniformly on the whole space T.

In general for the spaces $\mathscr{L}_1\mathfrak{M}(\boldsymbol{m})$ and $\mathscr{L}_1\mathfrak{I}(\boldsymbol{m})$ no analogs of classical convergence and separability theorems are valid, consider Example 7″ in [4]. Somewhat better the space $\mathscr{L}_1\mathfrak{I}_s(\boldsymbol{m})$ behaves. For this space we now state some analogs of important classical convergence theorems. Since they may be proved in the usual way, we omit their proofs. We note that the complete analogs of these classical convergence theorems are valid only for the space $\mathscr{L}_1(\boldsymbol{m})$, see Theorems 16 and 17 below. In Theorem 19 we prove that only the space $\mathscr{L}_1(\boldsymbol{m})$ may be separable.

We start with an analog of the Vitali convergence theorem, see Theorem 5 in [1], Theorems III.3.6 and III.6.15 in [6] and Theorem C of § 26 in [9].

Theorem 13. *Let* $f_n \in \mathscr{L}_1\mathfrak{I}_s(\boldsymbol{m})$, $n = 1, 2, \ldots$, *and let* f *be a measurable function. Then the following conditions are necessary and sufficient for the convergence of the sequence* $\{f_n\}_{n=1}^{\infty}$ *in the* L_1 *norm to the function* f:

(i) *The sequence* $\{f_n\}_{n=1}^{\infty}$ *converges in the semivariation* $\hat{\boldsymbol{m}}$ *on each set* $E \in \mathscr{P}$ *to the function* f,

(ii) *the set functions* $\hat{\boldsymbol{m}}(f_n, .)$, $n = 1, 2, \ldots$ *are uniformly absolutely* $\hat{\boldsymbol{m}}$ *continuous on* $\mathscr{P}$,

(iii) *for every* $\varepsilon > 0$ *there is a set* $A \in \mathscr{P}$ *such that* $\hat{\boldsymbol{m}}(f_n, T - A) < \varepsilon$ *for every* $n = 1, 2, \ldots$.

From here we immediately have the following

Corollary. *If* $f \in \mathscr{L}_1\mathfrak{I}_s(\boldsymbol{m})$, *then its* L_1 *norm* $\hat{\boldsymbol{m}}(f, .)$ *is absolutely* $\hat{\boldsymbol{m}}$ *continuous on* $\mathfrak{S}(\mathscr{P})$ *and to every* $\varepsilon > 0$ *there is a set* $A \in \mathscr{P}$ *such that* $\hat{\boldsymbol{m}}(f, T - A) < \varepsilon$.

Let us note that if $\mathscr{L}_1\mathfrak{I}(\boldsymbol{m}) \neq \mathscr{L}_1\mathfrak{I}_s(\boldsymbol{m})$, then it may happen that none of these properties is valid for a function $f \in \mathscr{L}_1\mathfrak{I}(\boldsymbol{m})$, see the paragraph after the Corollary of Theorem 5 above.

From Theorem 13 we easily obtain the following version of Lebesgue dominated convergence theorem in $\mathscr{L}_1\mathfrak{I}_s(\mathfrak{m})$, see Theorem 6 in [1], Theorems III.3.7 and III.6.18 in [6] and Theorem D of § 26 in [9].

Theorem 14. *Let* $f_n \in \mathscr{L}_1\mathfrak{I}_s(\boldsymbol{m})$, $n = 1, 2, \ldots$ *and let this sequence converge in the semivariation* $\hat{\boldsymbol{m}}$ *on each set* $E \in \mathscr{P}$ *to a measurable function* f. *Further let there exist a function* $g \in \mathscr{L}_1\mathfrak{I}_s(\boldsymbol{m})$ *such that* $|f_n(t)| \leqq |g(t)|$ *almost everywhere* $\boldsymbol{m}$ *for every* n. *Then* $f \in \mathscr{L}_1\mathfrak{I}_s(\boldsymbol{m})$ *and the sequence* $\{f_n\}_{n=1}^{\infty}$ *converges in the* L_1 *norm to the function* f.

It is worth noting that if $g \in \mathscr{L}_1\mathfrak{I}_s(\boldsymbol{m})$, $f \in \mathscr{L}_1\mathfrak{I}(\boldsymbol{m})$ and $|f(t)| \leqq |g(t)|$ almost everywhere $\boldsymbol{m}$, then it may happen that $f \notin \mathscr{L}_1\mathfrak{I}_s(\boldsymbol{m})$, see the function f in Example 7″ in [4].

The following version of bounded convergence theorem in $\mathscr{L}_1\mathfrak{I}_s(\boldsymbol{m})$ is a simple consequence of the preceding theorem, see also Theorem 7 in [1].

Theorem 15. *Let* $E \in \mathscr{P}$, *let a sequence* $f_n \in \mathscr{L}_1\mathfrak{I}_s(\boldsymbol{m})$, $n = 1, 2, \ldots$ *converge in the semivariation* $\hat{\boldsymbol{m}}$ *on* E *to a measurable function* f *and let there be an* $M > 0$ *such that* $|f_n(t)| \leqq M$ *almost everywhere* $\boldsymbol{m}$ *on* E *for every* n. *Then* $f . \chi_E \in \mathscr{L}_1\mathfrak{I}_s(\boldsymbol{m})$ *and the sequence* $\{f_n . \chi_E\}_{n=1}^{\infty}$ *converges in the* L_1 *norm to* $f . \chi_E$.

We now turn to the space $\mathscr{L}_1(\boldsymbol{m})$. By the Corollary of Theorem 4, $\mathscr{L}_1(\boldsymbol{m})$ generalizes the classical L_1 spaces. What is more important, we now show that the complete analogs of Vitali and Lebesgue convergence theorems are valid for $\mathscr{L}_1(\boldsymbol{m})$, see Theorems 16 and 17. Further we show that also complete analogs of separability and density theorems are valid for $\mathscr{L}_1(\boldsymbol{m})$, see Theorems 20, 21 and 22. At the same time the space $\mathscr{L}_1(\boldsymbol{m})$ is in general substantially larger then the space of Bochner integrable functions $\mathscr{L}_1(v(\boldsymbol{m}, .), X)$, the latter being treated in [3]. We note that if some of the spaces $\mathscr{L}_1\mathfrak{M}(\boldsymbol{m})$, $\mathscr{L}_1\mathfrak{I}(\boldsymbol{m})$ and $\mathscr{L}_1\mathfrak{I}_s(\boldsymbol{m})$ is different from $\mathscr{L}_1(\boldsymbol{m})$, then it does not share these properties.

We begin with the Vitali convergence theorem in $\mathscr{L}_1(\boldsymbol{m})$, see Theorem III.6.15 in [6] and Theorem C of § 26 in [9].

Theorem 16. Vitali convergence theorem in $\mathscr{L}_1(\boldsymbol{m})$. *Let a sequence $f_n \in \mathscr{L}_1(\boldsymbol{m})$, $n = 1, 2, \ldots$ converge almost everywhere $\boldsymbol{m}$ or in the measure $\boldsymbol{m}$ to a measurable function f. Then this sequence converges in the L_1 norm to the function f if and only if the L_1 norms $\hat{\boldsymbol{m}}(f, .)$, $n = 1, 2, \ldots$ are uniformly continuous on $\mathfrak{S}(\mathscr{P})$. In this case clearly $f \in \mathscr{L}_1(\boldsymbol{m})$.*

Proof. Since every sequence of measurable functions converging in the measure $\boldsymbol{m}$ to a measurable function contains a subsequence converging almost everywhere $\boldsymbol{m}$ to this function, see section 1.3 in [4], it is sufficient to prove the theorem for this case.

The necessity of the condition may be proved in the usual way, see for example the proof of Theorem C in § 26 in [9].

We now turn to the proof of sufficiency. Let the L_1 norms $\hat{\boldsymbol{m}}(f_n, .)$, $n = 1, 2, \ldots$ be uniformly continuous on $\mathfrak{S}(\mathscr{P})$ and let $\varepsilon > 0$. Since $F = \bigcup_{n=0}^{\infty} \{t \in T, |f_n(t)| > 0\} \in \mathfrak{S}(\mathscr{P})$, $f_0 = f$, see section 1.2 in [4], there is an increasing sequence of sets $F_k \in \mathscr{P}$, $k = 1, 2, \ldots$ with $\bigcup_{k=1}^{\infty} F_k = F$. Owing to the uniform continuity of the L_1 norms $\hat{\boldsymbol{m}}(f_n, .)$, $n = 1, 2, \ldots$ on $\mathfrak{S}(\mathscr{P})$ there is a k_0 such that $\hat{\boldsymbol{m}}(f_n, F - F_{k_0}) < \varepsilon/6$ for every $n = 1, 2, \ldots$.

Theorem 5 implies that for every $n = 1, 2, \ldots$ there is a finite non negative countably additive measure λ_n on $\mathfrak{S}(\mathscr{P})$ such that $\lambda_n(E) \leq \hat{\boldsymbol{m}}(f_n, E)$ and $\lim_{\lambda_n(E) \to 0} \hat{\boldsymbol{m}}(f_n, E) = 0$, $E \in \mathfrak{S}(\mathscr{P})$. Since the L_1 norms $\hat{\boldsymbol{m}}(f_n, .)$, $n = 1, 2, \ldots$ are uniformly continuous on $\mathfrak{S}(\mathscr{P})$, it follows that the measures λ_n, $n = 1, 2, \ldots$ are uniformly countably additive on $\mathfrak{S}(\mathscr{P})$. But then by Theorem 3.10 in [7], which by a slight modification of the proof given there (instead of λ_i consider the measures $\lambda_i/(1 + \sup_{A \in \Sigma} v(\lambda_i, A))$ is valid without the assumption of boundedness, there is a finite non negative countably additive measure λ on $\mathfrak{S}(\mathscr{P})$ such that $\lambda(E) \leq \sup \lambda_n(E)$, $E \in \mathfrak{S}(\mathscr{P})$, and the measures λ_n, $n = 1, 2, \dots$ are uniformly absolutely λ continuous on $\mathfrak{S}(\mathscr{P})$. Thus if $\lambda(N) = 0$, $N \in \mathfrak{S}(\mathscr{P})$, then $\lambda_n(N) = 0$ for every n, and therefore also $\hat{\boldsymbol{m}}(f_n, N) = 0$ for every n. From here and from the uniform continuity of the L_1 norms $\hat{\boldsymbol{m}}(f_n, .)$, $n = 1, 2, \dots$ on $\mathfrak{S}(\mathscr{P})$ it easily follows that the L_1 norms $\hat{\boldsymbol{m}}(f_n, .)$, $n = 1, 2, \dots$ are uniformly absolutely λ continuous on $\mathfrak{S}(\mathscr{P})$ (this may be proved in just the same way as Lemma 2 in [5]). Hence there is a $\delta > 0$ such that $\lambda(E) < \delta$, $E \in \mathfrak{S}(\mathscr{P})$ implies $\hat{\boldsymbol{m}}(f_n, E) < \varepsilon/6$ for every $n = 1, 2, \dots$.

If now $\|\boldsymbol{m}\|(N) = 0$, $N \in \mathfrak{S}(\mathscr{P})$, then obviously $\hat{\boldsymbol{m}}(f_n, N) = 0$ for every n, hence $\lambda_n(N) = 0$ for every n, and therefore $\lambda(N) = 0$. From here, since the sequence $\{f_n\}_{n=1}^{\infty}$ converges almost everywhere $\boldsymbol{m}$ to the function f, it converges also almost everywhere λ to f. According to Egoroff's theorem there is a set $A \in \mathfrak{S}(\mathscr{P})$ with $\lambda(A) < \delta$ such that on $F_{k_0} - A$ the sequence $\{f_n\}_{n=1}^{\infty}$ converges uniformly to the function f. Hence there is an n_0 such that for $n \geqq n_0$,

$$\|f_n - f\|_{F_{k_0}} < \frac{\varepsilon}{6} \cdot \frac{1}{1 + \hat{\boldsymbol{m}}(F_{k_0})}\,.$$

Using these inequalities, by Theorems 1 and 3 we obtain that for $n_1, n_2 \geqq n_0$ it holds $\hat{\boldsymbol{m}}(f_{n_1} - f_{n_2}, T) = \hat{\boldsymbol{m}}(f_{n_1} - f_{n_2}, F) \leqq \hat{\boldsymbol{m}}(f_{n_1}, F - F_{k_0}) + \hat{\boldsymbol{m}}(f_{n_2}, F - F_{k_0}) + \hat{\boldsymbol{m}}(f_{n_1} - f_{n_2}, F_{k_0} - A) + \hat{\boldsymbol{m}}(f_{n_1}, A) + \hat{\boldsymbol{m}}(f_{n_2}, A) \leqq \frac{2}{6}\varepsilon + \frac{2}{6}\varepsilon + \frac{2}{6}\varepsilon = \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we obtain that the sequence $\{f_n\}_{n=1}^{\infty}$ is fundamental in the L_1 norm. But then by Theorem 7 there is a functions $\boldsymbol{g} \in \mathscr{L}_1(\boldsymbol{m})$ to which this sequence converges in the L_1 norm. By Lemma 4 the sequence $\{f_n\}_{n=1}^{\infty}$ converges in the semivariation $\hat{\boldsymbol{m}}$ to the function $\boldsymbol{g}$, and therefore there is a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ converging almost everywhere $\boldsymbol{m}$ to the function $\boldsymbol{g}$, see section 1.3 in [4]. Since the sequence $\{f_{n_k}\}_{k=1}^{\infty}$ converges also almost everywhere $\boldsymbol{m}$ to the function f, $f = \boldsymbol{g}$ almost everywhere $\boldsymbol{m}$. Thus the theorem is proved.

From this theorem we easily obtain the complete analog of the Lebesgue dominated convergence theorem for $\mathscr{L}_1(\boldsymbol{m})$, see Theorem D of § 26 in [9] and Theorem III.6.18 in [6].

Theorem 17. Lebesgue dominated convergence theorem in $\mathscr{L}_1(\boldsymbol{m})$. *Let a sequence $\{f_n\}_{n=1}^{\infty}$ of measurable functions converge almost everywhere $\boldsymbol{m}$ or in the measure $\boldsymbol{m}$ to a measurable function f, let $\boldsymbol{g} \in \mathscr{L}_1(\boldsymbol{m})$ and let $|f_n(t)| \leqq |\boldsymbol{g}(t)|$ almost everywhere $\boldsymbol{m}$ for every $n = 1, 2, \dots$ Then for every $n = 0, 1, 2, \dots$ $f_n \in \mathscr{L}_1(\boldsymbol{m})$, $f_0 = f$, and the sequence $\{f_n\}_{n=1}^{\infty}$ converges in the L_1 norm to the function f.*

Proof. By assertion d) of Theorem 1 $\hat{\boldsymbol{m}}(f_n, E) \leqq \hat{\boldsymbol{m}}(\boldsymbol{g}, E)$ for every set $E \in \mathfrak{S}(\mathscr{P})$ and for every $n = 1, 2, \dots$ Since by the definition of $\mathscr{L}_1(\boldsymbol{m})$ the L_1 norm $\hat{\boldsymbol{m}}(\boldsymbol{g}, .)$ is continuous on $\mathfrak{S}(\mathscr{P})$, the L_1 norms $\hat{\boldsymbol{m}}(f_n, .)$, $n = 1, 2, \dots$ are uniformly continuous on $\mathfrak{S}(\mathscr{P})$. Thus $f_n \in \mathscr{L}_1(\boldsymbol{m})$ for every $n = 1, 2, \dots$, and according to Theorem 16 the sequence $\{f_n\}_{n=1}^{\infty}$ converges in the L_1 norm to the function f. But then $f \in \mathscr{L}_1(\boldsymbol{m})$, and the theorem is proved.

We now turn to separability properties of our spaces. For $E, F \in \mathscr{P}$ let us put $\varrho(E, F) = \hat{m}(E \triangle F)$, where $E \triangle F = (E - F) \cup (F - E)$ is the symmetric difference of the sets E and F. Then obviously $(\mathscr{P}, \varrho)$ is a semimetric space. Let us note here that only the semimetric space $(\mathscr{P}_1, \varrho)$ is complete. Here, as we now, $\mathscr{P}_1$ denotes the δ-ring of all sets from $\mathfrak{S}(\mathscr{P})$ with finite semivariation $\hat{m}$, see section 1.1 in [4].

We begin with two interesting theorems.

Theorem 18. *Let the semimetric space $(\mathscr{P}, \varrho)$ be separable. Then the semivariation $\hat{m}$ is continuous on $\mathscr{P}$.*

Proof. Suppose that the semivariation $\hat{m}$ is not continuous on $\mathscr{P}$. Then in the proof of $*$-Theorem in section 1.1 in [4] we find and $\varepsilon > 0$ and a sequence of disjoint sets $B_n \in \mathscr{P}$, $n = 1, 2, \ldots$, $\bigcup_{n=1}^{\infty} B_n \in \mathscr{P}$, such that $\hat{m}(B_n) > \varepsilon$ for every n ($B_n = A_{k_{n-1}} - A_{k_n}$ in the notation of the proof of $*$-Theorem). For any subset J of the set of all positive integers N let us put $B_J = \bigcup_{n \in J} B_n$. Obviously $B_J \in \mathscr{P}$ for any $J \subset N$ and $\varrho(B_{J_1}, B_{J_2}) = \hat{m}(B_{J_1} \triangle B_{J_2}) > \varepsilon$ for $J_1 \neq J_2$. From here, since $\{B_J, J \subset N\}$ is an uncountable family of elements of $\mathscr{P}$, the semimetric space $(\mathscr{P}, \varrho)$ cannot be separable, a contradiction. In this way the theorem is proved.

In the same way we may prove the following

Theorem 19. *If any of the spaces $\mathscr{L}_1\mathfrak{M}(\boldsymbol{m})$, $\mathscr{L}_1\mathfrak{I}(\boldsymbol{m})$ or $\mathscr{L}_1\mathfrak{I}_s(\boldsymbol{m})$ is separable, then this space is equal to $\mathscr{L}_1(\boldsymbol{m})$.*

Consequently, only the space $\mathscr{L}_1(\boldsymbol{m})$ may be separable. In Theorem 8 we proved that to each function $\boldsymbol{f} \in \mathscr{L}_1(\boldsymbol{m})$ there is a sequence of $\tilde{\mathscr{P}}$-simple functions converging to it in the L_1 norm ($\tilde{\mathscr{P}}$ was defined before Theorem 8). Using this fact we immediately have:

Theorem 20. *A non trivial space $\mathscr{L}_1(\boldsymbol{m})$ is separable if and only if the spaces $(\tilde{\mathscr{P}}, \varrho)$ and $\boldsymbol{X}$ are separable.*

From Theorem D of § 13 in [9] and from the second part of $*$-Theorem in section 1.1 in [4] we easily obtain the following result concerning the separability of the space $(\tilde{\mathscr{P}}, \varrho)$:

Theorem 21. *Let the δ-ring $\tilde{\mathscr{P}}$ be generated by a countable family of sets. Then the semimetric space $(\tilde{\mathscr{P}}, \varrho)$ is separable.*

Let now T be a locally compact Hausdorff topological space. By $\mathfrak{Q}$ we denote the set of all functions of the form $\boldsymbol{f} = \sum_{i=1}^{r} \varphi_i \boldsymbol{x}_i$, where $\boldsymbol{x}_i \in \boldsymbol{X}$ and φ_i is a continuous scalar function with compact support in T, for every $i = 1, 2, \ldots, r$. $C_0(T, X)$ denotes the closure of $\mathfrak{Q}$ in the uniform norm $\|\,.\,\|_T$ in the space of all bounded X valued functions on T, see Theorem 1 in [5]. In connection with the representation theorems of § 2 in [5] the following density theorem is of importance:

Theorem 22. *Let $\boldsymbol{m}$ be a Baire operator valued measure and let its semivariation $\hat{\boldsymbol{m}}$ be continuous on $\mathscr{B}_0$. Then $\mathfrak{Q}$ and therefore also $C_0(T, X)$ is dense in $\mathscr{L}_1(\boldsymbol{m})$.*

Proof. By Theorem 8 it is enough to prove that to every $\mathscr{B}_0$-simple function $f = \sum_{i=1}^{r} x_i \,.\, \chi_{E_i}$, $E_i \in \mathscr{B}_0$, $x_i \in X$, $i = 1, 2, \ldots, r$ and to every $\varepsilon > 0$ there is a function $f_1 \in \mathfrak{Q}$ such that $\hat{\boldsymbol{m}}(f_1 - f, T) < \varepsilon$. We proceed as in the proof of Theorem 1 in [5]. According to Theorem D of § 50 in [9] there is a relatively compact open Baire set U such that $\bigcup_{i=1}^{r} \bar{E}_i \subset U$, where $\bar{E}_i$ denotes the closure of E_i in T. Denote by $\mathscr{S}_1$ the σ-ring of all sets of the form $E = U \cap F$, where $F \in \mathscr{B}_0$. Since by assumption of the theorem the semivariation $\hat{\boldsymbol{m}}$ is continuous on $\mathscr{S}_1$, by Lemma 2 in [5] there is a finite non negative countably additive measure λ on $\mathscr{S}_1$ with the properties: $\lambda(E) \leqq \hat{\boldsymbol{m}}(E)$ and $\lim_{\lambda(E)\to 0} \hat{\boldsymbol{m}}(E) = 0$, $E \in \mathscr{S}_1$. Take $\delta > 0$ such that $\lambda(E) < \delta$, $E \in \mathscr{S}_1$ implies

$$\hat{\boldsymbol{m}}(E) < \frac{\varepsilon}{r \,.\, (\|f\|_T + 1)}\,.$$

Since the measure λ is regular on the measurable space $(U, \mathscr{S}_1)$, see Theorem G of § 52 in [9], there are compact in the relative topology of U sets C_i and open (in T, since U is open) set U_i, $C_i \subset E_i \subset U_i \subset U$, with $\lambda(U_i - C_i) < \delta$ for every $i = 1, 2, \ldots, r$. According to Theorem B of § 50 in [9] for every $i = 1, 2, \ldots, r$ there is a continuous real function φ_i on T, $0 \leqq \varphi_i(t) \leqq 1$ for every $t \in T$, such that $\varphi_i(t) = 1$ for $t \in C_i$ and $\varphi_i(t) = 0$ for $t \in T - U_i$. If we now put $f_1 = \sum_{i=1}^{r} \varphi_i \,.\, x_i$, then $f_1 \in \mathfrak{Q}$ and it is easy to see that $\hat{\boldsymbol{m}}(f_1 - f, T) \leqq \sum_{i=1}^{r} \hat{\boldsymbol{m}}(U_i - C_i) < \varepsilon$. Thus the theorem is proved.

At the end of section 1.3 in [4] we deduced that if $\boldsymbol{m}$ is a regular Borel operator valued measure countably additive in the uniform operator topology, then to every Borel measurable function f there is a Baire measurable function f_1 such that $f = f_1$ almost everywhere $\boldsymbol{m}$. From here we immediately have:

Theorem 23. *Let $\boldsymbol{m}$ be a regular Borel operator valued measure countably aditive in the uniform operator topology. Denote by $\boldsymbol{m}_0$ its Baire restriction. Then $L_1(\boldsymbol{m}) = L_1(\boldsymbol{m}_0)$ and the same is true for the spaces $L_1\mathfrak{M}(\boldsymbol{m})$, $L_1\mathfrak{I}(\boldsymbol{m})$ and $L_1\mathfrak{I}_s(\boldsymbol{m})$.*

It remains an open problem if the conclusions of this theorem are valid also when $\boldsymbol{m}$ is a regular Borel operator valued measure countably additive only in the strong operator topology.

In this short section we indicate how to generalize the preceding theory to the theory of L_p spaces for $1 \leqq p < +\infty$ and to the theory of Orlicz spaces. Without mentioning we shall suppose that $1 \leqq p, q < +\infty$ and $1/p + 1/q = 1$. We begin with the definition of the L_p norm, see Definition 1.

Definition 1′. *Let $\boldsymbol{g}$ be a measurable function and let $E \in \mathfrak{S}(\mathscr{P})$. Then the L_p norm of the function $\boldsymbol{g}$ on the set E, which will be denoted by $\hat{\boldsymbol{m}}_p(\boldsymbol{g}, E)$, is defined by the equality:*

$$\hat{\boldsymbol{m}}_p(\boldsymbol{g}, E) = \sup \left\{ \left| \int_E f \, \mathrm{d}\boldsymbol{m} \right|^{1/p}, \; f \in \mathfrak{J}_s, \; |f(t)| \leqq |\boldsymbol{g}(t)|^p \text{ for each } t \in E \right\}.$$

The L_p norm of $\boldsymbol{g}$ is defined by $\hat{\boldsymbol{m}}_p(\boldsymbol{g}, E) = \sup\limits_{E \in \mathfrak{S}(\mathscr{P})} \hat{\boldsymbol{m}}_p(\boldsymbol{g}, E)$.

Obviously $\hat{\boldsymbol{m}}_1(.,.) = \hat{\boldsymbol{m}}(.,.)$. From this definition we immediately see that $\hat{\boldsymbol{m}}_p(\boldsymbol{g}, E) = (\hat{\boldsymbol{m}}(|\boldsymbol{g}|^{p-1} . \boldsymbol{g}, E))^{1/p}$. Hence the properties of the L_p norm $\hat{\boldsymbol{m}}_p(.,.)$ may be deduced from the properties of the L_1 norm $\hat{\boldsymbol{m}}(.,.)$. Thus for example we have the following sort of Tschebyscheff inequality, see the Corollary of Theorem 1:

$$\hat{\boldsymbol{m}}(\{t \in E, \; |\boldsymbol{g}(t)| \geqq a > 0\}) \leqq \frac{(\hat{\boldsymbol{m}}_p(\boldsymbol{g}, E))^p}{a^p}.$$

Further, from this equality we immediately have the following useful generalization of Theorem 4:

Theorem 4′. *For every measurable function $\boldsymbol{g}$ and every set $E \in \mathfrak{S}(\mathscr{P})$*

$$\hat{\boldsymbol{m}}_p(\boldsymbol{g}, E) = \sup_{|y^*| \leqq 1} \left(\int_E |\boldsymbol{g}|^p \, \mathrm{d}v(\boldsymbol{y}^*\boldsymbol{m}, .) \right)^{1/p}, \quad \boldsymbol{y}^* \in \boldsymbol{Y}^*.$$

Using this equality and the classical triangle inequality for scalar L_p spaces we immediately have the triangle inequality $\hat{\boldsymbol{m}}_p(\boldsymbol{f} + \boldsymbol{g}, E) \leqq \hat{\boldsymbol{m}}_p(\boldsymbol{f}, E) + \hat{\boldsymbol{m}}_p(\boldsymbol{g}, E)$ for our L_p norms.

Using these facts it is easy to verify that the whole above given theory of L_1 spaces may be generalized to the theory of L_p spaces. Namely, if everywhere in § 2 we write $\hat{\boldsymbol{m}}_p(.,.)$ instead of $\hat{\boldsymbol{m}}(.,.)$ and $\mathscr{L}_p(L_p)$ instead of $\mathscr{L}_1(L_1)$, then all lemmas and theorems are valid except two facts. First, if $\mathscr{L}_p\mathfrak{J}(\boldsymbol{m})$ denotes the closure of the set of all integrable functions with finite L_p norms in $\mathscr{L}_p\mathfrak{M}(\boldsymbol{m})$, then for the general case the author has not succeeded in proving that every element of $\mathscr{L}_p\mathfrak{J}(\boldsymbol{m})$ is an integrable function when $p > 1$ (i.e., that the assertion of the generalized Theorem 7 concerning integrable functions is in general valid when $p > 1$). Second, the generalized Theorem 12 will have the following form:

Theorem 12′. *Let* φ *be a measurable scalar function and let there be an* $x \in X$, $x \neq 0$ *such that* $\varphi . x \in \mathscr{L}_q \mathfrak{M}(m)$. *Let further* $f \in \mathscr{L}_p \mathfrak{M}(m)$ *or* $f \in \mathscr{L}_p(m)$. *Then* $\varphi . f \in \mathscr{L}_1 \mathfrak{M}(m)$ *or* $\varphi . f \in \mathscr{L}_1(m)$ *respectively. Further, if there is an* $x \in X$, $x \neq 0$ *such that* $\varphi . x \in \mathscr{L}_q \mathfrak{J}_s(m)$ *and* $f \in \mathscr{L}_p \mathfrak{J}_s(m)$, *then* $\varphi . f \in \mathscr{L}_1 \mathfrak{J}_s(m)$.

This theorem easily follows from its classical scalar version by the Theorem 4′ stated above. The author again does not know what is the situation when $f \in \mathscr{L}_p \mathfrak{J}(m)$.

Using these methods it is easy to verify that the whole above given theory of L_p spaces may be further generalized to the theory of Orlicz spaces. We omit the details, see the scalar case for example in [16].

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